A Simple Algorithm to Estimate the Order Time Interval with a Linear Demand

Kou-Huang Chen, Yung-Ning Cheng

Abstract—**We improve the approximated solution for the inventory model with a linear trend in demand such that there is no necessary to adjust the last or the first time interval. Our formula is easier to compute. We estimate our approximated result to the optimal solution of the equal time interval constraint, then our result is measured with an error of less than two-thirds. For the second inventory system examined in this paper, we consider an inventory system with two retailers and a manufacturer. Through the algebraic process, we obtain the optimal profit that is missing in the previously published paper. We show the elegant character of the algebraic process.**

Index Terms—**Optimal solution, Inventory, Optimization, Formulated solution, Approximated solution**

I. Introduction

For the inventory model with linear demand, Mitra et al. [1] developed a simple algorithm to approximate the optimal solution by the analytic method of Donaldson [2] and then compared it to the heuristic procedure, that was constructed by Silver and Meal [3] and presented in Silver [4]. Mitra et al. [1] showed that their method is easy to apply and the total cost will suffer only a small portion increase. We try to improve Mitra et al. [1] from a different approach such that we will have a uniform time interval for each order. Therefore we avoid the adjustment for the last or the first time interval of Mitra et al. [1]. Meanwhile, the computation is easier to execute.

Recall of Mitra et al. [1], the total demand, M, satisfies

$$M = \int_0^H (a+bt)dt$$
$$= aH + (bH^2/2) = D^*H, \tag{1.1}$$

where H is the time horizon under consideration and D(t) is the demand rate, a linear function of time such that $D(t) = a + bt$.

Moreover, D^* is the equivalent constant demand date and T^* is the order interval with respect to the constant demand dateD^*.

Therefore, $D^* = a + (bH/2)$ and $T^* = \sqrt{2A/rD^*}$, where r is the inventory carrying cost, in \$ per unit per unit of time, and A is the fixed replenishment cost per order, in \$.

Mitra et al. [1] used T^* as the time between orders, so over the time horizon, there are $[M/D^*T^*]^+$ replenishments. The notation $[x]^+$ denotes the least natural number greater than or equal to x. T^* is used to express the duration period between two consecutive orderings, except for the first and last period, according to the following rules. If the quotient of H/T^* is an integer, then they will order H/T^* times such that each period has the same time interval T^*.

On the other hand, if the quotient of H/T^* is not an integer, so $H/T^* = I + \varepsilon$, with I is an integer and $0 < \varepsilon < 1$.

They will consider two alternatives:

(A1) Take I+1 time's order; for the first I times, the duration period equals T^* and the last period time interval is $H - IT^* = \varepsilon T^*$.

(A2) Have I time's order; for the first period, the time interval equals $(1+\varepsilon)T^*$, and the rest time interval is T^*.

By computing the total cost over the time horizon, they will choose the alternative with the smaller total cost. In Example 1 from Mitra et al. [1], they use εT^* for the last period and in Example 2 of Mitra et al. [1], they utilize $(1+\varepsilon)T^*$ for the first period.

We take different approaches such that first, we decide how many times we will order during the time horizon. Therefore, each period has the same time interval, hence we do not need to adjust the first or the last time interval.

II. Mathematical formulation

If we want the time intervals between orders to be equal then the number of orders over the time horizon will be approximated by the following,

$$\frac{M}{D^*T^*} = \frac{H}{T^*}. \tag{2.1}$$

Hence we take the order number, n, by the following rule

$$D^* = a + (bH/2), \tag{2.2}$$

and

$$T^* = \sqrt{\frac{2A}{rD^*}}, \tag{2.3}$$

if we suppose

$$y = \frac{H}{T^*}, \tag{2.4}$$

then

$$y = \sqrt{\frac{(2a+bH)r}{4A}H^2} = m + \theta, \tag{2.5}$$

with m is an integer and $0 < \theta < 1$.

Our strategy for the order number is constructed as follows:

For $0 < \theta < (1/3)$, we take

$$n = m. \tag{2.6}$$

Otherwise, if $(1/3) \le \theta < 1$, then we take

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$$n = m + 1. \tag{2.7}$$

The retailer orders the amount that will be consumed during the planning horizon. Since the order points are jH/n, for $j = 0,1,\dots,n-1$, so the order quantity for the time interval $[jH/n, ((j+1)H/n)]$, says Q_j, satisfies

$$Q_j = \int_{jH/n}^{((j+1)H/n)} (a + bt)dt$$
$$= aH/n + b((2j+1)/2n^2)H^2. \tag{2.8}$$

Hence, we compute the inventory level, which says $I_j(t)$, for $t \in [jH/n, ((j+1)H/n)]$, is denoted as

$$I_j(t) = \frac{a(j+1)H}{n} - at - \frac{b(j+1)^2H^2}{2n^2} - \frac{bt^2}{2}. \tag{2.9}$$

Therefore, the inventories carrying cost for the time interval $[jH/n, ((j+1)H/n)]$, says R_j, is expressed as follows:

$$R_j = r\int_{jH/n}^{((j+1)H/n)} I_j(t)dt$$
$$= \frac{arH^2}{2n^2} + \frac{br(3j+2)H^3}{6n^3}. \tag{2.10}$$

The set-up cost over the horizon is expressed as nA.
Since

$$\sum_{j=0}^{n-1}(3j+2) = \frac{n(3n+1)}{2}, \tag{2.11}$$

we know that the total cost, says $C(n)$, for order n times, over the time horizon, is denoted as

$$C(n) = nA + \frac{rH^2}{12n^2}[6an + b(3n+1)H]. \tag{2.12}$$

We recall that of Mitra et al. [1], their total cost over the horizon H as

$$C = nA + R_n + \frac{r}{2}\left\{a(n-1)(T^*)^2 + b\left(\frac{(3n-2)(n-1)}{6}\right)(T^*)^3\right\}, \tag{2.13}$$

where

$$R_n = r\left[\frac{aH^2}{2} + \frac{bH^3}{3} - T^*\left(aH + \frac{bH^2}{2}\right)(n-1), \frac{((n-1)T^*)^3 b}{6} + \frac{((n-1)T^*)^2 a}{2}\right]. \tag{2.14}$$

Hence, our total cost function is easier to compute. We let

$$C(x) = xA + \frac{rH^2}{12}\left(\frac{6a}{x} + b\frac{3x+1}{x^2}H\right), \tag{2.15}$$

for $x > 0$, then

$$\frac{dC}{dx} = A - \frac{rH^2}{12}\left(\frac{6a}{x^2} + bH\left(\frac{3}{x^2} + \frac{2}{x^3}\right)\right), \tag{2.16}$$

and

$$\frac{d^2C}{dx^2} = \frac{rH^2}{2}\left(\frac{a}{x^3} + bH\left(\frac{1+x}{x^4}\right)\right). \tag{2.17}$$

From $\frac{d^2C}{dx^2} > 0$, for $x > 0$, so $C(x)$ is concave up on the domain $(0,\infty)$. Since

$$\lim_{x\to 0^+} C(x) = \infty, \tag{2.18}$$

and

$$\lim_{x\to\infty} C(x) = \infty. \tag{2.19}$$

We know that $\frac{dC}{dx} = 0$ has a unique solution that is the absolute minimum point.

Let x^* be the unique solution of $\frac{dC}{dx} = 0$, since

$$\frac{dC}{dx}(y) = \frac{-brH^3}{6y^3} < 0, \tag{2.20}$$

where y is defined by Equation (2.5). By Equation (2.17), $\frac{dC}{dx}$ is an increasing function with the variable x, therefore, $y < x^*$, so we conclude the following lemma.

Lemma 1. If we treat the number of orders as a continuous variable, in the situation the equal time intervals are preferable, then taking $y = \sqrt{\frac{r(2a+bH)H^2}{4A}}$ will underestimate the number of orders.

Since

$$\frac{dC}{dx}\left(y + \frac{1}{3}\right) = \frac{A + 9yA + 9arH^2}{(3y+1)^3} > 0, \tag{2.21}$$

using Equation (2.17) and Lemma 1, then

$$y + \frac{1}{3} > x^* > y. \tag{2.22}$$

Combine Equations (2.7) and (2.22), we get

Lemma 2. The proposed order number n is very close to the continuous optimal solution for C(x), such that the error is less than $2/3$.

Mitra et al. [1] derived the cycle period T^*, and then showed that

$$n = [M/D^*T^*]^+. \tag{2.23}$$

We take a different approach, set n according to Equation (2.7), then the cycle period becomes H/n, therefore, we have a simpler expression for the total cost. Moreover, since we have a uniform cycle period, we do not need to consider whether or not to adjust the last or the first cycle period.

III. Numerical Examples

We use the same numerical examples as Mitra et al. [1]. The data for the examples appear in above Table 1. By Equation (2.7), we find y and n, then we solve for C(n). The results are listed in the following Table 1.

Since these two examples were considered by Silver [4] and Donaldson [2], we write down the total cost for these two examples for every different method. Moreover, we list the percentages of relative error with respect to the best optimal solution that is obtained from the Donaldson method in the following Table 2.

IV. A Related Problem

Table 1. Data for examples and our result

	A	r	D(t)	H	y	n	C(n)
Example 1	9	0.1	6+t	11	2.78	3	51.42
Example 2	9	2	900t	1	7.07	7	130.35

Table 2.Total cost and relative error

	Total cost				PRE		
	Ours	Mitra	Silver	Donaldson	Ours	Mitra	Silver
Ex. 1	51.42	51.23	51.20	49.88	3.1%	2.7%	2.6%
Ex. 2	130.35	129.80	126.60	125.70	3.7%	3.3%	0.7%

PRE: Percentages of relatives error concerning the analytic optimal solution of Donaldson [2]

We will study a related inventory problem with two retailers and a manufacturer. Ali and Nakade [5] examined a supply chain model for production, pricing, and service strategies with disruptions with one supplier and one retailer. They applied revenue-sharing contracts to consider two issues: demand and service sensitivity coefficient to obtain win-win positions for the partners in the supply chain. Konur and Geunes [6] developed pricing decisions for a supplier who produces an item through a retail channel that can be influenced by the competition among practitioners in the channel with decentralized or centrally managed.

Tan et al. [7] pointed out that traditional supply chain contract models cannot handle the present digital industries. For example, the industry will result in an increase in the price of digital goods. They studied a digital goods supply chain with the agency model for with two competing retailers and one supplier to show that the agency model is better than the traditional wholesale contract. In a closed-loop supply chain, Huang and Wang [8] considered cost disruptions of remanufactured and new products.

Li et al. [9] studied the influences of horizontal and vertical operational models. MacKenzie and Apte [10] developed a system with disruption management strategies. Their mathematical model studied relationships among parameters to find the optimal safety stock, and the duration time to deteriorate to not fresh products. Their model helps the manager of a supply chain to examine the balance among different disruption management strategies.

V. ALGEBRAIC PROCEDURE

To help researchers without knowledge of analytic analysis, researchers developed algebraic methods to solve inventory models, among many other papers. We may classify them into the following three categories:

(i) By the ordinary algebraic method to complete the square for the linear term as to change from

$$f(x) = x^2 - 6x + 15, \tag{5.1}$$

to

$$f(x) = (x-3)^2 + 6, \tag{5.2}$$

and then obtain the minimum point is $x = 3$. For example, Ronald et al. [11], and Lin et al. [12].

(ii) To complete the square for the constant term as to change from

$$f(x) = 9x + (4/x), \tag{5.3}$$

to

$$f(x) = \left(3\sqrt{x} - \left(2/\sqrt{x}\right)\right)^2 + 6, \tag{5.4}$$

and then imply the minimum point is $x = 2/3$. For example, Chang et al. [13], and Lan et al. [14].

(iii) From an operational research view, without developing a system of differential equations, their solution procedure becomes very short and compact. For example, Lin et al. [15], and Tuan and Chu [16].

VI. REVIEW OF XIAO AND QI

Following this research trend, we will examine Xiao and Qi [17] which is an inventory system with two retailers and one manufacturer. The two retailers are competing with each other to seize the share of the market by price strategy.

To be compatible with Xiao and Qi [17], we will use the same assumptions and notation. Xiao and Qi [17] examined an inventory model with two retailers and a manufacturer. The manufacturer produced items with a unit cost and then sold items to retailers. After buying items from the manufacturer, each retailer added some prices to the item with a unit cost c_i, then retailers decided their retailer price p_i to sell the item. The two retailers are followers and the manufacturer is the leader. They assumed the demand for retailer i is expressed as

$$q_1(p_1, p_2) = a - p_1 + d\ p_2, \tag{6.1}$$

and

$$q_2(p_1, p_2) = a - p_2 + d\ p_1, \tag{6.2}$$

where d is the substitutability coefficient of the two items provided by two retailers and a is the market scale, that is the maximum possible demand.

Xiao and Qi [17] wanted to maximize

$$\pi(p_1, p_2) = (p_1 - c_1 - c_0)(a - p_1 + dp_2) + (p_2 - c_2 - c_0)(a - p_2 + dp_1) \tag{6.3}$$

by analytic approach to finding that

$$\frac{\partial}{\partial p_1}\pi(p_1, p_2) = -2p_1 + 2dp_2 + a + c_0 + c_1 - d(c_0 + c_2), \tag{6.4}$$

and

$$\frac{\partial}{\partial p_2}\pi(p_1, p_2) = 2dp_1 - 2p_2 + a + c_0 + c_2 - d(c_0 + c_1). \tag{6.5}$$

They solved the system of $\frac{\partial}{\partial p_1}\pi(p_1, p_2) = 0$ and $\frac{\partial}{\partial p_2}\pi(p_1, p_2) = 0$ to yield that

$$p_1 = \frac{\det\begin{pmatrix} d(c_0 + c_2) - a - c_0 - c_1 & 2d \\ d(c_0 + c_1) - a - c_0 - c_2 & -2 \end{pmatrix}}{\det\begin{pmatrix} -2 & 2d \\ 2d & -2 \end{pmatrix}}, \tag{6.6}$$

and

$$p_2 = \frac{\det\begin{pmatrix} -2 & d(c_0+c_2)-a-c_0-c_1 \\ 2d & d(c_0+c_1)-a-c_0-c_2 \end{pmatrix}}{\det\begin{pmatrix} -2 & 2d \\ 2d & -2 \end{pmatrix}}. \quad (6.7)$$

Hence, they found that

$$p_1 = \frac{a}{2(1-d)} + \frac{c_1+c_0}{2}, \quad (6.8)$$

and

$$p_2 = \frac{a}{2(1-d)} + \frac{c_2+c_0}{2}. \quad (6.9)$$

However, they did not derive the maximum value.

In the following, we will find the maximum value and the optimal solution of p_1 and p_2 by applying an algebraic procedure.

VII. Algebraic procedure

We consider Equation (6.3) to express it in a decreasing trend of the variable p_1 to obtain that

$$\pi(p_1,p_2) = -p_1^2 + [(a+c_1+c_0)+d(2p_2-c_2-c_0)]p_1 - (c_1+c_0)(a+dp_2)+(p_2-c_2-c_0)(a-p_2). \quad (7.1)$$

By completing the square of p_1, we derive

$$\pi(p_1,p_2) = -\left[p_1 - \frac{a+c_1+c_0+d(2p_2-c_2-c_0)}{2}\right]^2 + \frac{[(a+c_1+c_0)+d(2p_2-c_2-c_0)]^2}{4} - (c_1+c_0)(a+dp_2)+(p_2-c_2-c_0)(a-p_2). \quad (7.2)$$

We observe that the coefficient of a positive term, $\left[p_1 - \frac{a+c_1+c_0+d(2p_2-c_2-c_0)}{2}\right]^2$ is -1, which is a negative number. To maximize the result, researchers will derive that

$$p_1 = \frac{a+c_1+c_0-d(c_2+c_0)}{2} + dp_2. \quad (7.3)$$

By substituting the findings of Equation (7.3) into Equation (7.2), and then by a decreasing sequence of the variable p_2, we show that

$$\pi(p_2) = [a+c_2+c_0+da-d^2(c_2+c_0)]p_2 - (1-d^2)p_2^2 - a(c_2+c_1-2c_0) + \frac{1}{4}[a+c_1+c_0-d(c_2+c_0)]^2. \quad (7.4)$$

By completing the square of p_2, we obtain

$$p_2 = \frac{a}{2(1-d)} + \frac{c_2+c_0}{2}. \quad (7.5)$$

By plugging the findings of Equation (7.5) into Equation (7.3), we derive

$$p_1 = \frac{a}{2(1-d)} + \frac{c_1+c_0}{2}. \quad (7.6)$$

Results of Equations (7.5) and (7.6) are identical to that mentioned in Xiao and Qi [17] by an analytic approach that is expressed as cited in Equations (6.8) and (6.9).

In the following, we try to locate the maximum value.

$$\begin{aligned}\pi(p_1^*,p_2^*) &= \left(\frac{a}{2}+\frac{d(c_2+c_0)-(c_1+c_0)}{2}\right)\left(\frac{a}{2(1-d)}-\frac{c_1+c_0}{2}\right) + \left(\frac{a}{2(1-d)}-\frac{c_2+c_0}{2}\right)\left(\frac{a}{2}+\frac{d(c_1+c_0)-(c_2+c_0)}{2}\right) \\ &= \frac{a^2}{2(1-d)} + \frac{(c_1+c_0)^2}{4} + \frac{(c_2+c_0)^2}{4} - \frac{a}{2}(c_1+c_2+2c_0) - \frac{d}{2}(c_1+c_0)(c_2+c_0) \\ &= \frac{1}{4}(a-c_1-c_0)^2 + \frac{1}{4}(a-c_2-c_0)^2 + \frac{d}{2(1-d)}a^2 - \frac{d}{2}(c_1+c_0)(c_2+c_0). \end{aligned} \quad (7.7)$$

During our derivation procedure, we need two extra criteria:

$$a > c_1 + c_0, \quad (7.8)$$

and

$$a > c_2 + c_0. \quad (7.9)$$

According to our restrictions of Equations (7.8) and (7.9), we further simplify Equation (7.7) as follows

$$\pi(p_1^*,p_2^*) = \frac{1}{4}(a-c_1-c_0)^2 + \frac{1}{4}(a-c_2-c_0)^2 + \frac{d}{2}\left[a^2-(c_1+c_0)(c_2+c_0)\right] + \frac{d^2}{2(1-d)}a^2. \quad (7.10)$$

From our derivation of Equation (7.10), the positivity of the maximum value is verified.

VIII. A numerical example of the second model

According to Xiao and Qi [17], the following data for parameters are used: $d = 0.5$, $c_2 = 2$, $c_1 = 3$, $c_0 = 5$, and $a = 20$, such that we derive that

$$a = 20 > c_0 + c_1 = 8, \quad (8.1)$$

and

$$a = 20 > c_0 + c_2 = 7\,, \tag{8.2}$$

with

$$p_1 = 24\,, \tag{8.3}$$

and

$$p_2 = 23.5\,. \tag{8.4}$$

We point out that our two criteria of Equations (7.8) and (7.9) are examined by the above-mentioned example proposed by Xiao and Qi [17].

IX. A RELATED IMPROVEMENT

We study Tung et al. [18] to present an improvement for their derivations.

In the beginning, we recall that Tung et al. [18] obtained that

$$\frac{k}{\sqrt{k^2+1}} = \frac{\alpha_4 - \alpha_3(1+\beta)Q}{\alpha_4 + \alpha_3(1-\beta)Q}\,. \tag{9.1}$$

Based on equation (9.1), because k is the safe factor under the restriction of $k \ge 0$, Tung et al. [18] found an upper bound for the lot size, Q. Consequently, Tung et al. [18] asserted that

$$\alpha_4 \ge \alpha_3(1+\beta)Q\,. \tag{9.2}$$

We observe equation (9.1) to know that owing to $k \ge 0$, such that their derivations of equation (9.2) is right. However, by the first partial derivative system, the solution is to locate an interior optimal solution. Hence, we imply that $k \ge 0$ should be revised to $k > 0$.

Consequently, we find an upper bound for the lot size, Q, in the following,

$$\alpha_4 > (1+\beta)\alpha_3 Q\,. \tag{9.3}$$

We apply the findings of equation (9.3) to the denominator of the first derivative and then the unpleasant result of

$$\frac{1}{\left(\alpha_4 - (1+\beta)\alpha_3 Q\right)^2} = \frac{1}{0}\,, \tag{9.4}$$

occurred on the boundary $\alpha_4 = (1+\beta)\alpha_3 Q$, will not happen.

Our revision will not change the structure of Tung et al. [18], because we compare two upper bounds: $\sqrt{\alpha_1+\alpha_2}$ and $\dfrac{\alpha_4}{(1+\beta)\alpha_3}$ proposed by Tung et al. [18], and then we obtain that

$$\frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1+\alpha_2}\,. \tag{9.5}$$

In the following derivation, we use the upper bound of $\sqrt{\alpha_1+\alpha_2}$.

We recall the findings of Tung et al. [18] to using $\alpha_4 - \beta\alpha_3 Q \ge \alpha_3 Q$ and then Tung et al. [18] obtained the following finding,

$$\sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}} \le 1\,, \tag{9.6}$$

such that Tung at al. [18] implied the second upper bound for Q, as follows

$$Q \le \sqrt{\alpha_1+\alpha_2}\,. \tag{9.7}$$

By our improvement, after we know that $\alpha_4 > (1+\beta)\alpha_3 Q$, then we can improve the above findings as follows.

By equation (9.3) again, it yields that $\alpha_4 - \beta\alpha_3 Q > \alpha_3 Q$ such that

$$\sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}} < 1\,, \tag{9.8}$$

and then owing to derivation of Tung et al. [18], we derive the second upper bound for Q,

$$Q < \sqrt{\alpha_1+\alpha_2}\,. \tag{9.9}$$

Therefore, we can simplify the condition of equation (9.3) into one result as equation (9.9).

X. A RELATED PROBLEM

We aim to present an alternative approach to the parameter constraints in a minimum problem, which have previously been explored by Luo and Chou [22], Chiu et al. [41], Lau et al. [42], Chang et al. [13], and Ronald et al. [11]. While their findings are significant for the advancement of inventory systems, certain aspects of their research lack rigorous mathematical analysis, or their derivations include questionable results. These factors serve as a valid motivation for our renewed investigation into this subject. In addition to addressing the aforementioned limitations, our research seeks to extend the existing results by offering a more comprehensive analysis of the parameter constraints within the minimum problem. We strive to provide a solid mathematical foundation for our derivations, ensuring their accuracy and reliability. By doing so, we aim to contribute to the field of inventory systems and enhance the understanding and implementation of optimal solutions. Through our study, we anticipate uncovering valuable insights that can lead to practical improvements in inventory management, ultimately benefiting businesses and organizations.

XI. OUR DERIVATION FOR PARAMETER C

During their examination of an inventory model with two backorder costs, Chang et al. [13] posed an open question regarding the search for an abstract extension of a minimum problem. This question remains unresolved and represents an opportunity for further exploration and investigation.

For the given goal function proposed by Chang et al. [13],

$$f(x) = \sqrt{ax^2+bx+c} - x\,, \tag{11.1}$$

our objective is to find the minimum point and minimum value for the variable $x \in (0,\infty)$ under several reasonable conditions for the parameters: a, b, and c, with their interrelationships.

That is, our goal is to establish criteria that guarantee the existence of a unique minimum solution for positive values with $f(x) > 0$, satisfying the condition for $x \in (0,\infty)$.

To ensure the expression of $\sqrt{ax^2+bx+c}$ is meaningful, researchers already know the next two conditions:

$$a \geq 0, \tag{11.2}$$

and

$$b^2 - 4ac \leq 0. \tag{11.3}$$

We can derive the following equations with respect to the first and the second derivations:

$$f'(x) = \frac{2ax + b - 2\sqrt{ax^2+bx+c}}{2\sqrt{ax^2+bx+c}}, \tag{11.4}$$

and

$$f''(x) = \frac{4ac - b^2}{4\left(ax^2+bx+c\right)^{1.5}}. \tag{11.5}$$

Using equation (11.4), we can deduce the increasing and decreasing interval for the objective function.

From the left hand boundary limit, $\lim_{x\to 0^+} f(x) = \sqrt{c}$, we imply that $c \geq 0$. We will begin to show that $c = 0$ is not acceptable for this minimum problem.

If $c = 0$, then the minimum point would be $x^* \to 0^+$, which is not acceptable in the original inventory model. This is because when the replenishment cycle duration time approaches zero, the average setup cost increases infinitely, rendering unacceptable, $x^* \to 0^+$, as the minimum point. Hence, we conclude that

$$c > 0, \tag{11.6}$$

which is the first derivation by our approach.

XII. Our Result for Parameter a

We can rewrite $f(x)$ as:

$$f(x) = \frac{(a-1)x^2 + bx + c}{\sqrt{ax^2+bx+c} + x}, \tag{12.1}$$

to work on our second result of the condition,

$$a > 1. \tag{12.2}$$

from equation (11.2), $a \geq 0$.

If $a < 1$, then the left hand side limit, $\lim_{x\to\infty} f(x) = -\infty$ that is unreasonable for an inventory model having the minimum value approaching to negative infinite so we revise our restriction for the parameter, a, from

$$a \geq 0, \tag{12.3}$$

to

$$a > 1. \tag{12.4}$$

Next, we compare the numerator of $f'(x)$, in the equation (11.4), we try to compare the following two auxiliary functions: $g(x)$ with $h(x)$, where we defined that

$$g(x) = 2ax + b, \tag{12.5}$$

and

$$h(x) = 2\sqrt{ax^2+bx+c}. \tag{12.6}$$

For the parameter, b, we divide this into two cases: (i) $b \geq 0$ and (ii) $b < 0$.

Under case (i), with $b \geq 0$, we know that $2ax + b$ is always positive, and then $g(x) \geq h(x)$ is equavelent to $g^2(x) \geq h^2(x)$.

We can compute

$$g^2(x) - h^2(x) = b^2 - 4c + 4b(a-1)x + 4a(a-1)x^2. \tag{12.7}$$

Under case (ii), with $b < 0$, we observe that $g(x) \leq 0$ for $x \in (0, -b/2a)$, and $g(x) > 0$, for $x \in (-b/2a, \infty)$.

For the first sub-restriction, if $x \in (0, -b/2a)$, we know the first derivation is negative, making $f(x)$ a decreasing function within the sub-domain $x \in (0, -b/2a)$.

For the second sub-domain, with $x \in (-b/2a, \infty)$, we know that the value of g(x) is positive, and thus $g(x) \geq h(x)$ is equavelent to $g^2(x) \geq h^2(x)$.

We will divide into the following four cases: (A) $a = 1$, $b \geq 0$ and $b^2 - 4ac < 0$, (B) $a = 1$, $b < 0$ and $b^2 - 4ac < 0$, (C) $a = 1$, $b^2 - 4ac = 0$, $b = 2\sqrt{c}$, and (D) $a = 1$, $b^2 - 4ac = 0$, $b = -2\sqrt{c}$, and then to conclude that $a > 1$. The reasoning for our partition will be explained in the following.

Let us consider the cases with $a = 1$. Under the restriction that $b^2 - 4ac \leq 0$, we split it into two sub cases: $b^2 - 4ac < 0$, and $b^2 - 4ac = 0$. When $a = 1$ and $b^2 - 4ac < 0$, we further divide it into case (A) $a = 1$, $b \geq 0$ and $b^2 - 4ac < 0$, and case (B) $a = 1$, $b < 0$ and $b^2 - 4ac < 0$. When $a = 1$ and $b^2 - 4ac = 0$, we further divide it into case (C) $a = 1$, $b^2 - 4ac = 0$, $b = 2\sqrt{c}$, and (D) $a = 1$, $b^2 - 4ac = 0$, $b = -2\sqrt{c}$.

Under case (A), when $a = 1$, $b \geq 0$ and $b^2 - 4ac < 0$, according to Equation (4), we find that $g(x) < h(x)$, so $f'(x) < 0$, indicating that $f(x)$ is a decreasing function with minimum point as $x^* \to \infty$. However, having an infinite during the replenishment time cycle is unreasonable for our inventory model.

Under case (B), if $a = 1$, $b < 0$ and $b^2 - 4ac < 0$, to find the minimum value, we simplify our domain from $x \in (0,\infty)$ to $x \in (-b/2a, \infty)$. Referring to Equation (4), we see that $g(x) < h(x)$, which implies the minimum point happens at $x^* \to \infty$. However, this is unreasonable for our original inventory system.

Next, we consider the problem with $a = 1$ and $b^2 - 4ac = 0$. We further split it into two cases: case (C) $b = -2\sqrt{c}$, and case (D) $b = 2\sqrt{c}$.

Under case (C), when $a = 1$, $b = -2\sqrt{c}$, we find that

$$f(x) = \sqrt{\left(x - \sqrt{c}\right)^2} - x = \left|x - \sqrt{c}\right| - x, \quad (12.8)$$

such that for $x \in \left(0, \sqrt{c}\right)$, $f(x) = \sqrt{c} - 2x$ and for $x \in \left(\sqrt{c}, \infty\right)$, $f(x) = -\sqrt{c}$ to deduce the minimum value is $-\sqrt{c}$ which is a negative value for our minimum problem. It is unreasonable for any inventory system.

Under case (D), when $a = 1$, and $b = 2\sqrt{c}$, we observe that $f(x) = \sqrt{c}$ is a constant function, and therefore, the minimum problem is solved.

Based on the above examination, our problem leads to two outcomes: (a) it is unreasonable for our inventory system, or (b) it results in a constant objective function. Therefore, we can conclude that is not within the scope of our examination. As a result, we obtain that the appropriate condition should be denoted as:

$$a > 1. \quad (12.9)$$

XIII. A Relation among a, b, and c

In this section, we will begin to consider the criterion regarding the condition of $b^2 - 4ac \le 0$, we first examine the condition of $b^2 = 4ac$, then we split it into two cases: case (E) $b = 2\sqrt{ac}$, and case (F) $b = -2\sqrt{ac}$. Our goal is to derive a restricted condition of $b^2 - 4ac < 0$.

Under case (E), when $b = 2\sqrt{ac}$, we can simplify the equation (11.4) as:

$$f'(x) = \frac{2\left(\sqrt{a} - 1\right)\left(\sqrt{a}x + \sqrt{c}\right)}{2\left(\sqrt{a}x + \sqrt{c}\right)} = \sqrt{a} - 1 > 0. \quad (13.1)$$

This implies that the minimum as $x^* \to 0^+$, which is unreasonable for our inventory system.

Under case (F), when $b = -2\sqrt{ac}$, we rewrite equation (11.4) by the following two cases: case (G) $x > \sqrt{c}/\sqrt{a}$, and case (H) $x < \sqrt{c}/\sqrt{a}$:

For case (G), under the restriction $\sqrt{c}/\sqrt{a} < x$,

$$f'(x) = 2\left(\sqrt{a}x - \sqrt{c}\right)\left(\sqrt{a} - 1\right), \quad (13.2)$$

For case (H), under the restriction $x < \sqrt{c}/\sqrt{a}$,

$$f'(x) = 2\left(\sqrt{a}x - \sqrt{c}\right)\left(\sqrt{a} + 1\right). \quad (13.3)$$

This indicates that for $x < \sqrt{c}/\sqrt{a}$, $f'(x) < 0$ and for $x > \sqrt{c}/\sqrt{a}$, $f'(x) > 0$.

We derive that the minimum point is

$$x^* = \sqrt{c}/\sqrt{a}. \quad (13.4)$$

However, we discover that

$$f\left(x^*\right) = -\sqrt{c}/\sqrt{a} < 0, \quad (13.5)$$

that is unreasonable for an inventory system.

Based on the above examination, when $b^2 = 4ac$, we encounter two unreasonable findings. Therefore, we can assume that

$$b^2 - 4ac < 0, \quad (13.6)$$

is a reasonable condition in the future examinations.

Based on our derivations, we propose a novel approach for constraining the parameters a and c, as well as establishing a relationship among a, b, and c.

XIV. Directions for Future Research

To help researchers to realize the possible directions for future research, we mention some related papers that also handle operation research problems by algebraic methods. Yen [19] constructed an intuitive method to solve inventory models that had been examined by Luo and Chou [22], Chang et al. [13], Ronald et al. [11], Cárdenas-Barrón [21], and Grubbström and Erdem [20].

Yen [23] studied (a) Çalışkan [24], (b) Çalışkan [25], (c) Wee [26], (d) Çalışkan [27], and (e) Minner [28], to point out their questionable findings and then provided revisions. Wang and Chen [29] indicated that the findings of Aguaron and Moreno-Jimenez [30] are questionable and then showed improvements. On the other hand, Wang and Chen [29] offered their enhancements for Yen [19]. Yang and Chen [31] examined Çalışkan [32], Osler [33], Çalışkan [34], and Yen [19] and then mentioned their improvements.

Moreover, we also refer to several important articles that were recently published: Wichapa and Sodsoon [35], Ojo et al. [36], Wang et al. [37], Pappalardo et al. [38], Sun et al. [39], and Zhang et al. [40] to help researchers locate possible directions for future study.

XV. Conclusion

We find a simpler algorithm to estimate the optimal solution for the inventory model with a linear trend in demand. Our method will infer the equal order time interval, hence we induce the new procedure without facing the problem to adapt the last or the first time interval. From the numerical examples, we ascertain that the relative error of the total cost with respect to the analytic optimal solution of Donaldson [2] is moderate.

On the other hand, we study the inventory model proposed by Xiao and Qi [17]. They applied an analytic procedure to locate the optimal point for the price strategy for two retailers. We demonstrate that this kind of inventory system can be solved through an algebraic process. Our proposed process not only locates the optimal price pair for two retailers but also finds the maximum profit for the inventory system. Our results demonstrate the distinct feature of algebraic methods.

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