Discussion for Consistency Test in Analytic Hierarchy Process

Chao-Min Wang, Yu-Lan Wang, Ming-Li Chen

Abstract—**We aim to address an unresolved question raised in a previously published paper regarding the 1-9 bounded scale and consistency index. The objective of this study is to demonstrate that researchers need not be concerned about the inconsistency phenomenon highlighted in the aforementioned article. This paper has a four-fold purpose. Firstly, we illustrate that by employing an alternative ordering approach for the entries of a comparison matrix, the issue of inconsistency can be eliminated. Secondly, we emphasize that, by accepting Murphy's assumptions, we can present an efficient method for deriving the priority vector without relying on the comparison matrix. This reveals that Murphy's assumptions are excessively stringent for ordinary researchers to accept. Additionally, we offer a simplified version to establish the condition for the optimal solution in an inventory model with a temporary price discount, which was previously proposed in another published paper. Consequently, an additional condition put forth by the aforementioned paper becomes redundant. We have made revisions to a related paper discussing local stability intervals in the Analytic Hierarchy Process.**

Index Terms—**Comparison matrix, Consistency test, Decision making, Analytic Hierarchy Process, Inventory, Discount, Supplier-restricted**

I. Introduction

ANALYTIC hierarchy process has been extensively used as a multi-criteria decision-making approach in the area of applications such as personal, manufacturing, industry, social, education, etc. The paper tries to solve real-world problems in complicated environments with many criteria and several decision variables.

During the Analytic Hierarchy Process, the decision-maker designs several layer hierarchies and then makes judgments to compare alternatives under the criterion or to compare criteria under the higher layer criterion to obtain relative weights and then synthesizes the final weights to distribute the source to each alternative or choice the best alternative. A paper written by Vaidya et al. [1] provides an excellent overview of applications of the Analytic Hierarchy Process by reviewing a total of 154 articles, which have been published in international journals of high repute, regards it as a weight estimation technique in many fields such as selection, evaluation, benefit-cost analysis, allocations, planning and development, priority and ranking, decision making, forecasting, and so on.

Despite this, the issue of the inherent suitability and completeness of the Analytic Hierarchy Process is still of interest and is discussed by some researchers. For example, Yang et al. [2] showed that the method of Bernhard and Canada [3] is incomplete, and proposes some revisions of it. An article by Chao et al. [4] clarified that the diagonal procedure of Finan and Hurley [5] did not pass the consistency test of Saaty [6]. In addition, The present paper looks into a short communication paper by Murphy [7] intending to examine whether his point of limits on the Analytic Hierarchy Process from its consistency index is correct or not, and furthermore, we will try to provide a further discussion to point out the intrinsic problem for decision as regards entries of a comparison matrix as well as a method of deriving the priority vector.

There is a paper by Kwiesielewicz and van Uden [8] that cited Murphy [7] in their references. Kwiesielewicz and van Uden [8] concentrated on the contradiction phenomenon in a comparison matrix to point out that sometimes researchers cannot derive a consistent relation among alternatives.

However, Kwiesielewicz and van Uden [8] did not aware of the questionable results that will be discussed in this article.

So, the objective of this paper is to show that if we use another order to decide the entries of a comparison matrix then the inconsistency will disappear.

Based on the assumptions presented by Murphy [7], simultaneously, the present paper can provide an efficient method to derive the priority vector without using a comparison matrix.

In section 2 of this paper, we review the Murphy approach and outline how his revisions enhance the Analytic Hierarchy Process.

In section 3, we present the inconsistency caused by Murphy's approach and try to adopt our proposed method to consider another ordering to decide entries of a comparison matrix through which the inconsistency disappears.

Section 4 offers an easy solution method to derive the priority vector without reference to the comparison matrix of the Analytic Hierarchy Process for Murphy's problem. Finally, the conclusions recommend that researchers need not worry about the inconsistency phenomenon raised by Murphy [7].

II. Review of Previous Approach

Deng et al. [9] studied Murphy [7] for the comparison

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matrix with the following entries: $a_{11}=1$, $a_{12}=x$, $a_{13}=y$, $a_{21}=\frac{1}{x}$, $a_{22}=1$, $a_{23}=9$, $a_{31}=\frac{1}{y}$, $a_{32}=\frac{1}{9}$, and $a_{33}=1$, to derive the maximum eigenvalue, denoted as v_{max}, then

$$v_{max}=3, \tag{2.1}$$

and the normalized principal right eigenvector that is expressed as $(P_1(x), P_2(x), P_3(x))$, where

$$P_1(x)=\frac{9x}{10+9x}, \tag{2.2}$$

$$P_2(x)=\frac{9}{10+9x}, \tag{2.3}$$

and

$$P_3(x)=\frac{1}{10+9x}, \tag{2.4}$$

such that Deng et al. [9] pointed out their questionable results and then provided improvements.

Hsueh [10] studied the three-by-three comparison matrix proposed by Murphy [7], with $a_{11}=1$, $a_{12}=a$, $a_{13}=(ab+x)$, $a_{21}=\frac{1}{a}$, $a_{22}=1$, $a_{23}=b$, $a_{31}=\frac{1}{ab+x}$, $a_{32}=\frac{1}{b}$, and $a_{33}=1$, and then Hsueh [10] examined the monotonic properties of the eigenvector for the comparison matrix that was proposed by Murphy [7] to amend his paper.

Hsueh et al. [11] considered the three-by-three comparison matrix proposed by Murphy [7], with $a_{11}=1$, $a_{12}=x$, $a_{13}=y$, $a_{21}=\frac{1}{x}$, $a_{22}=1$, $a_{23}=9$, $a_{31}=\frac{1}{y}$, $a_{32}=\frac{1}{9}$, and $a_{33}=1$, and then Hsueh et al. [11] decided on the estimation of priority vectors for the comparison matrix that was published in the Journal of Discrete Mathematical Sciences and Cryptography to develop properties to compare alternatives and the rank reversal problems to provide a further examination of Murphy [7].

Let us recall the approach of Murphy [7]. There are four alternatives, A_i, for $i=1,2,3,4$. Then he tried to create a comparison matrix, say $\left[a_{ij}\right]_{4\times 4}$ under the condition that A_i is strongly favored (5 on the semantic scale) over A_{i+1}. Hence, he assumed that $a_{12}=a_{23}=a_{34}=5$.

He wanted the resultant comparison matrix to be as consistent as possible and still follow the 1-9 bounded scale proposed by Saaty [12].

Therefore, to keep consistent, he knew $a_{13}=a_{12}a_{23}$ and $a_{14}=a_{12}a_{23}a_{34}$ and then he assumed that $a_{13}=25$.

By the same approach, in the beginning, he assumed that $a_{24}=25$, and $a_{14}=125$.

However, the above assignment violates the 1-9 bounded scale proposed by Saaty [12] in that he took the values for a_{13}, a_{14} and a_{24} to be as big as possible, which then yields $a_{13}=a_{14}=a_{24}=9$.

By the reciprocal rule, $a_{ij}a_{ji}=1$, the following comparison matrix was constructed by Murphy [7]

$$A=\begin{bmatrix} 1 & 5 & 9 & 9 \\ 1/5 & 1 & 5 & 9 \\ 1/9 & 1/5 & 1 & 5 \\ 1/9 & 1/9 & 1/5 & 1 \end{bmatrix}, \tag{2.5}$$

with the maximum eigenvalue,

$$\lambda_{\max}=4.47, \tag{2.6}$$

that is more than the bound of

$$\lambda_{\max}=4.24, \tag{2.7}$$

that was proposed by Vargas [13] for 4 by 4 pairwise comparison matrices to imply that A is inconsistent. Based on this kind of approach, he suggested that his revisions enhance the Analytic Hierarchy Process.

In the next two sections, we will first demonstrate in Section 3 that the inconsistency is caused by Murphy's approach. If we adopt other orderings to decide the value of a_{ij} then the inconsistency disappears.

In Section 4, under the assumptions of Murphy [7], we offer two direct results for the best selection problem and allocation problem, respectively. This indicates that it is unnecessary to create a comparison matrix and the discussion of the consistency problem in Murphy [7] is redundant.

III. Our Proposed Method

For the same problem, we still try to construct a 4×4 comparison matrix under the following principle:

(1) There are four alternatives, A_i, for $i=1,2,3,4$, such that A_i is strongly favored over A_{i+1}.

(2) Under the 1-9 bounded scale proposed by Saaty [12].

(3) The resultant matrix will be created as consistently as possible, except violated (2).

(4) The order of determination is the first row from right to left so that a_{14} will be decided first, then a_{13}, and then a_{12}.

The difference between our method and Murphy's approach is that the ordering by his approach is to decide a_{12}, a_{23} and a_{34}.

The followings are the details of our method. First, we construct a consistent comparison matrix without the 1-9 bounded scale then

$$A=\left[a_{ij}\right]_{4\times 4}=\begin{bmatrix} 1 & x & x^2 & x^3 \\ 1/x & 1 & x & x^2 \\ 1/x^2 & 1/x & 1 & x \\ 1/x^3 & 1/x^2 & 1/x & 1 \end{bmatrix}. \tag{3.1}$$

Owing to the fact that A_4 is absolutely more important than A_1 so we assign $a_{14}=9$.

From $x^3=9$ and x will be selected for the 1-9 bounded scale, so we take $x=2$ to imply that $a_{13}=4$, and $a_{12}=2$. Hence, we construct that

$$A=\begin{bmatrix} 1 & 2 & 4 & 9 \\ 1/2 & 1 & 2 & 4 \\ 1/4 & 1/2 & 1 & 2 \\ 1/9 & 1/4 & 1/2 & 1 \end{bmatrix}, \tag{3.2}$$

with the maximum eigenvalue,

$$\lambda_{\max}=4.0017. \tag{3.3}$$

that is less than the bound of

$$\lambda_{\max} = 4.24\,, \tag{3.4}$$

that was proposed by Vargas [13] so that by our method the comparison matrix in equation (3.2) is consistent.

Based on the above discussion, we point out that the inconsistency may be caused by Murphy's approach to deciding the upper diagonal, $a_{i,i+1}$, for $i = 1,...,n-1$. If we use another algorithm, for example, our proposed method, then inconsistency disappears.

IV. An Easy Solution for Previous Approach

On the other hand, we will begin to discuss the fundamental issue in Murphy's approach. To simplify the expression, we assume that w_i is the weight for the alternative A_i.

Based on Murphy's assumption, the decision maker already makes up his/her mind that the following holds for $i = 1,...,n-1$,

$$\frac{w_i}{w_{i+1}} = b > 1\,. \tag{4.1}$$

For the best selection problem, for example, Carnero [14], from $w_1 > w_2 > ... > w_n$, so the optimal alternative is A_1.

Moreover, it yields that for i = 1, 2, ..., n,

$$\frac{w_{i+1}}{w_i} = c^i\,. \tag{4.2}$$

with $c = 1/b$ so that for allocation problems, for example, Saaty et al. [15], we know the relative ratio among alternatives

$$\left(\frac{w_1}{w_1}, \frac{w_2}{w_1}, ..., \frac{w_n}{w_1}\right) = \left(1, c, ..., c^{n-1}\right). \tag{4.3}$$

After normalization, the priority vector for alternatives is expressed as

$$\left(\frac{1}{\sum_{k=0}^{n-1} c^k}, \frac{c}{\sum_{k=0}^{n-1} c^k}, ..., \frac{c^{n-1}}{\sum_{k=0}^{n-1} c^k}\right). \tag{4.4}$$

That can be directly derived without reference to the comparison matrix of the Analytic Hierarchy Process.

As we explained, there is neither a need to create any comparison matrix nor to worry about the consistency test.

V. Further Applications for Our Results

Many papers tried to provide revisions for the Analytic Hierarchy Process. For example, Barzilai [16] tried to use the geometric mean to replace the maximum eigenvalue and eigenvector method. Chang et al. [17] pointed out that the component-wise operation of matrices proposed by Barzilai [16] sometimes will imply unreasonable findings. And then, Chang et al. [17] presented their revisions and improvements for the theorems in Barzilai [16]. At last, Chang et al. [17] showed that the numerical examples constructed by Barzilai [16] that committed calculation mistakes. Therefore, the conclusion proposed by Barzilai [16] is invalid to challenge the eigenvector method proposed by Saaty [6].

Macharis et al. [18] considered connecting (i) the Analytic Hierarchy Process with (ii) the preference ranking organization method for enrichment evaluations to develop a new approach to deal with multicriteria decision-making problems. To construct a consistent comparison matrix that satisfies the opinion provided by an expert, Macharis et al. [18] developed a new approach to construct comparison matrices where the changing order is arranged as a decreasing sequence of columns. Chu et al. [19] pointed out the ordering of changes is questionable that should be revised to a decreasing sequence of rows. Moreover, Chu et al. [19] proved that after their revisions, they can verify that the developed matrix is consistent and preserved all opinions of the expert. Lin [20] showed that the construction of Chu et al. [19] did not follow the rule of Macharis et al. [18] such that Lin [20] developed a new procedure that followed the rule proposed by Macharis et al. [18]. Moreover, Lin [20] provided analytical proof that the numerical example proposed by Chu et al. [19] can be treated abstractly to show that dilemma will happen.

Based on our above discussion, and the examination of this paper, we can claim that more open questions to criticize the Analytic Hierarchy Process will provide more motivation for researchers to realize and utilize this valuable tool for operation research.

VI. A Related Problem

In this section, we will provide a short communication on "Note on supplier-restricted order quantity under temporary price discounts". We study the inventory model with temporary price discounts under supplier-restricted order quantities. We examine a published paper to point out that their proof is too complicated and their assumption of the finite union of closed intervals is redundant. In this short communication, we offer simpler proof to show that for the supplier-restricted inventory model, the optimal replenishment time is the minimum inventory level during the on-sale period.

We make the same assumptions as Ardalan [21], Aull-Hyde [22], and Chu et al. [23] that in most practical contexts the sale period is typically short relative to the regular inventory cycle, lead time is zero, and shortages are not allowed. Moreover, to develop the inventory model, the following notation is used.

T_S is the time interval between the receipt of special order and the next replenishment time;

TC_S is the total cost of the special order during the time interval T_S;

TC_r is the total cost of the regular ordering policy during the time interval T_S;

$g = TC_S - TC_r$: is the profit for applying for the special order during the time interval T_S;

t_F is the finish time of the sale period;

t_B is the start time of the sale period;

q is the level of remnant inventory when the special order is placed, with $q \geq 0$;

C is the ordering cost per order;

F is the annual inventory carrying cost, as a percentage of unit cost;

R is the annual demand in units;

d is the discount on unit price; P is the unit price per item;

$t^{\#}$ is the time at which $Q^{\#}$ is placed, assuming zero lead time;

t^{*} is the time at which Q^{*} is placed;

t_R is the time of the next regular scheduled replenishment after t_B;

$Q^{\#}$ is the optimal restricted special order quantity;

X is the set of restricted special orders available from suppliers at a discounted price $P-d$;

Q^{*} is the optimal unrestricted special order quantity;

$Q_O = \dfrac{2CR}{FP}$ is the regular optimal order quantity with the full unit price P;

$Q_L = \dfrac{2CR}{F(P-d)}$ is the regular optimal order quantity using the sale price;

Q is the unrestricted special order quantity;

$Q_S = \dfrac{dR}{F(P-d)} + \dfrac{PQ_O}{P-d}$, (Tersine [24]): is the special optimal order quantity when $q=0$.

VII. Review of Previous Results

Ardalan [21] constructed an inventory system to evaluate the benefit of an unconditional environment, by adopting a special order during the time interval T_S,

$$g(q,Q) = Qd + \frac{FPQQ_O}{R} - \frac{qFQ(P-d)}{R} - \frac{FQ^2(P-d)}{2R} - C \qquad (7.1)$$

to maximize the profit where the remnant inventory level q is fixed, and the following quantity

$$Q_{Or}(q) = \frac{dR}{F(P-d)} + \frac{PQ_O}{P-d} - q = Q_S - q, \qquad (7.2)$$

is the special ordering quantity that was proposed by Aull-Hyde [22].

Based on Equation (7.2) of Aull-Hyde [22], Chu et al. [23] tried to point out that the special order is dependent on q so they used $Q(q)$ to replace Q and then rewrote the result of her Equation (7.1) as

$$g(q,Q(q)) = \frac{C}{Q_L^2}\left[(Q_{Or}(q))^2 - Q_L^2 - (Q(q) - Q_{Or}(q))^2\right]. \qquad (7.3)$$

The optimal replenishment strategy for the unconditional environment is to adopt $Q(q) = Q_{Or}(q)$ and reduce the amount of q to its minimum, because of two conditions: $0 \leq q \leq Q_0$ and $Q_0 < Q_s$.

For the restricted model, Chu et al. [23] faced the following problem

$$\max g(q,Q(q)) \qquad (7.4)$$

for $q = q(t)$ with $t_B \leq t \leq t_F$ and $Q(q) \in X$. By Equation (7.3), when q is fixed, we define that

$$g(q,Q_X(q)) = \max\{g(q,Q(q)) : Q(q) \in X\} \qquad (7.5)$$

where $Q_X(q)$ satisfies

$$|Q_X(q) - Q_{Or}(q)| = \min\{|Q(q) - Q_{Or}(q)| : Q(q) \in X\}. \qquad (7.6)$$

Here, Chu et al. [23] pointed out that the condition for X being a closed set must be put in to insure that Equation (7.6) has solutions, However, in Chu et al. [23], for technical reasons, they further assumed that X is a union of finite numbers of closed intervals as $X = \bigcup_{k=1}^{n}[Q_{2k-1}, Q_{2k}]$ with $Q_{2k-1} \leq Q_{2k}$ and $Q_{2k} < Q_{2k+1}$. The on-sale period is $[t_B, t_F]$ and the regular order times are expressed as t_{R-1}, t_R and t_{R+1}. In our proof, we do not need to divide it into two cases $t_F \leq t_R$ and $t_F > t_R$ as Chu et al. [23] did, therefore for the case $t_F > t_R$, their sophisticated extension of on sale period from $[t_B, t_F]$ to $[t_B, t_F + \delta]$ is shown to be unnecessary. In Chu et al. [23], they wanted to prove that

"During the on-sale period, the maximum of $g(q,Q_X(q))$ occurs when q attains its minimum value" (7.7)

The key of their proof is to show that $g(q,Q_X(q))$ is a decreasing function of q. Their procedure is to prepare to derive the explicit formulation of $g(q,Q_X(q))$. From private communication, they have worked out the simplest case with $X = [Q_1, Q_2]$. However, for the more difficult case of $X = [Q_1, Q_2] \cup [Q_3, Q_4]$, there are too many different cases beyond the control of ordinary people. Hence, they develop the locally explicit formulation of $g(q,Q_X(q))$ for $X = \bigcup_{k=1}^{n}[Q_{2k-1}, Q_{2k}]$. However, the main purpose of their note is to show that Equation (7.7) is valid not to derive the explicit formula of $g(q,Q_X(q))$. We will show that their procedure contains too many redundant materials and the restriction of $X = \bigcup_{k=1}^{n}[Q_{2k-1}, Q_{2k}]$ is unnecessary.

VIII. Our Improved Verification

Given q_1 and q_2 with $q_1 < q_2$, we will directly prove that

$$g(q_1, Q_X(q_1)) \geq g(q_2, Q_X(q_2)). \qquad (8.1)$$

In the following, we will divide the problem into two cases: $Q_X(q_1) \geq Q_S - q_1$, and $Q_X(q_1) < Q_S - q_1$.

For case $Q_X(q_1) \geq Q_S - q_1$

According to $Q_S - q_1 > Q_S - q_2$ and Equation (7.6), we know that $X \cap (2(Q_S - q_1) - Q_X, Q_X(q_1)) = \phi$, then we have $Q_X(q_2) \in [Q_X(q_1), \infty)$ or $Q_X(q_2) \in [0, 2(Q_X - q_1) - Q_X(q_1)]$ hence, we have two situations: (a) $Q_X(q_1) = Q_X(q_2)$ and (b) $Q_X(q_1) - [Q_S - q_1] \leq [Q_S - q_1] - Q_X(q_2)$.

For situation (a), if $Q_X(q_1) = Q_X(q_2)$, then it follows that

$$g(q_1, Q_X(q_1)) - g(q_2, Q_X(q_2)) = 2Q_X(q_1)(q_2 - q_1) > 0. \tag{8.2}$$

For situation (b), we have

$$2(Q_S - q_1) \geq Q_X(q_1) + Q_X(q_2). \tag{8.3}$$

By Equation (7.6), we derive that,

$$\begin{aligned} Q_X(q_1) - (Q_S - q_2) &\geq |(Q_S - q_2) - Q_X(q_2)| \\ &\geq (Q_S - q_2) - Q_X(q_2), \end{aligned} \tag{8.4}$$

such that

$$Q_X(q_1) + Q_X(q_2) \geq 2(Q_S - q_2). \tag{8.5}$$

Combining Equations (8.3) and (8.5) yields

$$\begin{aligned} g(q_1, Q_X(q_1)) &= 2(Q_S - q_1)Q_X(q_1) - (Q_X(q_1))^2 \\ &\geq Q_X(q_1)Q_X(q_2) \\ &\geq 2(Q_S - q_2)Q_X(q_2) - (Q_X(q_2))^2 \\ &= g(q_2, Q_X(q_2)). \end{aligned} \tag{8.6}$$

For case $Q_X(q_1) < Q_S - q_1$

First, we need the following lemma.

Lemma 1. $Q_X(q_1) \geq Q_X(q_2)$.

(proof of Lemma 1)

If $Q_S - q_2 \leq Q_X(q_1) < Q_S - q_1$, by Equation (7.6), then it implies $Q_X(q_1) \geq Q_X(q_2)$. By way of contradiction, we assume that $Q_X(q_1) < Q_X(q_2)$, and $Q_X(q_1) < Q_S - q_2$.

By Equation (7.6), then it implies

$$Q_X(q_2) > Q_S - q_1 > Q_S - q_2 > Q_X(q_1), \tag{8.7}$$

and

$$Q_X(q_2) - (Q_S - q_1) \geq (Q_S - q_1) - Q_X(q_1), \tag{8.8}$$

so we compute that

$$\begin{aligned} Q_X(q_2) - (Q_S - q_2) &> Q_X(q_2) - (Q_S - q_1) \\ &\geq (Q_S - q_1) - Q_X(q_1) \\ &> (Q_S - q_2) - Q_X(q_1) \\ &= |(Q_S - q_2) - Q_X(q_1)|. \end{aligned} \tag{8.9}$$

Equation (8.9) leads to a contradiction with Equation (7.6). Hence, we finish the proof for our claim of $Q_X(q_1) \geq Q_X(q_2)$.

Under this case $Q_X(q_1) < Q_S - q_1$, we assume an auxiliary function,

$$g(q_1, y) = 2(Q_S - q_1)y - y^2, \tag{8.10}$$

then $g(q_1, y)$ is an increasing function for $y < Q_S - q_1$. Therefore, using

$$Q_S - q_1 > Q_X(q_1) \geq Q_X(q_2), \tag{8.11}$$

we yield that

$$\begin{aligned} &g(q_1, Q_X(q_1)) = \\ &2(Q_S - q_1)Q_X(q_1) - (Q_X(q_1))^2 \\ &\geq 2(Q_S - q_1)Q_X(q_2) - (Q_X(q_2))^2 \\ &= g(q_1, Q_X(q_2)). \end{aligned} \tag{8.12}$$

Moreover, we have

$$g(q_2, Q_X(q_2)) = 2(Q_s - q_2)Q_X(q_2) - (Q_X(q_2))^2 < g(q_1, Q_X(q_2)). \tag{8.13}$$

Combining Equations (8.2), (8.6), (8.12) and (8.13) in the above discussion, we have proved that Equation (8.1) is valid.

Our proof only requires that X, the restricted special order quantity provided by the supplier under discounted price environment is reasonable which must be a closed set. Chu et al. [23] required an extra condition of X, to be a union of finite numbers of closed intervals so that they could locally extend the domain. Our proof avoids this unnecessary requirement.

IX. Application to Inventory Systems

We apply our previously derived findings to inventory systems. These problems had been studied by Teng [25], Wee et al. [26], Cárdenas-Barrón [27], Wee et al. [28], Wee and Chung [29], Sphicas [30], Ronald et al. [31], Minner [32], Grubbström and Erdem [33], Grubbström [34], Chung and Wee [35], Chang et al. [36], and Cárdenas-Barrón [37].

Cárdenas-Barrón [27] published an article to revise Teng [25], and also to point out three necessary conditions to apply Arithmetic-Geometric Mean: (i) When all functions are assumed to be identical, then the system of all proposed equations is solvable to find the minimum solution, (ii) Multiplication of all proposed functions should be a constant, and (iii) All terms should be positive functions. First, we recall the Economic Ordering Quantity model with linear backorder cost. Where Q is denoted as the ordering quantity, and r is the fill rate that was proposed by Wee et al. [26], under the condition that the optimal fill rate satisfied

$$r = \frac{v}{h + v}, \tag{9.1}$$

however, Wee et al. [26] did not provide any reason to support their assertion.

Wee et al. [26] mentioned that they will apply the Cost-difference Comparisons Method to derive the optimal fill rate. In Minner [32], he tried to balance of holding cost and backorder cost, then Minner [32] obtained that

$$hrQ = v(1-r)Q \tag{9.2}$$

to imply that

$$r = \frac{v}{h+v}. \tag{9.3}$$

We know that $B = (1-r)Q$ is the back order quantity, and then rQ is the beginning inventory lever. We must point out that Cárdenas-Barrón [27] did not explain how did he derive the optimal solution,

$$Q^* = \sqrt{\frac{2Ad(h+v)}{hv}}. \tag{9.4}$$

Moreover, Cárdenas-Barrón [27] did not write down his objective function.

X. Our Improvements

We recall the inventory model,

$$TC(Q,B) = \frac{Ad}{Q} + \frac{hr^2Q}{2} + \frac{vB^2}{Q}, \tag{10.1}$$

under two conditions

$$B = (1-r)Q, \tag{10.2}$$

and

$$r = \frac{v}{h+v}. \tag{10.3}$$

We plug two conditions of Equations (10.2) and (10.3) into Equation (10.1) to yield that

$$TC(Q) = \frac{Ad}{Q} + \frac{hr^2}{2}Q + \frac{v(1-r)^2}{2}Q,$$

$$= \frac{Ad}{Q} + \frac{hv}{2(h+v)}Q. \tag{10.4}$$

Based on Equation (10.4), then the optimal solution of Equation (9.4) can be easily derived.

Based on our above discussions, we can apply our findings to other inventory models that had been examined by Cárdenas-Barrón [27] with questionable derivations.

XI. Application to Economic Production Quantity Model

Next, we examine the Economic Production Quantity model with a linear backorder cost. We used the same expressions proposed by Cárdenas-Barrón [27]. We express that $T_1 + T_2 + T_3 + T_4 = T$ are the four phases in a replenishment cycle.

Under the following conditions, $dT = Q = p(T_2 + T_3)$ is the total demand, and the full rate r is the ratio of $\frac{T_3 + T_4}{T}$, and then it shows that

$$hrT = v(1-r)T \tag{11.1}$$

is the balance between backorder cost and holding cost, then according to Equation (11.1), we obtained that

$$r = \frac{v}{h+v}. \tag{11.2}$$

Based on the continuity of the objective function balance of stock and shortages, researchers derived that

$$rT = T_3 + T_4 = \frac{p}{d}T_3 = \frac{p}{p-d}T_4, \tag{11.3}$$

$$(1-r)T = T_1 + T_2 = \frac{p}{p-d}T_1, \tag{11.4}$$

$$T = \frac{p}{(p-d)r}T_4 = \frac{p}{(p-d)(1-r)}T_1, \tag{11.5}$$

$$rT_1 = (1-r)T_4, \tag{11.6}$$

$$(p-d)T_3 = dT_4, \tag{11.7}$$

$$(p-d)T_2 = dT_1, \tag{11.8}$$

$$h(T_3 + T_4) = v(T_1 + T_2), \tag{11.9}$$

$$h\frac{v}{h+v}\frac{Q}{d} = v\left(\frac{B}{d} + \frac{B}{p-d}\right), \tag{11.10}$$

and

$$B = \left(\frac{p-d}{p}\right)\frac{h}{h+v}Q. \tag{11.11}$$

Cárdenas-Barrón [27] did not explain how did he derive the optimal solution,

$$Q^* = \sqrt{\frac{2Ad(h+v)}{(1-(d/p))hv}}. \tag{11.12}$$

We can say that solving the problem of the Economic Production Quantity model under the restrictions of Equations (11.2-11.11) to derive the optimal solution of Equation (11.12) that will be an interesting research topic for future practitioners.

XII. We Study a Related Problem

Recently, Wang and Chen [38] discussed local stability intervals in Analytic Hierarchy Process that was proposed by Aguaron and Moreno-Jimenez [39] to show several questionable results in their derivations. Following this research trend, we will study a related paper of Aguaron and Moreno-Jimenez [40] to examine the geometric consistency index in Analytic Hierarchy Process.

First, we recall their definition for ε as

$$\varepsilon = \max_{i,j}\{|\varepsilon_{ij}|\}. \tag{12.1}$$

However, $(a_{ij})_{n\times n}$, is a positive reciprocal matrix and then the row geometric mean is defined as

$$w_j = \left(\prod_{k=1}^{n} a_{jk}\right)^{1/n}, \tag{12.2}$$

such that Aguaron and Moreno-Jimenez [40] defined that

$$e_{ij} = \frac{w_j}{w_i}a_{ij}, \tag{12.3}$$

and

$$\varepsilon_{ij} = \log e_{ij}. \tag{12.4}$$

We recall that $(a_{ij})_{n\times n}$, is a positive reciprocal matrix, and then it follows that

$$a_{ij}a_{ji} = 1. \tag{12.5}$$

Owing to Equation (12.5), we compute that

$$e_{ij}e_{ji} = \left(\frac{w_i}{w_j}a_{ij}\right)\left(\frac{w_j}{w_i}a_{ji}\right) = 1, \quad (12.6)$$

and then we obtain that

$$\ln(a_{ij}) + \ln(a_{ji}) = \ln(e_{ij}e_{ji}) = \ln(1) = 0. \quad (12.7)$$

We refer to Equation (12.4), it follows that

$$\varepsilon_{ij} + \varepsilon_{ji} = 0. \quad (12.8)$$

Therefore, ε_{ij} and ε_{ji} are additive inverse for each other such that

$$\left|\varepsilon_{ij}\right| = \left|\varepsilon_{ji}\right| = \max\{\varepsilon_{ij}, \varepsilon_{ji}\}. \quad (12.9)$$

Consequently, we may define ε without the absolute sign to simplify the expressions.

They need to prove that

$$\sum_i \varepsilon_{ij} = 0. \quad (12.10)$$

We will provide the following new lemma.

Lemma 1. $\prod_{i=1}^{n} e_{ij} = 1$, for $j = 1,2,\cdots,n$.

(Proof) In the following, we provide a direct verification.

$$\prod_{i=1}^{n} e_{ij} = \prod_{i=1}^{n} \frac{w_j}{w_i} a_{ij} = \prod_{i=1}^{n} \frac{\left(\prod_{k=1}^{n} a_{jk}\right)^{1/n}}{\left(\prod_{k=1}^{n} a_{ik}\right)^{1/n}} a_{ij}$$

$$= \frac{\prod_{k=1}^{n} a_{jk}}{\left(\prod_{i,k=1}^{n} a_{ik}\right)^{1/n}} \prod_{i=1}^{n} a_{ij}, \quad (12.11)$$

since the comparison matrix, $\left(a_{ij}\right)_{n\times n}$, is a positive reciprocal matrix with $a_{ij}a_{ji} = 1$ such that

$$\prod_{i,j=1}^{n} a_{ij} = 1, \quad (12.12)$$

and

$$\prod_{i=1}^{n} a_{ij} = \prod_{i=1}^{n} \frac{1}{a_{ji}} = \frac{1}{\prod_{i=1}^{n} a_{ji}}. \quad (12.13)$$

Thus, we obtain that $\prod_{i=1}^{n} e_{ij} = 1$ to finish the proof of Lemma 1.

Based on our findigns of Lemma 1, we develop the following corollary for further examination.

Corollary 1. $\sum_i \varepsilon_{ij} = 0$, for $j = 1,2,\cdots,n$.

(Proof) Since $\varepsilon_{ij} = \log e_{ij}$ and the property of logarithmic function $\log(ab) = \log a + \log b$ and $\log 1 = 0$, from Lemma 1, we derive the Corollary 1.

From Equation (A.11) of Aguaron 2003, we improve the computation as follows.

$$n\lambda_{\max} = \sum_{i,j} e_{ij}\left(1 + d_j\right)$$

$$= \sum_{i,j}\left(1 + \varepsilon_{ij} + \frac{1}{2}\varepsilon_{ij}^2 + o\left(\varepsilon^3\right)\right)\left(1 + d_j\right)$$

$$= \sum_{i,j}\left(1 + \varepsilon_{ij} + \frac{1}{2}\varepsilon_{ij}^2 + d_j + \varepsilon_{ij}d_j + \frac{1}{2}\varepsilon_{ij}^2 d_j\right) + o\left(\varepsilon^3\right). \quad (12.14)$$

We accept their results of (i) owing to $e_{ij}e_{ji} = 1$, with $e_{ij} = \frac{w_j}{w_i}a_{ij}$ and $\varepsilon_{ij} = \log e_{ij}$ then $\varepsilon_{ij} + \varepsilon_{ji} = 0$ such that $\sum_{i,j} \varepsilon_{ij} = 0$ and (ii) $\sum_j d_j = 0$.

Moreover, by our Corollary 1, we can explain the following computation

$$\sum_{i,j} \varepsilon_{ij} d_j = \sum_j d_j \sum_i \varepsilon_{ij} = \sum_j d_j 0 = 0. \quad (12.15)$$

Now, we state the main contribution of this section. From Equation (A.10), Aguaron and Moreno-Jimenez [40] derived that

$$-(n-1)\left(e^{\varepsilon} - 1\right) \le d_i \le e^{\varepsilon} - 1. \quad (12.16)$$

so d_i is of the order $e^{\varepsilon} - 1 = o(\varepsilon)$. Recall that $\varepsilon = \max_{i,j}\{\varepsilon_{ij}\}$, it yields that $\sum_{i,j} \varepsilon_{ij}^2 d_j$ is of the order $o\left(\varepsilon^3\right)$.

We conclude our discussions as follows,

(a) the explanations of Aguaron and Moreno-Jimenez [40] to estimate

$$\lambda_{\max}\left(1 + d_i\right) = n + o(\varepsilon). \quad (12.17)$$

in which contained questionable result of $\sum_j d_j \varepsilon_{ij}$ is of the order $o(\varepsilon)$.

(b) according to Equation (A.12), they claimed that

$$\lambda_{\max} = n + o(\varepsilon), \quad (12.18)$$

without proper explanation.

(c) the estimation of

$$d_i = o(\varepsilon). \quad (12.19)$$

in Equation (A.13) of Aguaron and Moreno-Jimenez [40] is unnecessary.

Based on our above examinations, we provide a further study for Aguaron and Moreno-Jimenez [40].

XIII. Directions for Future Research

There are several recently published papers that are worthy to mention to help researchers look for possible directions for future studies. Yang and Chen [41] studied the questionable results of four published papers and then they provided revisions. Wang and Chen [38] examined the local stability intervals in Analytic Hierarchy Process that was proposed by Aguaron and Moreno-Jimenez [39]. Yen [42] demonstrated solving inventory systems by an intuitive algebraic process. Wang et al. [43] developed natural heuristic algorithms to solve feature selection problems. Basapur et al. [44] considered constraints-relaxed functional dependency for data privacy preservation systems. Liu et al. [45] constructed a data-based compensation system through a gold cyanide leaching process to locate an optimal operation setting. For vibration fault diagnosis, Gelman and Patel [46] adopted Novel Intelligent Data Processing Technology to study nonstationary nonlinear wavelet bispectrum. Based on small samples, Li et al. [47] applied an improved generative adversarial network for fault diagnosis. For bus passengers, Sooknum and Pochai [48] used a mathematical system to estimate airborne infection risk. The above-mentioned nine papers can help researchers realize the current academic trend with their related topics.

XIV. Conclusion

The main contribution of this paper is to present an easy method for deriving the priority vector for alternatives without creating any comparison matrix in the Analytic Hierarchy Process based on the above discussion.

As a result, we may advise researchers not to worry about the inconsistency phenomenon raised by Murphy. The Analytic Hierarchy Process is a useful method in multi-criteria decision-making problems.

We have provided simple proof to show that for the supplier-restricted inventory model, the optimal replenishment time is the minimum inventory level during the on-sale period. Our proof avoids the difficult expression of a decreasing function under various conditions and demonstrates the beauty of analytical proof.

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