Capability Process on a Two-Sided Extended EWMA Control Chart for Moving Average with Exogenous Factors Model

Yupaporn Areepong and Saowanit Sukparungsee

Abstract**—**Extended Exponentially Weighted Moving Average (Extended EWMA) control chart was developed to detect small changes and effectively detect changes quickly, especially with correlated data. Average Run Length (*ARL*) is a crucial feature measure used to measure the performance of the control chart. This research presents explicit formulas for in-control ARL (ARL_0) and out-of-control ARL (ARL_1) for a two-sided Extended EWMA control chart for moving average with exogenous factors (MAX) model. In addition, the results of the *ARL* obtained by the explicit formula were checked against the *ARL* calculated from the numerical integral equation (NIE) method. The results showed that both methods showed very consistent *ARL* values. The performance of the Extended EWMA control chart was compared with Exponentially Weighted Moving Average (EWMA) and Cumulative Sum (CUSUM) control charts. The performance comparison showed that the Extended EWMA control chart outperformed because of the lowest ARL_1 at each change level. Crude palm oil data prices were applied to the control charts to illustrate their effectiveness in detecting changes.

Index Terms**—Exponential white noise, explicit formula, average run length, numerical integral equation**

I. Introduction

STATISTICAL process control (SPC) is a quality control method used to monitor and control a process, to maintain its stability and consistency over time. SPC uses statistical methods to analyze data, detect any changes or variations in the process, and take corrective action if necessary. SPC can be applied to any process that produces measurable data, including manufacturing, service industries, and administrative processes. Control charts are one of the critical tools in SPC widely used in a wide range of industries and processes, including manufacturing, healthcare, service industries, and administrative processes [1], [2], and [3]. A Cumulative Sum (CUSUM) [4] and an Exponentially Weighted Moving Average (EWMA) [5] control charts are used in statistical process control (SPC) to detect small and gradual shifts in the process output. It is beneficial for detecting changes that occur slowly over time rather than sudden or abrupt changes. It is handy for detecting small to gradual shifts in the process mean or average. Later, an Extended Exponential Weighted Moving Average (Extended EWMA) control chart, which Naveed [6], introduced is a statistical process control tool that uses to monitor a process over time for any out-of-control signals. The control chart is designed to detect small shifts in the process mean or changes in the variability of the process.

Autocorrelation refers to the tendency of consecutive data points in a time series to correlate. Control charts are statistical tools used to monitor a process and detect when it goes out of control. One approach to address autocorrelation in a control chart is to use specialized methods considering the correlation between consecutive data points. For example, the autoregressive and moving average (ARMA) control chart is a method that uses a combination of time-series models and control charts to monitor a process with autocorrelated data. Besides, the exogenous variables can include any external factors that may influence the dependent variable, such as economic indicators [7] and [8]. Examples of a time series model are autoregressive with exogenous factors model *(ARX(p,r))* and moving average with exogenous factors model *(MAX(q,r))*.

The average run length (*ARL*) measures the expected number of observations before a control chart signals a change in the monitoring process. There are two components: ARL_0 (in-control *ARL*) and ARL_1 (out-of-control *ARL*). ARL_0 refers to the expected number of observations or samples that will be collected before a control chart signals a false alarm and should be as large as possible, while ARL_1 represents the average number of samples that will be collected before the control chart signals that the process has changed and should be as small as possible. Nowadays, there are various methods to calculate *ARL,* such as Monte Carlo simulation, Markov chain approach, numerical integration equation (NIE), and *ARL* explicit formulas [9]. Roberts [5] proposed the exponentially weighted moving average (EWMA) control chart by using Monte Carlo simulation to estimate the *ARL*, which helps detect the smaller as compared to the Shewhart control chart. Champ and Rigdon [10] approximated *ARL* by the Markov chain approach and numerical integral equation using the midpoint rule for *ARL* on CUSUM and EWMA control charts. Sunthornwat and Areepong [11] derived

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explicit expressions of the average run length (*ARL*). They evaluated the results against the numerical integral equation (NIE) method on the cumulative sum (CUSUM) control chart for seasonal and non-seasonal moving average processes with exogenous variables. In addition, the explicit formula method is extremely useful for finding optimal parameters and applied with the empirical data of a stock price from the stock exchange of Thailand. The resulting performance efficiency is compared with an exponentially weighted moving average (EWMA) control chart. Supharakonsakun et al. [12] investigated explicit formulas of *ARL* on a modified exponentially weighted moving average (EWMA) control chart for *MA(1)* process with exponential white noise. The accuracy of the *ARL* obtained with the modified EWMA control chart was compared to the numerical integral equation method, and compare the performance with the EWMA control chart. The results show that the *ARL* obtained by the explicit formulas and the numerical integral equation method are in close agreement. Phanyaem [13] solved explicit formulas of average run length (*ARL*) and developed numerical integration for the *ARL* of the EWMA control chart on an autoregressive integrated moving average, *ARIMA(p,d,q)* model. The accuracy of the proposed formulas is presented by comparing them to the numerical integration method. The results show that in terms of computational time, the explicit formula can reduce the computational time better than the numerical integration. Recently, Kochaporn and Areepong [14] proposed *ARL* explicit formulas on an Extended EWMA control chart for a seasonal autoregressive process of order p $(SAR(p)_L)$ with exponential white noise. The efficiency of the extended EWMA control chart was also compared with the EWMA control chart utilizing the explicit formulas technique for the *ARL*. Supharakonsakun and Areepong [15] compared the performance of the modified Exponentially Weighted Moving Average (modified EWMA) and EWMA control charts via average run lengths (*ARLs*) computed by using explicit formulas and the numerically integrated equation (NIE) technique for detecting shifts in the mean of an autoregressive process with exogenous variables *(ARX(p,r))* model. Kotchaporn et al. [16] derived the explicit formulas for the *ARL* and compared them to the numerical integral equation (NIE) method on a two-sided extended EWMA control chart for the trend *AR(p)* model with exponential white noise. Peerajit and Areepong [17] proposed formulas for the average run length (*ARL*) on the modified EWMA control chart for detecting small-to-moderate shifts in the process mean of an autoregressive fractionally integrated *(ARFI(p, d))* process with exponential white noise.

Therefore, it is interesting to study the explicit formula of ARL when the data has a *MAX(q,r)* model on the Extended EWMA control chart. The Fredholm-type integral equations were applied to derive explicit formulas for ARL_0 and ARL_1. This paper is organized as follows. Control charts and their properties are given in Section II. The derivation of *ARL* by explicit formulas and the NIE method of the *MAX(q,r)* process on the Extended EWMA control chart is proposed in Section III. Next, the numerical results for the ARL of the *MAX(q,r)* processes with exponential white noise by using explicit formulas and the NIE method are presented in Section IV. The efficiency performance between CUSUM, EWMA, and Extended EWMA control charts are also compared in Section IV. Furthermore, the applications of the proposed explicit formulas with real data is reported in Section V. Finally, conclusions are given in Section VI.

II. Control Charts and Their Properties

A. The CUSUM Control Chart

A Cumulative Sum control chart, or a CUSUM control chart, is a statistical tool that monitors the process performance over time. It is used to detect small shifts in the process mean or variance. The CUSUM control chart can be defined as

$$C_t = \max\{0,\, C_{t-1} + Y - a\};\; t = 1,2,3,\,..., \tag{1}$$

where C_t is the CUSUM statistic, Y_t is the sequence of the *MAX(q,r)* process with exponential white noise, a is usually called a reference value. $C_0 = u$ is the initial value when $u \in [e, f]$, where e and f are the upper control limit (UCL) and lower control limit (LCL).

The stopping time of the two-sided CUSUM control chart (τ) is given by

$$\tau = \inf\{t \ge 0 : C_t < e \text{ or } C_t > f\}.$$

B. The Exponentially Weighted Moving Average Control Chart (EWMA)

An Exponentially Weighted Moving Average (EWMA) control chart is a statistical process control tool used to monitor a process over time. It is similar to a traditional Shewhart control chart but emphasizes recent data more by giving more weight to the most recent observations. The EWMA control chart can be expressed by the recursive equation

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t,\; t = 1,2,... \tag{2}$$

where Z_t is the EWMA statistic, λ is an exponential smoothing parameter with $0 < \lambda < 1$ and Z_0 is the initial value of EWMA statistics, $Z_0 = u.$ The upper control limit (*UCL*) and lower control limit (*LCL*) of EWMA control charts are given by

$$UCL = d = \mu_0 + L_1\sigma\sqrt{\frac{\lambda}{2-\lambda}},\; LCL = b = \mu_0 - L_1\sigma\sqrt{\frac{\lambda}{2-\lambda}},$$

where μ_0 is the target mean, σ is the process standard deviation, and L_1 is the suitable control limit width. The stopping time of the two-sided EWMA control chart (τ) is given by $\tau = \inf\{t \ge 0 : Z_t < b \text{ or } Z_t > d\}$.

C. The Extended Exponentially Weighted Moving Average Control Chart (Extended EWMA)

The Extended EWMA control chart was presented by Naveed et al. [6]. It is developed from the EWMA control chart and effectively monitors and detects small changes in the process mean. The Extended EWMA control chart can be expressed by the recursive equation below.

$$E_t = (1-\lambda_1+\lambda_2)E_{t-1} + \lambda_1 Y_t - \lambda_2 Y_{t-1},\; t = 1,2,... \tag{3}$$

where λ_1 and λ_2 are exponential smoothing parameters with ($0<\lambda_1\le 1$) and $(0\le\lambda_2<\lambda_1)$, and the initial value is a constant, $E_0=u$. The upper control limit (*UCL*) and lower control limit (*LCL*) of the Extended EWMA control chart are given by

$$UCL=l=\mu_0+L_2\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

$$LCL=h=\mu_0-L_2\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

where μ_0 is the target mean, σ is the process standard deviation, and L_I is the suitable control limit width.

The stopping time of the two-sided Extended EWMA control chart (τ) is given by $\tau=\inf\{t\ge 0: Z_t<h \text{ or } Z_t>l\}$, where τ is the stopping time.

III. Explicit Formulas for the Average Run Length of MAX Process On Extended EWMA Control Chart

A. The MAX(q, r) Process

A moving average with exogenous variables is a statistical model that estimates the relationship between a dependent variable and one or more independent variables. It is commonly used in time series analysis to account for the influence of external factors on the dependent variable and is defined as

$$Y_t=\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}+\sum_{j=1}^{r}\beta_jX_{jt};\ t=1,2,3,... \quad (4)$$

where μ is a constant, θ_i is moving average coefficient as a real-valued constant for $i=1,2,...,q$, ε_t are independent and identically distributed *(i.i.d.)* observations from exponential distribution $(\varepsilon_t\sim Exp(\alpha))$, X_{it} are explanatory variables of Y_t. The initial value for the *MAX(q,r)* process is $\varepsilon_{t-1},\varepsilon_{t-2},...,\varepsilon_{t-q}$.

B. Explicit Formulas for the ARL of MAX(q,r) Process

From the recursion of Extended EWMA statistic in Equation 2 as follows

$$E_t=(1-\lambda_1+\lambda_2)E_{t-1}+\lambda_1Y_t-\lambda_2Y_{t-1}.$$

The Extended EWMA statistic for *MAX(q,r)* process can be written as,

$$E_t=(1-\lambda_1+\lambda_2)E_{t-1}+\lambda_1(\mu+\varepsilon_t-\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}+\sum_{j=1}^{r}\beta_jX_{jt})-\lambda_2Y_{t-1}.$$

Consider the in-control process, given *LCL = h, UCL = l* initial value $E_0=u$ and

$$h<E_t<l$$

$$h<(1-\lambda_1+\lambda_2)u+\lambda_1(\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}+\sum_{j=1}^{r}\beta_jX_{jt})+\lambda_1\varepsilon_t-\lambda_2Y_{t-1}<l$$

$$\frac{h-(1-\lambda_1+\lambda_2)u+\lambda_2Y_{t-1}}{\lambda_1}-\mu+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\beta_jX_{jt}<\varepsilon_t$$

$$<\frac{l-(1-\lambda_1+\lambda_2)u+\lambda_2Y_{t-1}}{\lambda_1}-\mu+\sum_{i=1}^{q}\theta_i\varepsilon_{t-i}-\sum_{j=1}^{r}\beta_jX_{jt}.$$

The stopping time $\tau=\inf\{t>0;\ E_t<h \text{ or } E_t>l\}$ then the *ARL* is defined as,

$$ARL=M(u)=E_\infty(\tau).$$

We study the change-point time at $t=1$, then we set $Y_0=v$ and $\varepsilon_0=e$. Therefore, $M(u)$ can be expressed by Fredholm integral equation of the second kind as follows,

$$M(u)=1+\int_{\frac{h-(1-\lambda_1+\lambda_2)u+\lambda_2v}{\lambda_1}-\mu+\theta_1e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_jX_{j1}}^{\frac{l-(1-\lambda_1+\lambda_2)u+\lambda_2v}{\lambda_1}-\mu+\theta_1e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_jX_{j1}}M(E_1)f(\varepsilon_1)d\varepsilon_1. \quad (5)$$

Let,

$$w=E_1=(1-\lambda_1+\lambda_2)u-\lambda_2v+\lambda_1(\mu-\theta_1e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_jX_{j1})+\lambda_1\varepsilon_1,$$

then $\frac{dw}{d\varepsilon_1}=\lambda_1,\ d\varepsilon_1=\frac{1}{\lambda_1}dw.$

After changing the variable, Equation (5) can be rewritten as

$$M(u)=1+\frac{1}{\lambda_1}\int_h^l M(w)f(\varepsilon_1)dw$$
$$=1+\frac{1}{\lambda_1}\int_h^l M(w)f(\frac{w-(1-\lambda_1+\lambda_2)u+\lambda_2v}{\lambda_1}-\mu$$
$$+\theta_1e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_jX_{j1})dw. \quad (6)$$

Since we determine $\varepsilon_1\sim Exp(\alpha)$, then $f(x)=\frac{1}{\alpha}e^{-\frac{x}{\alpha}}$.

Thus,

$$M(u)=1+\frac{1}{\lambda_1}\int_h^l M(w)\frac{1}{\alpha}e^{\frac{-w+(1-\lambda_1+\lambda_2)u-\lambda_2v}{\alpha\lambda_1}+\frac{\mu-\theta_1e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_jX_{j1}}{\alpha}}dw$$

$$=1+\frac{1}{\alpha\lambda_1}e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2v}{\alpha\lambda_1}+\frac{\mu-\theta_1e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_jX_{j1}}{\alpha}}\int_h^l M(w)e^{\frac{-w}{\alpha\lambda_1}}dw. \quad (7)$$

Let,

$$D(u)=e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2v}{\alpha\lambda_1}+\frac{\mu-\theta_1e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_jX_{j1}}{\alpha}},\ N=\int_h^l M(w)e^{\frac{-w}{\alpha\lambda_1}}dw.$$

Then,

$$M(u)=1+\frac{D(u)}{\alpha\lambda_1}N. \quad (8)$$

Consider,

$$N=\int_h^l M(w)e^{\frac{-w}{\alpha\lambda_1}}dw=\int_h^l\left(1+\frac{D(w)}{\alpha\lambda_1}N\right)e^{\frac{-w}{\alpha\lambda_1}}dw$$

$$=-\alpha\lambda_1e^{\frac{-w}{\alpha\lambda_1}}\Big|_h^l+\frac{N}{\alpha\lambda_1}\int_h^l e^{\frac{(1-\lambda_1+\lambda_2)w-\lambda_2v}{\alpha\lambda_1}+\frac{\mu-\theta_1e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_jX_{j1}}{\alpha}}\cdot e^{\frac{-w}{\alpha\lambda_1}}dw$$

$$=-\alpha\lambda_1(e^{\frac{-l}{\alpha\lambda_1}}-e^{\frac{-h}{\alpha\lambda_1}})+\frac{N}{\alpha\lambda_1}e^{\frac{-\lambda_2v}{\alpha\lambda_1}+\frac{\mu-\theta_1e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_jX_{j1}}{\alpha}}\int_h^l e^{\frac{-(\lambda_1-\lambda_2)w}{\alpha\lambda_1}}dw$$

$$N=-\alpha\lambda_1(e^{\frac{-l}{\alpha\lambda_1}}-e^{\frac{-h}{\alpha\lambda_1}})-\frac{N}{(\lambda_1-\lambda_2)}e^{\frac{-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}(e^{\frac{-(\lambda_1-\lambda_2)l}{\alpha\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha\lambda_1}}).$$

Then, it can be written as

$$N=\frac{-\alpha\lambda_1(\lambda_1-\lambda_2)(e^{\frac{-l}{\alpha\lambda_1}}-e^{\frac{-h}{\alpha\lambda_1}})}{(\lambda_1-\lambda_2)+e^{\frac{-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}(e^{\frac{-(\lambda_1-\lambda_2)l}{\alpha\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha\lambda_1}})}.$$

Substituting N in Equation (8), we have

$$M(u)=1-\frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\alpha\lambda_1}}(e^{\frac{-l}{\alpha\lambda_1}}-e^{\frac{-h}{\alpha\lambda_1}})}{(\lambda_1-\lambda_2)e^{\frac{\lambda_2 v}{\alpha\lambda_1}-(\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha})}+(e^{\frac{-(\lambda_1-\lambda_2)l}{\alpha\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha\lambda_1}})}. \quad (9)$$

Therefore, the explicit formulas for the *ARL* of the *MAX(q,r)* process running on the Extended EWMA control chart can be solved using Fredholm integral equations of the second kind. When the process is in the in control state with exponential parameter $\alpha=\alpha_0$, the explicit solution ARL_0 can be obtained as follows:

$$ARL_0=1-\frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\alpha_0\lambda_1}}(e^{\frac{-l}{\alpha_0\lambda_1}}-e^{\frac{-h}{\alpha_0\lambda_1}})}{(\lambda_1-\lambda_2)e^{\frac{\lambda_2 v}{\alpha_0\lambda_1}-(\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha_0})}+(e^{\frac{-(\lambda_1-\lambda_2)l}{\alpha_0\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha_0\lambda_1}})}. \quad (10)$$

Additionally, when the process is in an out of control state with an exponential parameter $\alpha=\alpha_1$, the explicit solution can be written as

$$ARL_1=1-\frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\alpha_1\lambda_1}}(e^{\frac{-l}{\alpha_1\lambda_1}}-e^{\frac{-h}{\alpha_1\lambda_1}})}{(\lambda_1-\lambda_2)e^{\frac{\lambda_2 v}{\alpha_1\lambda_1}-(\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha_1})}+(e^{\frac{-(\lambda_1-\lambda_2)l}{\alpha_1\lambda_1}}-e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha_1\lambda_1}})}. \quad (11)$$

Theorem 1: Banach's Fixed-Point Theorem

Let $T: X \to X$ represent a mapping of contractions with the contraction constant $g\in[0,1)$, and let X represent a whole metric space. There is a unique $M(\cdot)\in X$, and then $T(M(u))=M(u)$, i.e., a unique fixed-point in *X*. Next step, M_1, M_2 is given to be a solution to Equation (5) for all $M_1, M_2 \in X$, $\|T(M_1)-T(M_2)\|\le g\|M_1-M_2\|$ as is proved below.

Proof: Let T be a contraction mapping as specified in Equation (5) for all $M_1, M_2\in u[h,l]$

Thus, $\|T(M_1)-T(M_2)\|\le g\|M_1-M_2\|$, $\forall M_1,M_2\in u[h,l]$ with $g\in[0,1)$ under the norm $\|M\|_\infty=\sup_{u\in[h,l]}|M(u)|$, so

$$\|T(M_1)-T(M_2)\|_\infty$$

$$=\sup_{u\in[h,l]}\left|\frac{1}{\alpha\lambda_1}e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\int_h^l M(w)e^{\frac{-w}{\alpha\lambda_1}}dw \int_h^l (M_1(w)-M_2(w))e^{-\frac{b}{\lambda_1\alpha}}dw\right|$$

$$\le \sup_{u\in[h,l]}\left|\|M_1-M_2\|\frac{1}{\alpha\lambda_1}e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\cdot(-\alpha\lambda_1)(e^{-\frac{l}{\alpha\lambda_1}}-e^{-\frac{h}{\alpha\lambda_1}})\right|$$

$$=\|M_1-M_2\|_\infty \sup_{u\in[h,l]}\left|e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\right|\cdot\left|e^{-\frac{l}{\alpha\lambda_1}}-e^{-\frac{h}{\alpha\lambda_1}}\right|\le g\|M_1-M_2\|_\infty .$$

Therefore, as confirmed by applying Banach's fixed-point theorem, the unique solution exists.

C. The Numerical Integral Equation for the ARL of MAX(q,r) Process

From the integral Equation (4)

$$M(u)=1+\frac{1}{\lambda_1}\int_h^l F(w)f(\frac{w-(1-\lambda_1+\lambda_2)u+\lambda_2 v}{\lambda_1}.$$

The approximation for an integral is evaluated by the quadrature rule as follows

$$\int_h^l f(x)dx\approx\sum_{k=1}^{n}w_k f(a_k)$$

where a_k is a point and w_k is a weight that is determined by the different rules.

Using the quadrature formula, we obtain

$$\tilde{M}(a_c)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k F(a_k)f(\frac{a_k-(1-\lambda_1+\lambda_2)a_c+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}).$$

The system of n linear equations is as follows;

$$\tilde{M}(a_1)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k F(a_k)f(\frac{a_k-(1-\lambda_1+\lambda_2)a_1+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1})$$

$$\tilde{M}(a_2)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k F(a_k)f(\frac{a_k-(1-\lambda_1+\lambda_2)a_2+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1})$$

$$\vdots$$

$$\tilde{M}(a_n)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k F(a_k)f(\frac{a_k-(1-\lambda_1+\lambda_2)a_n+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1})$$

This system can be shown as

$$\mathbf{M}_{n\times1}=\mathbf{1}_{n\times1}+\mathbf{R}_{n\times n}\mathbf{L}_{n\times1},\ \mathbf{I}_n-\mathbf{R}_{n\times n}=\mathbf{1}_{n\times1}$$

or
$$\mathbf{M}_{n\times1}=(\mathbf{I}_n-\mathbf{R}_{n\times n})^{-1}\mathbf{1}_{n\times1},$$

where $\mathrm{M}_{n\times1}=\begin{bmatrix}\tilde{M}(a_1)\\ \tilde{M}(a_2)\\ \vdots\\ \tilde{M}(a_n)\end{bmatrix}$, $\mathrm{I}_n=diag(1,1,...,1)$

and $\mathbf{1}_{n\times1}=\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}$.

Let $\mathbf{R}_{n\times n}$ is a matrix and define the n to n^{th} is an element of the matrix $\mathbf{R}$ as follows

$$[R_{ck}]\approx\frac{1}{\lambda_1}w_k f(\frac{a_k-(1-\lambda_1+\lambda_2)a_c+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}).$$

If $\left(\mathbf{I}\text{-}\mathbf{R}\right)^{-1}$ exists the numerical approximation for the integral equation is the term of the matrix, $\mathbf{M}_{n\times 1}=\left(\mathbf{I}_{n\times 1}-\mathbf{R}_{n\times n}\right)^{-1}\mathbf{1}_{n\times 1}$. Finally, we substitute a_b by u in $\tilde{M}\left(a_b\right)$, the approximation of numerical integral for a function $M\left(u\right)$ is,

$$\tilde{M}\left(u\right)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k M(a_k)f\left(\frac{a_k-\left(1-\lambda_1+\lambda_2\right)u+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\right) \tag{12}$$

IV. Numerical results

Here, the details of the simulation study to compare the efficacies of the NIE method $(\tilde{M}(u))$ and the explicit formulas $(M(u))$ for the *ARL* of a *MAX(q,r)* process on the Extended EWMA control chart are provided.

The numerical procedure for calculating the ARL values for the MAX process can be summarized as follows.

Step 1: Set the value of the parameters of the *MAX(q,r)* model, the parameter of exponential white noise (α_0) for the in control process, and the smoothing parameters; λ_1, λ_2.

Step 2: Determine the initial value of the MAX model, the initial value of the Extended EWMA statistic.

Step 3: Select an acceptable in-control value of ARL_0 and decide on the change parameter value $\alpha_1=(1+\delta)\alpha_0$ for an out-of-control state.

Step 4: Specify the lower control limit (*h*) equals 0.001 or 0.005 and compute the upper control limit (*l*) that yields thedesired average run length for in control process by Equations 7 and 10.

Step 5: Compute the solution of ARL_1 for shift sizes in the monitoring process where $\alpha_1=(1+\delta)\alpha_0$ by Equation 8.

Step 6: Compare the *ARL* of the explicit formula and the NIE method.

This research uses criteria to compare the results obtained by the explicit formula and the numerical integral equation method by the absolute percentage difference (*Diff%*) and it is defined as

$$Diff\,(\%)=\frac{\left|M(u)-\tilde{M}(u)\right|}{M(u)}\times 100 \tag{13}$$

Moreover, the relative mean index (*RMI*) [19] is employed to evaluate each control chart's effectiveness under different λ conditions. The *RMI* is calculated using the formula shown below:

$$RMI(r)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_i(r)-Min\left[ARL_i(s)\right]}{Min\left[ARL_i(s)\right]}\right) \tag{14}$$

where *ARLi(r)* is the *ARL* of the control chart for the shift size in a row *i* and *Min[ARLi(s)]* denotes the smallest *ARL* of the three control charts compared to the shift size in a row *i*, for *i = 1, 2,..., n.* The criterion is that the control chart with the lowest *RMI* is the most effective for detecting changes.

Additionally, average extra quadratic loss (*AEQL*) is another criterion that can be used to measure the performance of a control chart [20]. The *AEQL* is calculated in (15) as follows:

$$AEQL=\frac{1}{\Delta}\sum_{\delta_i=\delta_{\min}}^{\delta_{\max}}\left(\delta_i^{\,2}\times ARL(\delta_i)\right) \tag{15}$$

where δ_i is change level value at each level *i*, $ARL(\delta_i)$ is the *ARL* value of the control chart for the amount of shift δ_i, and Δ is the total numbers of shift levels from $\delta_{\min}$ to $\delta_{\max}$. It has similar criteria to the *RMI* which the control chart with the lowest *AEQL* is the most effective for detecting changes.

Table I provides the upper control limits of Extended EWMA with *MAX(q,r)* processes ARL_0=370 for the simulation study. For example, if the parameter values were set as $\lambda_1=0.5$, $\lambda_2=0.3\lambda_1$ the upper control limit would be equal to 0.00553453 for *MAX(1,1)* for $\mu=0.5$, $\theta_1=0.1$, $\beta_1=0.5.$ In addition, Tables II and III compared the *ARL* results from the explicit formula and NIE method for *MAX* (2,1) and (3,2) models for different choices, θ, respectively.

The results show that the *ARL* derived by the explicit formula is close to the *ARL* obtained by the NIE method, with the numerical estimate having an absolute percentage difference of less than 0.000001%. According to Tables IV and V, the comparison of the *ARL* for *MAX(q,r)* processes on CUSUM, EWMA, and Extended EWMA control charts are presented. The parameter values were set as $ARL_0=370$, $\lambda_1=0.05,\ 0.1,\ 0.2$; in-control parameter $\alpha_0=1$, and the shift size varied as 0.001, 0.003, 0.005, 0.007, 0.01, 0.03, 0.05, 0.07, 0.1, 0.3, and 0.5. Equations (10) and (11) were used to evaluate the *ARL* on Extended EWMA of the *MAX(q,r)* processes. The *ARL* values derived from the explicit formulas for the Extended EWMA control chart were less than those for the CUSUM and EWMA control charts for all shift sizes and all values of λ_2. In addition, as λ_2 increases, the result shows that the ARL_1 values decreased accordingly. When the *ARL* values obtained from each chart shown in Tables IV and V were used to find the *RMI* and *AEQL* values to see the performance of each chart shown in Table VI, it was found that the Extended EWMA control chart had the best performance because it gave the lowest *RMI* and *AEQL* at $\lambda_2=0.9\lambda_1$. Therefore, it also can be concluded that the Extended EWMA control chart performs better than the EWMA and CUSUM control charts in all situations.

V. Practical Applications with Real Data

We applied the explicit formulas for the *ARL* of a *MAX(q,r)* process on CUSUM, EWMA, and Extended EWMA control charts using 60 real-world data observations of the price of crude palm oil in the south of Thailand. The exogenous factor is the export value of vegetable oils and fats from January 2018 to December 2022. The model has an improvement pattern with two *MAX* processes, i.e., *MAX(1,1)* and *MAX (2,1)*, so these two models should be included in the model estimation, as shown in Table VII. Consequently, the *MAX(2,1)* has the lowest RMSE and Normalized Bayesian Information Criteria (Normalized BIC) value, implying that the best model is the *MAX(2,1)*, as shown in Table VIII. Based on the final result of a coefficient parameter in Table VII, we get the *MAX(2,1)* coefficient parameters as follows: $\hat{\theta}_1 = -1.323, \hat{\theta}_2 = -0.794,$ and $\hat{\beta} = 0.004.$ The in control parameter is equal to 9.6501, as shown in Table IX. Through the parameter of this prediction model, we get the equation for *MAX (2,1)* model as follows:

$$\hat{Y}_t = 1.323\varepsilon_{t-1} + 0.794\varepsilon_{t-2} + 0.004X_t$$

According to Tables X and Fig. 1, it is evident from the results that the *ARL* values for the explicit formulas method on the Extended EWMA control chart were less than those for the CUSUM and EWMA control charts for all shift sizes. Additionally, we compared the detection of shifts in the process means for the *MAX(1,1)* process with real data on the two types of EWMA control charts only, the results for which are shown in Fig 2 (a) and (b). The results showed that the Extended EWMA control chart could detect a change in the price of crude palm oil for the first time at the first observation while the EWMA control chart has no observations outside the control limit, can therefore be concluded that the Extended EWMA control chart is more efficient in detecting mean changes than the EWMA control chart.

VI. Conclusions

We have definitively proved the formula for the *ARL* of the *MA(q,r)* process with exponential white noise on the Extended EWMA control chart and used simulated data to validate it by comparing the differences. The results showed that although the *ARL* was very similar between the explicit formula and numerical integral equation methods, the explicitly formulated method took much less computation time. This research also compared performance with CUSUM and EWMA control charts using *ARL* values. It was found that the Extended EWMA control chart was more effective than CUSUM and EWMA control charts in all situations considering the lowest *RMI* and *AEQL* criteria. In addition, the Extended EWMA control chart also has been applied to the real data. For further study, the Extended EWMA chart can be applied to other types of models such as ARMAX, ARIMAX, ARFIMAX, Etc. or various applications for real-world data. Besides, using explicit formulas of *ARL*, the optimal value of the control chart could be considered.

TABLE I
CONTROL LIMITS OF EXTENDED EWMA CONTROL CHART WITH MAX PROCESSES

Models	Coefficients						$\lambda_1 = 0.05$			
	μ	θ_1	θ_2	θ_3	β_1	β_2	$\lambda_2 = 0.3\lambda_1$	$\lambda_2 = 0.5\lambda_1$	$\lambda_2 = 0.7\lambda_1$	$\lambda_2 = 0.9\lambda_1$
MAX(1,1)	0.5	0.1			0.5		0.00553453	0.00266576	0.00161246	0.001225246
MAX(1,2)	0.5	0.1			0.5	1	0.00266637	0.00161262	0.001225296	0.0010828623
MAX(2,1)	0.5	0.1	0.2		0.5		0.00654056	0.003034762	0.00174808	0.0012751180
MAX(2,2)	0.5	0.1	0.2		0.5	1	0.00303559	0.001748285	0.00127518	0.0011012089
MAX(3,1)	0.5	0.1	0.2	0.3	0.5		0.00848442	0.00374718	0.00200985	0.0013713730
MAX(3,2)	0.5	0.1	0.2	0.3	0.5	1	0.00374850	0.00201015	0.00137146	0.0011366180
Models	Coefficients						$\lambda_1 = 0.01$			
	μ	θ_1	θ_2	θ_3	β_1	β_2	$\lambda_2 = 0.3\lambda_1$	$\lambda_2 = 0.5\lambda_1$	$\lambda_2 = 0.7\lambda_1$	$\lambda_2 = 0.9\lambda_1$
MAX(1,1)	0.5	0.1			0.5		0.010083450	0.00433289	0.002225030	0.001450497
MAX(1,2)	0.5	0.1			0.5	1	0.004334665	0.00222543	0.001450607	0.001165725
MAX(2,1)	0.5	0.1	0.2		0.5		0.012102700	0.00507159	0.002496320	0.001550243
MAX(2,2)	0.5	0.1	0.2		0.5	1	0.005074080	0.00249685	0.001550380	0.001202418
MAX(3,1)	0.5	0.1	0.2	0.3	0.5		0.016008150	0.00649815	0.003020000	0.001742755
MAX(3,2)	0.5	0.1	0.2	0.3	0.5	1	0.006502250	0.00302081	0.001742959	0.001273237

TABLE II
COMPARISON OF ARL USING EXPLICIT FORMULAS AND NIE METHODS WITH MAX (2,1) PROCESS FOR DIFFERENT CHOICES OF θ WITH $\mu = 1, \beta = 0.5, \lambda_1 = 0.05$ AND $h = 0.001$

λ_2	Coefficients θ_1	θ_2	l	Methods	Shift sizes (δ) 0.001	0.003	0.005	0.01	0.03	0.05	0.1
$0.5\lambda_1$	0.1	0.1	0.002116418	Explicit	132.900	58.574	37.745	20.199	7.441	4.783	2.774
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	132.900	58.574	37.745	20.199	7.441	4.783	2.774
				CPU$_{NIE}$	(2.574)	(2.578)	(2.625)	(2.687)	(2.797)	(2.594)	(2.640)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.1	0.00191399433	Explicit	129.608	56.695	36.462	19.487	7.182	4.623	2.691
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	129.608	56.695	36.462	19.487	7.182	4.623	2.691
				CPUNIE	(2.562)	(2.563)	(2.563	(2.610)	(2.875)	(2.734)	(2.609)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.2	0.2	0.0023636886	Explicit	136.327	60.567	39.112	20.962	7.720	4.955	2.864
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	136.327	60.567	39.112	20.962	7.720	4.955	2.864
				CPU$_{NIE}$	(2.593)	(2.641)	(3.640)	(2.562)	(2.656)	(2.594)	(2.672)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.2	0.0019139944	Explicit	129.609	56.695	36.462	19.487	7.182	4.623	2.691
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	129.609	56.695	36.462	19.487	7.182	4.623	2.691
				CPU$_{NIE}$	(2.547)	(2.687)	(2.687)	(2.657)	(2.703)	(2.641)	(2.594)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$0.7\lambda_1$	0.1	0.1	0.0014105282	Explicit	117.819	50.186	32.061	17.061	6.305	4.082	2.410
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	117.819	50.186	32.061	17.061	6.305	4.082	2.410
				CPU$_{NIE}$	(2.625)	(2.625)	(2.782)	(2.687)	(2.609)	(2.641)	(2.516)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.1	0.0013361076	Explicit	115.177	48.777	31.117	16.544	6.120	3.968	2.351
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	115.177	48.777	31.117	16.544	6.120	3.968	2.351
				CPU$_{NIE}$	(2.547)	(2.547)	(2.609)	(2.672)	(2.641)	(2.562)	(2.547)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.2	0.2	0.0015014283	Explicit	120.574	51.676	33.062	17.610	6.503	4.204	2.473
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	120.574	51.676	33.062	17.610	6.503	4.204	2.473
				CPU$_{NIE}$	(2.750)	(2.656)	(2.578)	(2.546)	(2.546)	(2.609)	(2.718)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.2	0.0013361076	Explicit	115.177	48.777	31.117	16.544	6.120	3.968	2.351
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	115.177	48.777	31.117	16.544	6.120	3.968	2.351
				CPU$_{NIE}$	(2.656)	(2.703)	(2.578)	(2.687)	(2.547)	(2.578)	(2.640)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$0.9\lambda_1$	0.1	0.1	0.0011509855	Explicit	105.655	43.830	27.826	14.755	5.480	3.575	2.149
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	105.655	43.830	27.826	14.755	5.480	3.575	2.149
				CPU$_{NIE}$	(2.687)	(2.641)	(2.641)	(2.688)	(2.672)	(2.594)	(2.546)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.1	0.0011236162	Explicit	103.507	42.744	27.108	14.367	5.342	3.490	2.106
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	103.507	42.744	27.108	14.367	5.342	3.490	2.106
				CPU$_{NIE}$	(2.625)	(2.594)	(2.718)	(2.688)	(2.625)	(2.594)	(2.703)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.2	0.2	0.0011844146	Explicit	107.891	44.972	28.582	15.165	5.626	3.664	2.195
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	107.891	44.972	28.582	15.165	5.626	3.664	2.195
				CPU$_{NIE}$	(2.610)	(2.625)	(2.610)	(2.656)	(2.632)	(2.656)	(2.500)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.2	0.0011236162	Explicit	103.507	42.744	27.108	14.367	5.342	3.490	2.106
				CPU$_{Exp}$	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	103.507	42.744	27.108	14.367	5.342	3.490	2.106
				CPU$_{NIE}$	(2.688)	(2.718)	(2.546)	(2.719)	(2.547)	(2.516)	(2.765)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00

NOTE: THE NUMERICAL RESULTS IN PARENTHESES ARE COMPUTATIONAL TIMES IN SECONDS

TABLE III
COMPARISON OF ARL USING EXPLICIT FORMULAS AND NIE METHODS WITH MAX (3, 2) PROCESS
FOR DIFFERENT CHOICES OF θ WITH $\mu = 1,\ \theta_1 = 0.1,\ \beta_1 = 0.5,\ \beta_2 = 1,\ \lambda_1 = 0.05$ AND $h = 0.001$

λ_2	Coefficients			Methods	Shift sizes (δ)						
	θ_1	θ_2	l		0.001	0.003	0.005	0.01	0.03	0.05	0.1
$0.5\lambda_1$	0.2	0.3	0.001612618	Explicit	123.464	53.257	34.129	18.196	6.715	4.334	2.541
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	123.464	53.257	34.129	18.196	6.715	4.334	2.541
				CPU_{NIE}	(2.797)	(2.656)	(2.610)	(2.750)	(2.610)	(2.813)	(2.719)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.3	0.0013361862	Explicit	115.186	48.780	31.119	16.545	6.120	3.968	2.351
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	115.186	48.780	31.119	16.545	6.120	3.968	2.351
				CPUNIE	(2.672)	(2.828)	(2.641)	(2.656)	(2.813)	(2.735)	(2.766)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	-0.2	0.3	0.0014106272	Explicit	117.828	50.190	32.064	17.062	6.306	4.082	2.410
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	117.828	50.190	32.064	17.062	6.306	4.082	2.410
				CPU_{NIE}	(2.828)	(2.735)	(2.640)	(2.828)	(2.594)	(2.656)	(2.813)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.3	0.0012253455	Explicit	110.230	46.179	29.384	15.600	5.781	3.760	2.244
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	110.230	46.179	29.384	15.600	5.781	3.760	2.244
				CPU_{NIE}	(2.782)	(2.735)	(2.781)	(2.656)	(2.609)	(2.797)	(2.797)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$0.7\lambda_1$	0.2	0.3	0.0012252953	Explicit	110.222	46.176	29.382	15.599	5.781	3.760	2.244
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	110.222	46.176	29.382	15.599	5.781	3.760	2.244
				CPU_{NIE}	(2.797)	(2.656)	(2.765)	(2.641)	(2.812)	(2.765)	(2.750)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.3	0.0011236425	Explicit	103.514	42.746	27.110	14.368	5.342	3.490	2.106
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	103.514	42.746	27.110	14.368	5.342	3.490	2.106
				CPU_{NIE}	(2.594)	(2.656)	(2.641)	(2.625)	(2.703)	(2.641)	(2.656)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	-0.2	0.3	0.001151018	Explicit	105.660	43.832	27.827	14.756	5.480	3.575	2.149
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	105.660	43.832	27.827	14.756	5.480	3.575	2.149
				CPU_{NIE}	(2.641)	(2.625)	(2.797)	(2.719)	(2.765)	(2.625)	(2.781)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.3	0.00108287942	Explicit	99.462	40.725	25.779	13.650	5.088	3.334	2.026
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	99.462	40.725	25.779	13.650	5.088	3.334	2.026
				CPU_{NIE}	(2.672)	(2.828)	(2.797)	(2.796)	(2.671)	(2.750)	(2.703)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$0.9\lambda_1$	0.2	0.3	0.00108286215	Explicit	99.458	40.724	25.779	13.650	5.088	3.334	2.026
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	99.458	40.724	25.779	13.650	5.088	3.334	2.026
				CPU_{NIE}	(2.641)	(2.828)	(2.593)	(2.718)	(2.860)	(2.609)	(2.735)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.3	0.0010454756	Explicit	93.942	38.027	24.012	12.702	4.752	3.129	1.921
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	93.942	38.027	24.012	12.702	4.752	3.129	1.921
				CPU_{NIE}	(2.703)	(2.891)	(2.609)	(2.828)	(2.750)	(2.766)	(2.609)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	-0.2	0.3	0.00105554404	Explicit	95.709	38.885	24.573	13.003	4.858	3.194	1.954
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	95.709	38.885	24.573	13.003	4.858	3.194	1.954
				CPU_{NIE}	(2.734)	(2.796)	(2.719)	(2.812)	(2.688)	(2.641)	(2.609)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00
		-0.3	0.00103048316	Explicit	90.584	36.418	22.964	12.142	4.555	3.009	1.860
				CPU_{Exp}	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
				NIE	90.584	36.418	22.964	12.142	4.555	3.009	1.860
				CPU_{NIE}	(2.765)	(2.656)	(2.750)	(2.656)	(2.703)	(2.797)	(2.734)
				Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00

NOTE: THE NUMERICAL RESULTS IN PARENTHESES ARE COMPUTATIONAL TIMES IN SECONDS

TABLE IV
COMPARISON OF THE ARL FOR MAX(2,1) PROCESS ON CUSUM EWMA AND EXTENDED EWMA CONTROL CHARTS WITH $\mu = 0.5,\ \beta_1 = 0.5,\ \theta_1 = 0.1,\ \theta_2 = 0.2$ AND $h = 0.005$

λ_1	Shift	CUSUM	EWMA	Extended EWMA			
		$a = 2.5$	$\lambda = \lambda_1$	$\lambda_2 = 0.3\lambda_1$	$\lambda_2 = 0.5\lambda_1$	$\lambda_2 = 0.7\lambda_1$	$\lambda_2 = 0.9\lambda_1$
		d=5.4632	f=0.0302153042	l=0.01055612235	l=0.00703883655	l=0.0057489738	l=0.005275226137
0.05	0.000	370.024	370.000	370.000	370.000	370.000	370.000
	0.001	366.768	208.072	165.212	143.705	126.473	112.646
	0.003	360.359	111.294	78.741	64.966	54.929	47.441
	0.005	354.091	76.172	51.875	42.152	35.262	30.225
	0.01	338.995	42.863	28.233	22.669	18.822	16.057
	0.03	286.031	16.118	10.421	8.346	6.941	5.945
	0.05	207.815	10.235	6.636	5.343	4.474	3.860
	0.10	87.149	5.691	3.751	3.068	2.613	2.296
	0.30	51.990	2.594	1.825	1.570	1.408	1.299
	0.50	24.991	1.965	1.454	1.293	1.195	1.132
		d=5.4632	f=0.0560817852	l=0.0161338856	l=0.00907974825	l=0.00649811564	l=0.00555045983
0.1	0.000	370.024	370.000	370.000	370.000	370.000	370.000
	0.001	366.768	209.656	165.435	143.759	126.484	112.647
	0.003	360.359	112.648	78.891	64.999	54.936	47.442
	0.005	354.091	77.221	51.983	42.174	35.266	30.225
	0.01	338.995	43.513	28.295	22.682	18.824	16.058
	0.03	286.031	16.371	10.443	8.351	6.942	5.945
	0.05	207.815	10.391	6.649	5.346	4.474	3.861
	0.10	87.149	5.770	3.757	3.069	2.613	2.296
	0.30	51.990	2.619	1.827	1.570	1.408	1.299
	0.50	24.991	1.980	1.455	1.293	1.195	1.132
		d=5.4632	f=0.109909125	l=0.0273550128	l=0.01316781392	l=0.0079969031	l=0.0061009498
0.2	0.000	370.024	370.000	370.000	370.000	370.000	370.000
	0.001	366.768	212.916	165.883	143.868	126.506	112.649
	0.003	360.359	115.472	79.193	65.064	54.948	47.443
	0.005	354.091	79.421	52.199	42.220	35.274	30.226
	0.01	338.995	44.884	28.419	22.707	18.829	16.058
	0.03	286.031	16.905	10.488	8.360	6.943	5.946
	0.05	207.815	10.721	6.676	5.351	4.475	3.861
	0.10	87.149	5.938	3.770	3.071	2.614	2.296
	0.30	51.990	2.674	1.831	1.571	1.408	1.299
	0.50	24.991	2.012	1.457	1.293	1.195	1.132

TABLE V
COMPARISON OF THE ARL FOR MAX(3,2) PROCESS ON CUSUM EWMA AND EXTENDED EWMA CONTROL CHARTS WITH $\mu = 0.5,\ \beta_1 = 0.5,\ \beta_2 = 1,\ \theta_1 = 0.1,\ \theta_2 = 0.2,\ \theta_3 = 0.3$ AND $h = 0.005$

λ_1	Shift	CUSUM	EWMA	Extended EWMA			
		$a = 2.5$	$\lambda = \lambda_1$	$\lambda_2 = 0.3\lambda_1$	$\lambda_2 = 0.5\lambda_1$	$\lambda_2 = 0.7\lambda_1$	$\lambda_2 = 0.9\lambda_1$
		d=2.5795	f=0.021829073	l=0.00872182357	l=0.00636641964	l=0.005502030337	l=0.005184488311
0.05	0.000	370.059	370.000	370.000	370.000	370.000	370.000
	0.001	367.809	195.082	156.066	136.361	120.589	107.891
	0.003	363.379	100.608	72.701	60.587	51.684	44.974
	0.005	359.019	67.985	47.573	39.126	33.068	28.584
	0.01	348.418	37.814	25.751	20.970	17.613	15.165
	0.03	309.943	14.112	9.489	7.723	6.504	5.626
	0.05	276.937	8.961	6.054	4.957	4.204	3.665
	0.10	212.738	4.999	3.442	2.865	2.474	2.195
	0.30	90.680	2.314	1.708	1.497	1.360	1.266
	0.50	48.565	1.776	1.379	1.248	1.167	1.114
		d=2.5795	f=0.0389456436	l=0.0124533443	l=0.00773377102	l=0.006004136092	l=0.005368980016
0.1	0.000	370.059	370.000	370.000	370.000	370.000	370.000
	0.001	367.809	196.113	156.211	136.396	120.596	107.892
	0.003	363.379	101.425	72.794	60.608	51.688	44.974
	0.005	359.019	68.601	47.639	39.140	33.071	28.584
	0.01	348.418	38.186	25.789	20.978	17.615	15.166
	0.03	309.943	14.254	9.502	7.725	6.505	5.627
	0.05	276.937	9.047	6.062	4.958	4.205	3.665
	0.10	212.738	5.043	3.446	2.866	2.474	2.195
	0.30	90.680	2.328	1.709	1.497	1.360	1.266
	0.50	48.565	1.784	1.380	1.248	1.167	1.114
		d=2.5795	f=0.0740817523	l=0.0199456802	l=0.01047127404	l=0.00700857396	l=0.00573797361
0.2	0.000	370.059	370.000	370.000	370.000	370.000	370.000
	0.001	367.809	198.217	156.503	136.467	120.610	107.894
	0.003	363.379	103.107	72.982	60.649	51.696	44.975
	0.005	359.019	69.873	47.771	39.168	33.076	28.585
	0.01	348.418	38.956	25.863	20.993	17.617	15.166
	0.03	309.943	14.546	9.529	7.731	6.506	5.627
	0.05	276.937	9.227	6.078	4.962	4.205	3.665
	0.10	212.738	5.132	3.453	2.867	2.474	2.195
	0.30	90.680	2.356	1.712	1.498	1.360	1.266
	0.50	48.565	1.800	1.380	1.248	1.167	1.114

TABLE VI
RMI AND AEQL VALUES FOR IN THE INDICATED CAPABILITY OF CONTROL CHARTS

Control Charts		λ_1	0.05	0.1	0.2
		MAX(2,1)			
CUSUM	a=15	*RMI*	26.2968	26.2968	26.2968
		AEQL	1.4024	1.4024	1.4024
EWMA	$\lambda = \lambda_1$	*RMI*	1.3285	1.3580	1.4210
		AEQL	0.0921	0.0930	0.0948
Extended EWMA	$\lambda_2 = 0.3\lambda_1$	*RMI*	0.5996	0.6022	0.6078
		AEQL	0.0662	0.0663	0.0664
	$\lambda_2 = 0.5\lambda_1$	*RMI*	0.3252	0.3257	0.3267
		AEQL	0.0578	0.0578	0.0578
	$\lambda_2 = 0.7\lambda_1$	*RMI*	0.1360	0.1360	0.1362
		AEQL	0.0525	0.0525	0.0525
	$\lambda_2 = 0.9\lambda_1$	*RMI*	**0.0000**	**0.0000**	**0.0000**
		AEQL	**0.0490**	**0.0490**	**0.0490**
		MAX(3,2)			
CUSUM	a=15	*RMI*	42.3133	42.3133	42.3133
		AEQL	2.6054	2.6054	2.6054
EWMA	$\lambda = \lambda_1$	*RMI*	1.1744	1.1922	1.2289
		AEQL	0.0827	0.0831	0.0840
Extended EWMA	$\lambda_2 = 0.3\lambda_1$	*RMI*	0.5466	0.5483	0.5518
		AEQL	0.0623	0.0624	0.0624
	$\lambda_2 = 0.5\lambda_1$	*RMI*	0.2995	0.2998	0.3006
		AEQL	0.0554	0.0554	0.0554
	$\lambda_2 = 0.7\lambda_1$	*RMI*	0.1263	0.1264	0.1265
		AEQL	0.0509	0.0509	0.0509
	$\lambda_2 = 0.9\lambda_1$	*RMI*	**0.0000**	**0.0000**	**0.0000**
		AEQL	**0.0479**	**0.0479**	**0.0479**

NOTE: THE RESULTS ARE BOLD BECAUSE THEY HAVE THE LOWEST OF RMI AND AEQL VALUES.

TABLE VII
MAX ESTIMATE FOR CRUDE PALM OIL PRICE WITH AN EXPORT VALUE OF VEGETABLE OILS AND FATS AS THE EXOGENOUS VARIABLE

Data	Variable	Coefficient	Std. Error	t	Sig.
MAX(1,1)	MA(1) ($\hat{\theta}$)	-0.793	0.089	-8.928	0.000
	Export value ($\hat{\beta}$)	0.008	0.001	6.242	0.000
MAX(2,1)	MA(1) ($\hat{\theta}_1$)	-1.323	0.088	-14.959	0.000
	MA(2) ($\hat{\theta}_2$)	-0.794	0.093	-8.505	0.000
	Export value ($\hat{\beta}$)	0.004	0.001	3.691	0.001

TABLE VIII
MODEL FIT

Process	RMSE	Normalized BIC
MAX(1,1)	14.376	5.468
MAX(2,1)	**10.950**	**4.991**

TABLE IX

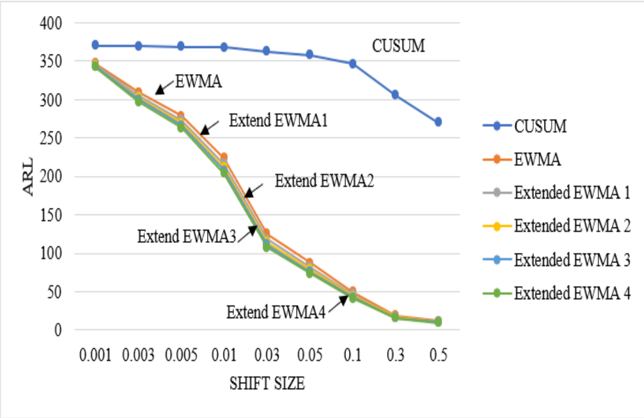
EXPONENTIAL WHITE NOISE OF RESIDUAL USING THE KOLMOGOROV-SMIRNOV GOODNESS-OF-FIT TEST

Process	Mean (α_0)	Kolmogorov-Smirnov Z	Sig.
MAX(2,1)	9.6501	1.265	0.081

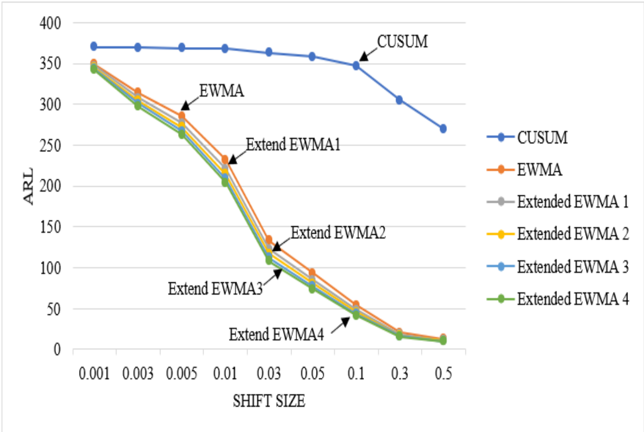
TABLE X

COMPARISON OF THE ARL FOR MAX(2,1) PROCESS ON CUSUM EWMA AND EXTENDED EWMA CONTROL CHARTS WITH $\theta_1 = -1.323, \theta_2 = -0.794,\ \beta_1 = 0.004,\ \alpha_0 = 9.6501$ AND $l = 0.005$

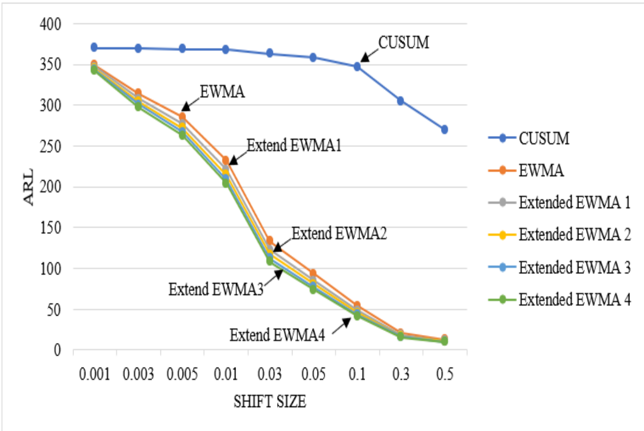
λ_1	Shift	CUSUM $a = 15$	EWMA $\lambda = \lambda_1$	Extended EWMA			
				$\lambda_2 = 0.3\lambda_1$	$\lambda_2 = 0.5\lambda_1$	$\lambda_2 = 0.7\lambda_1$	$\lambda_2 = 0.9\lambda_1$
		d=33.4673	*f*=0.399902065	*l*=0.340189535	*l*=0.305835312	*l*=0.275209009	*l*=0.247858507
0.05	0.00	370.009	370.000	370.000	370.000	370.000	370.000
	0.00	369.764	347.203	345.376	344.224	343.112	342.030
	0.00	369.275	309.135	304.829	302.153	299.597	297.137
	0.00	368.787	278.612	272.827	269.272	265.904	262.689
	0.01	367.571	223.520	216.177	211.753	207.624	203.735
	0.03	362.759	125.094	118.384	114.480	110.924	107.651
	0.05	358.029	87.072	81.731	78.663	75.894	73.367
	0.10	346.556	49.786	46.374	44.437	42.703	41.133
	0.30	305.212	18.941	17.552	16.771	16.075	15.448
	0.50	270.145	12.043	11.159	10.662	10.220	9.821
RMI		**6.7190**	**0.1361**	**0.0828**	**0.0528**	**0.0252**	**0.0000**
AEQL		**11.0825**	**0.6196**	**0.5751**	**0.5500**	**0.5276**	**0.5075**
		d=33.4673	*f*=0.8116347	*l*=0.683720595	*l*=0.61142703	*l*=0.547703473	*l*=0.491328735
0.1	0.00	370.009	370.000	370.000	370.000	370.000	370.000
	0.00	369.764	347.861	345.778	344.487	343.256	342.074
	0.00	369.275	310.701	305.769	302.761	299.927	297.236
	0.00	368.787	280.736	274.082	270.077	266.338	262.819
	0.01	367.571	226.260	217.754	212.748	208.152	203.892
	0.03	362.759	127.668	119.797	115.346	111.373	107.781
	0.05	358.029	89.140	82.845	79.340	76.241	73.466
	0.10	346.556	51.114	47.077	44.859	42.918	41.194
	0.30	305.212	19.474	17.830	16.936	16.158	15.472
	0.50	270.145	12.376	11.332	10.764	10.271	9.836
RMI		**6.7190**	**0.1551**	**0.0930**	**0.0583**	**0.0276**	**0.0000**
AEQL		**11.0825**	**0.6365**	**0.5839**	**0.5552**	**0.5303**	**0.5082**
		d=33.4673	*f*=1.69179475	*l*=1.397543292	*l*=1.23749819	*l*=1.09970567	*l*=0.98011674
0.2	0.00	370.009	370.000	370.000	370.000	370.000	370.000
	0.00	369.764	349.193	346.588	345.015	343.547	342.162
	0.00	369.275	313.909	307.675	303.988	300.593	297.437
	0.00	368.787	285.121	276.641	271.706	267.213	263.081
	0.01	367.571	232.000	220.994	214.772	209.220	204.206
	0.03	362.759	133.203	122.742	117.126	112.284	108.042
	0.05	358.029	93.631	85.181	80.733	76.947	73.667
	0.10	346.556	54.029	48.558	45.732	43.356	41.317
	0.30	305.212	20.653	18.418	17.279	16.328	15.519
	0.50	270.145	13.112	11.697	10.977	10.376	9.865
RMI		**6.7190**	**0.1964**	**0.1135**	**0.0698**	**0.0324**	**0.0000**
AEQL		**11.0825**	**0.6738**	**0.6026**	**0.5661**	**0.5357**	**0.5097**



(a)

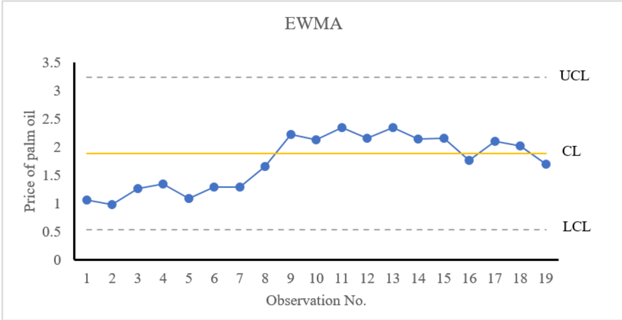


(b)

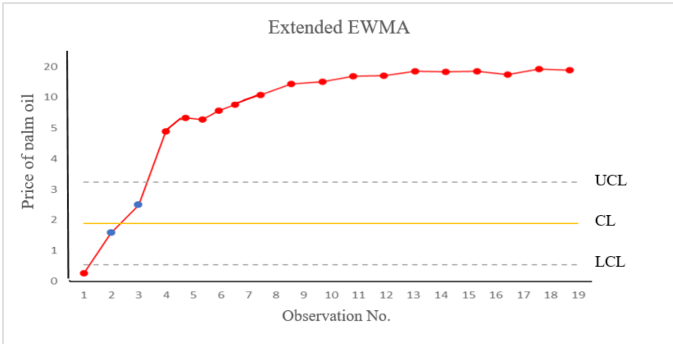


(c)

Fig. 1. *ARL* on the CUSUM, EWMA and Extended EWMA charts for MAX(2,1) process with ARL_0 = 370 and (a) $\lambda_1 = 0.05$, (b) $\lambda_1 = 0.1$, (c) $\lambda_1 = 0.2$.



(a)



(b)

Fig.2. Mean shift detection for MAX(2,1) process for the price of crude palm oil (a) EWMA control chart and (b) Extended EWMA control chart.

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