Study for Contradictory Pairwise Comparison Matrices

Yung-Ning Cheng, Kou-Huang Chen

Abstract—**The purpose of this paper is to point out the questionable results of a published paper with respect to the Analytical Hierarchy Process. They created contradictory matrices such that there are pairwise comparison matrices that pass the consistency test but are contradictory. The main purpose of this paper is to point out that their six cases for contradiction can be simplified as two cases. Moreover, we show that their contradictory phenomenon is a local property that will not influence the result of the consistent test.**

Index Terms—**Contradictory phenomenon, Consistency test, Pairwise comparison, Analytic hierarchy process**

I. Introduction

There are more than two hundred and seventy papers that cited the article of Kwiesielewicz and Uden [1] in their references. We just list a few of them in the following: Liang et al. [2], Ortiz-Urbina et al. [3], Jiang et al. [4], Wang and Xu [5], Lai et al. [6], Kou et al. [7], Lin et al. [8], Siraj et al. [9], Ishizaka and Labib [10], Koczkodaj and Szarek [11], Alonso and Lamata [12], Dytczak and Ginda [13], Alonso and Lamata [14], and Jaganathan, et al. [15], have referred to Kwiesielewicz and Uden [1] in their references. Those mentioned papers developed a new consistency approach, applied it to heating systems, and extended it to fuzzy AHP with statistical criteria.

However, none of them have noticed that there are some questionable results in Kwiesielewicz and Uden [1] that will be discussed in this paper. In AHP, the possible solutions for a decision problem are often named "alternatives" but in their paper, they used "factors". In this note, we will use "alternatives" and "factors" interchangeably.

II. Review of Kwiesielewicz and Uden [1]

Based on a lack of the transitivity property for comparison, they defined a matrix $R=\left(r_{ij}\right)_{n\times n}$ is called contradictory if there exists i,j,k in $1,2,\ldots n$ such that any of the detailed below six cases holds:

$$r_{ij}>1\wedge r_{ik}<1\wedge r_{jk}>1, \tag{2.1}$$

$$r_{ij}<1\wedge r_{ik}>1\wedge r_{jk}<1, \tag{2.2}$$

$$r_{ij}=1\wedge r_{ik}>1\wedge r_{jk}<1, \tag{2.3}$$

$$r_{ij}=1\wedge r_{ik}<1\wedge r_{jk}>1, \tag{2.4}$$

$$r_{ij}=1\wedge r_{ik}=1\wedge r_{jk}<1, \tag{2.5}$$

and

$$r_{ij}=1\wedge r_{ik}=1\wedge r_{jk}>1, \tag{2.6}$$

where $\wedge$ is the minimum operator.

They also mentioned that in terms of factors (alternatives), the above expressions are equivalent to the following conditions, to save precious space in this journal, we only quote one of them for the later explanation. Equation (1) is equivalent to

$$F_i >> F_j \text{ and } F_i << F_k \text{ and } F_k << F_j \tag{2.7}$$

where ">>" denotes "better", and "<<" denotes "worst".

They provided an example of a pairwise comparison matrix with acceptable CI, but which is contradictory. We quote their example in the following with only one exception, in Kwiesielewicz and Uden [1], $r_{23}=4$, and $r_{32}=1/4$ which are crisp numbers. For our later further explanation, we abstractly express $r_{23}=x$ and $r_{32}=1/x$, in the following comparison matrix.

$$R=\left(r_{ij}\right)_{8\times 8}=\begin{bmatrix} 1 & 2 & 1/2 & 2 & 1/2 & 2 & 1/2 & 2 \\ 1/2 & 1 & x & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 1/x & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \end{bmatrix}. \tag{2.8}$$

When $x=4$, in Kwiesielewicz and Uden [1], the consistency index for the above matrix is equal to $CI=0.07252$ to pass the consistency test. We recall their assertion that in the above mentioned matrix with $R(1,2)=2$, $R(1,3)=1/2$, and then researchers may predict that criterion F_2 is better than criterion F_1, criterion F_1 is better than criterion F_3. Consequently, researchers will assume that criterion F_2 is better than criterion F_3,

Manuscript received February 23, 2023; revised July 10, 2023.

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however $R(2,3)=4$, that implies criterion F_3 is better than criterion F_2.

Kwiesielewicz and Uden [1] considered from a combinatorial point of view to estimating ratios. For three factors, F_1, F_2 and F_3, there are six possible ranking:

(a) $F_3 >> F_2 >> F_1$,

(b) $F_3 >> F_1 >> F_2$,

(c) $F_2 >> F_3 >> F_1$,

(d) $F_2 >> F_1 >> F_3$,

(e) $F_1 >> F_3 >> F_2$,

and

(f) $F_1 >> F_2 >> F_3$.

Kwiesielewicz and Uden [1] provided a contradictory pairwise comparison matrix:

$$R=\begin{bmatrix} 1 & 2 & 1/4 \\ 1/2 & 1 & 2 \\ 4 & 1/2 & 1 \end{bmatrix}, \tag{2.9}$$

then they provided detailed explanation to demonstrate that there does not exist a ranking which satisfies all the relations in the above matrix.

We recall their assertion that according to the above constructed matrix, researchers can assume a contradictory matrix of Equation (2.9). Kwiesielewicz and Uden [1] asserted that there is impossible to arrange a ordering for criteria that fulfill all the data cited in the matrix of (2.9).

To be complete, we recall the conclusion of Kwiesielewicz and Uden [1] in the following. The comparison matrices will provide contradictory opinions. Moreover, Kwiesielewicz and Uden [1] challenged that the consistency test proposed by Saaty [16] is not enough for practitioners to decide the comparison matrix should be revised or accepted.

Kwiesielewicz and Uden [1] invited researchers to pay attention to the contradictory test that proposed by Kwiesielewicz and Uden [1].

III. Our revisions

First, we will show that their six cases, as equations (2.1-2.6) for contradictory pairwise comparison matrices can be simplified as two cases.

If we combine $r_{ij}>1$ and $r_{ij}=1$ as $r_{ij}\geq 1$, so we merge equations (2.1) and (2.4) as

$$r_{ij}\geq 1 \wedge r_{ik}<1 \wedge r_{jk}>1 . \tag{3.1}$$

Similarly, we put $r_{ij}<1$ and $r_{ij}=1$ together as $r_{ij}\leq 1$, then we unite equations (2.2) and (2.3) as

$$r_{ij}\leq 1 \wedge r_{ik}>1 \wedge r_{jk}<1 . \tag{3.2}$$

If we interchange i and j in equation (3.2) to imply that

$$r_{ji}\leq 1 \wedge r_{jk}>1 \wedge r_{ik}<1 . \tag{3.3}$$

Since in pairwise comparison matrix, $r_{ij}r_{ji}=1$, owing to $r_{ji}\leq 1$, it follows that $r_{ij}\geq 1$ such that we can rewrite equation (3.3) as equation (3.1). Consequently, we may join equations (2.1-2.4) into one expression of equation (3.1).

On the other hand, if we interchange k and j in equation (2.5) to yield

$$r_{ik}=1 \wedge r_{ij}=1 \wedge r_{kj}<1 . \tag{3.4}$$

Since in pairwise comparison matrix, $r_{kj}r_{jk}=1$, because of $r_{kj}<1$, it obtains that $r_{jk}>1$ such that equation (3.4) is rewritten as equation (2.6) to indicate that equations (2.5-2.6) can be shorten as equation (2.5).

Hence, for pairwise comparison matrices, their six cases for contradictory can be simplified two cases as equations (2.5) and (3.1).

From the above discussion, we may claim that they did not prepare a sounded definition of their contradictory pairwise comparison matrices.

Next, we recall their results in equations (2.1) and (2.7) to note that $r_{ij}>1$ is equivalent to $F_i >> F_j$ that is accepted by researchers in AHP. It indicates that r_{ij} stands for F_i/F_j as the relative ratio for F_i over F_j.

However, from $R(1,2)=2$, the corrected result should be F_1 is better than F_2. It implies that their results in Page 716, Lines 5-7, as we discussed previously to point out that the assertion of Kwiesielewicz and Uden [1] are wrong. For completeness, we present the corrected results for them.

Based on the matrix of equation (2.8), researchers find that $R(1,2)=2$, $R(1,3)=1/2$. Consequently, researchers say that criterion F_1 is better than criterion F_2, and criterion F_3 is better than criterion F_1. According to transitivity, researchers will predict that criterion F_3 is better than criterion F_2. However, $R(2,3)=4$, indicates that criterion F_2 is better than criterion F_3.

Third, we reconsider their example that we abstractly expressed in equation (2.8). For those values in the 1-9 bounded scale proposed by Saaty [16], the computation results are listed in the next Table 1. After our computation, it reveals that when $r_{23}\geq 4$ then the resulting pairwise comparison matrix cannot pass the consistency test. Hence, in the following table 1, we only list the some computation results to indicate the trend of $\lambda_{\max}$, and CI.

Table 1 reveals that when $r_{23}\leq 3$ then the corresponding value is less than 10% of mean consistency, with its value 1.4217. On the other hand, when $r_{23}\geq 4$, the resulting matrix will not pass the consistency test. Let us make a careful examination of $R=(r_{ij})_{8\times 8}$ in equation (2.8).

Table 1. The value of r_{23}, $\lambda_{\max}$ and CI.

r_{23}	1/9	1/7	1/5	1/4	1/2	1	2	3	4
$\lambda_{\max}$	8.06407	8.02991	8.00468	8	8.04643	8.20074	8.50773	8.78408	9.03387
CI	0.00915	0.00427	0.00067	0	0.00662	0.02868	0.07253	0.11201	0.14770

We assume that in the beginning, only the entries is the first row are given. It means that

$$r_{12}=2,\ r_{13}=1/2,\ r_{14}=2,$$
$$r_{15}=1/2,\ r_{16}=2,\ r_{17}=1/2,$$

and

$$r_{18}=2, \tag{3.5}$$

are already designed for us, then we try to create a consistent reciprocal matrix such that $a_{ij}=a_{i1}a_{ij}=a_{1j}/a_{1i}$, for $i,j=1,2,...,8$. We have checked all entries of $R=(r_{ij})_{8\times 8}$ in equation (2.8) to reveal that except that $r_{23}=x$ and $r_{32}=1/x$, all other entries satisfies the requirement of $a_{ij}=a_{i1}a_{ij}=a_{1j}/a_{1i}$, for $i,j=1,2,...,8$.

We may predict that the original purpose of $R=(r_{ij})_{8\times 8}$ in equation (2.8) is to construct an example of an 8×8 pairwise comparison matrix in which only two entries, r_{23} and r_{32} that are not only unsatisfied the consistent condition, but also implies contradictory proposed by Kwiesielewicz and Uden [1] such that the resulting matrix have the following two cases:

Case A: it cannot pass the consistency test,

Case B: it still passes the consistency test.

Kwiesielewicz and Uden [1] assumed that $r_{23}=4$, the consistency index for the above matrix is equal to $CI=0.07252$ to pass the consistency test. However, we find that when $r_{23}=4$, then $CI=0.14770$ cannot pass the consistency test.

It points out that an almost perfect consistent pairwise comparison matrix with only two entries not satisfied the consistent condition then the resulting matrix cannot pass the consistency test.

The purpose for the example is to demonstrate that contradictory can make severe damage to pairwise comparison matrix.

In Kwiesielewicz and Uden [1], they wanted the resulting matrix still passes the consistency test. We may predict that Case A is not their goal.

Hence, we may conclude that $r_{23}=4$ is a typo. Let us recall our Table 1, when $r_{23}=2$ we derive that $CI=0.07253$. Our result is very close to their result of $CI=0.07252$ with a possible round off difference.

It follows that Kwiesielewicz and Uden [1] wanted to create an almost perfect consistent pairwise comparison matrix with two exceptions, r_{23} and r_{32} such that the resulting matrix still passes the consistency test.

Let us recall $R=(r_{ij})_{8\times 8}$ in equation (2.8) again, from $r_{12}=2$ and $r_{13}=1/2$, then $r_{23}\geq 1$ will be a contradiction. If Kwiesielewicz and Uden [1] really wanted a contradictory matrix and still passed the consistency test, then they should consider that $r_{23}=1$. Owing to the eigenvalue function is a concave up function for one entry of a pairwise comparison matrix. For this example, the minimum values of $\lambda_{\max}$ and CI will occur at $r_{23}=1/4$. To create an example with the desired properties, Kwiesielewicz and Uden [1] should use $r_{23}=1$ (consequently $r_{32}=1$) to reveal that their knowledge of $\lambda_{\max}$, CI and the definition of "contradictory".

Fourth, we consider their combinatorial point of view to estimating ratios. They only provide a numerical example of a 3 by 3 pairwise comparison matrix for criteria, F_1, F_2 and F_3 to demonstrate that if contradictory happens than all $3!=6$ possible ranking for criteria, F_1, F_2 and F_3 cannot be satisfied by all entries in the contradictory pairwise comparison matrix.

We must point out that they implicitly assume that local property can stand for global property.

Under their approach, we can provide an analytic proof for arbitrary size contradictory pairwise comparison matrix for their purpose. We assume that $A=(a_{ij})_{n\times n}$ is a contradictory pairwise comparison matrix. By our simplified definition for their contradictory, then the least one of the following two cases will happen:

Case (a), $a_{ij}\geq 1$, $a_{ik}<1$ and $a_{jk}>1$, it is only some local properties for three entries. However, under the assumption that local property can stand for global property. It means that from $F_i\geq F_j$ locally for one entry to obtain that in the priority vector, $w_i\geq w_j$, for the global results. Hence, the ranking that satisfies these three conditions are

$$w_i\geq w_j,\ w_i<w_k \text{ and } w_j>w_k. \tag{3.6}$$

Owing to $w_j>w_k$ and $w_k>w_i$, then it implies that $w_j>w_i$. However, it is against $w_i\geq w_j$ in equation (3.6).

Case (b), $a_{ij}=1$, $a_{ik}=1$ and $a_{jk}\neq 1$, it is only some local properties for three entries. However, under the assumption that local property can stand for global property, then it means that from $F_i=F_j$ locally for one entry to obtain that in the priority vector, $w_i = w_j$, for the global results. Hence, the ranking that satisfies these three conditions are

$$w_i = w_j,\ w_i = w_k \text{ and } w_j \neq w_k. \tag{3.7}$$

According to $w_i = w_j$ and $w_i = w_k$, then it implies that $w_j = w_k$. However, it is against $w_j \neq w_k$ in equation (3.7).

By their approach, we can generalize their results from a numerical example to an abstract analytical proof. However, we must point out that something is still questionable.

For example, $a_{ij} \geq 1$ that only indicates when an expert compares F_i with F_j, then for that moment, he feels that $F_i \geq F_j$. The expert does not assume that in the priority vector that $w_i \geq w_j$.

Kwiesielewicz and Uden [1] misused a local property for $F_i \geq F_j$ of one comparison to imply that the global property, $w_i \geq w_j$.

There are $n(n-1)/2$ comparisons for a $n \times n$ pairwise comparison matrix. In AHP, inconsistent (or contradictory) are allowed. From all the information provided by all entries, through eigenvector algorithm, decision maker finds the priority vector.

From local property of one entry, then Kwiesielewicz and Uden [1] implied the global property, we may claim that their order relation among alternatives will be a linear order relation as the number system. However, AHP tried to handle the problems with relation as complicated real world problems in which the alternatives do not have a linear relation such that they overlooked the true meaning of AHP.

IV. Our Numerical Example

In this section, we prepare an example to illustrate that their contradictory test is unnecessary. If an expert tried to use AHP to derive the relative weights for three alternatives, A_1, A_2 and A_3 corresponding to a criterion, price. In his mind, he roughly estimate that the price for these three alternatives are \$17, \$13 and \$10, respectively. From $17/13 \approx 1.3$, and the 1-9 bounded scale proposed by Saaty [16], then he assumed that $a_{12} = 1$. Moreover, owing to $17/10 \approx 1.7$ and the 1-9 bounded scale proposed by Saaty [16], so he supposed that $a_{13} = 2$. Similarly, he took $a_{23} = 1$ to complete the following pairwise comparison matrix

$$A = \left(a_{ij}\right)_{3\times 3} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1/2 & 1 & 1 \end{pmatrix}, \tag{4.1}$$

with $\lambda_{\max} = 3.05362$, $CI = 0.02681$ to pass the consistency test, and the normalized priority vector

$$(w_1, w_2, w_3) = (0.41260, 0.32748, 0.25992). \tag{4.2}$$

We compute that without using AHP, the expert directly estimate the relative weight for these three alternatives will be

$$\begin{aligned}&(17/40, 13/40, 10/40),\\ &= (0.425, 0.325, 0.250).\end{aligned} \tag{4.3}$$

The resulting relative weights in equation (4.2) are compatible with the normalized roughly estimated price in expert's mind in equation (4.3) to illustrate that AHP is a useful tool. However, this pairwise comparison matrix, A, in equation (4.1) cannot pass the contradictory test proposed by Kwiesielewicz and Uden [1], according to equation (2.6) with $i = 2$, $j = 1$ and $k = 3$.

They advised researcher that the resulting priority vector of equation (4.2) is not acceptable.

There is a simple way to solve the dilemma that is to neglect the contradictory test of Kwiesielewicz and Uden [1].

V. Direction for Future Research

Ataei et al. [17] developed an ordinal priority method to handle multi-criteria decision making problems that rank and weight criteria, alternatives and experts under an incomplete information environment. Wu and Tu [18] considered ordinal consistency to obtain criteria to preserve the order and then they compared their results with others to show their method can solve the consistent of preferences under group decision making problems. Mubarik et al. [19] studied green supply chain management under green strategic sourcing to point out that delivery time and supplier's quality are the dominant factors for supplier selection.

VI. A New Approach to Solve Inventory Models

We study a famous paper that tried to solve the minimum by the algebraic method without derivatives according to two different approaches. To complete the square for the shortage level, they finished the work. However, to complete the square for inventory level, they cannot find the optimal solution so they proposed an open question to solve the minimum problem. In this paper, we not only answer their open question but also provide an alternative method to solve the minimum problem. According to our new method, which is based on Cauchy-Schwarz inequality, and not related to completing the square, their open question disappears naturally. Our work will provide a new feature for solving inventory models with algebraic methods.

We try to solve the open question in Chang et al. [20] and provide an alternative method to solve the minimum problem. Since Grubbström [21] uses the algebraic method to solve the traditional EOQ model, there is a trend to develop inventory models without referring to differential equations so that inventory models can be introduced to those practitioners who are not familiar with calculus. Grubbström and Erdem [22] considered inventory models with shortages.

Cardenas-Barron [23] developed the EPQ model with shortages. Ronald et al. [24] reconsidered the inventory models in Grubbström and Erdem [22] and Cárdenas-Barron [23] to point out some questionable methods in their papers and then provide a sophisticated but tedious procedure to derive the optimal replenishment policy by an algebraic method. Chang et al. [20] improved the lengthy procedure of

Ronald et al. [24] and then proposed an open question on how to solve a minimum problem with the square root. Chang [25] studied inventory models with variable lead times. Sphicas [26] studied the EOQ inventory model and EPQ production inventory model under fixed and linear backlogged costs by algebraic processes. Furthermore, there are several papers that have referred to Chang et al. [20]. We just mentioned a few. For example, Cardenas-Barron [27] demonstrated that without referring to analytic methods, using pure algebraic algorithms can solve supply chain inventory systems. Leung [28] examined the inventory models where the quantity back ordered and the quantity received are both uncertain. Leung [29] considered the model proposed by Montgomery et al. [30] to apply the complete squares method. Chung [31] revised the questionable results of Wu and Ouyang [32] regarding the incomplete proof for the integer solution to justify the algorithm described in Wu and Ouyang [32].

On the other hand, Minner [33] considered the finite planning horizon inventory problem and then studied the limiting behavior to derive the optimal solution to provide an alternative approach different from the traditional algebraic skill. However, they did not provide a solution for the open question proposed by Chang et al. [20]. Moreover, we propose a more challenging problem: Is a method that can avoid the problem that appears in Chang et al. [20]? In this paper, we not only solve the open question raised by Chang et al. [20] but also propose our new approach to consider the inventory model by Cauchy-Schwarz's inequality instead of the traditionally complete square method. Consequently, the open question in Chang et al. [20] will no longer pertain which demonstrates the robust character of our approach.

VII. Notation and Assumptions

There are two different representations in the series of papers concerning inventory models derived without calculus. For example, Grubbström and Erdem [22], and Ronald et al. [24], used B to stand the maximum backorder level and $Q+B$ is used for the order quantity. In Cardenas-Barron [23] and Chang et al. [20], also used B to stand the maximum backorder level but the economic production quantity is expressed as Q. To avoid chaos and Chang et al. [20] responded to Ronald et al. [24], so we use the same notation as Grubbström and Erdem [22], and Ronald et al. [24].

D is the demand rate per unit of time;

K is the ordering cost per replenishment cycle;

$Q+B$ is the order quantity;

Q is the maximum inventory level;

B is the maximum backorder level;

h is the inventory holding cost per unit and time unit;

b is the backlog cost per unit and time unit.

VIII. Review of previous results

We will briefly review the series of papers related to inventory models derived without calculus. In Grubbström and Erdem [22], they considered the following minimum problem for the average cost per unit of time

$$C(Q,B)=\frac{1}{Q+B}\left(\frac{h}{2}Q^2+\frac{b}{2}B^2+DK\right). \quad (8.1)$$

In Cardenas-Barron [23], he extended the inventory model from the classical economic order quantity model to the economic production quantity model.

We abstractly consider the problem in Grubbström and Erdem [22] and Cardenas-Barron [23], to minimize

$$f(x,y)=\frac{ax^2+by^2+c}{x+y}, \quad (8.2)$$

for $x>0$ and $y>0$ with $a>0$, $b>0$ and $c>0$. Both of them decomposed c as

$$c=c\left(\frac{b}{a+b}+\frac{a}{a+b}\right),$$
$$=a\frac{bc}{a(a+b)}+b\frac{ac}{b(a+b)}, \quad (8.3)$$

then assumed

$$E=\sqrt{\frac{bc}{a(a+b)}}, \quad (8.4)$$

and

$$F=\sqrt{\frac{ac}{b(a+b)}}, \quad (8.5)$$

such that we rewrite $f(x,y)$ as follows

$$f(x,y)=\frac{2aEx+2bFy}{x+y},$$
$$+\frac{a(x-E)^2+b(y-F)^2}{x+y}. \quad (8.6)$$

Since

$$2aE=2\sqrt{\frac{abc}{a+b}}=2bF, \quad (8.7)$$

it yields that

$$f(x,y)=\frac{a(x-E)^2+b(y-F)^2}{x+y}+2aE. \quad (8.8)$$

Based on equation (8.8), we derive the minimum point

$$(x^*,y^*)=(E,F), \quad (8.9)$$

and the minimum value

$$f(x^*,y^*)=2aE. \quad (8.10)$$

In Ronald et al. [24], they predicted the decomposition in equation (8.3) may be too sophisticated for ordinary readers. They used the method developed by Montgomery et al. [30] to find the minimum solution along each ray of $\{(x,mx): x>0\}$, say (x_m, mx_m) for $0<m<\infty$. Next, they considered the minimum problem that was denoted $f(x_m, mx_m)$ as a function of the variable m, for the domain $0<m<\infty$.

However, Ronald et al. [24] may still create another complicated procedure such that Chang et al. [20] provided an improved method to find the minimum problem for $C(Q,B)$ of Cardenas-Barron [23]. They first completed the square for the variable B to obtain the minimum point,

$$B = B(Q) = \frac{h\rho}{b+h}Q, \quad (8.11)$$

and then they solved for $C(Q, B(Q))$. They pointed out a possible direction for future researchers such that if they first completed the square for the variable Q to imply the minimum point,

$$Q = Q(B) = \sqrt{\frac{2}{h\rho}\left(DK + \frac{b+h}{2\rho}B^2\right)}, \quad (8.12)$$

then they would face the following minimum problem

$$C(Q(B), B) = cD - B + h\sqrt{\left(1+\frac{b}{h}\right)B^2 + \frac{2\rho DK}{h}}. \quad (8.13)$$

They ask an open question about how to solve

$$f(B) = \sqrt{(1+\alpha)B^2 + \beta} - B, \quad (8.14)$$

where

$$\alpha = \frac{b}{h}, \quad (8.15)$$

and

$$\beta = \frac{2\rho DK}{h}. \quad (8.16)$$

IX. Our improvements

We will first solve their open problem and then provide an easy alternative method to solve the same problem. We square $f(B)+B$ to express the results in the decreasing order of B where we treat $f(B)$ as a constant, then complete the square for B so equation (8.14) follows

$$\alpha\left(B - \frac{f(B)}{\alpha}\right)^2 = \left(1+\frac{1}{\alpha}\right)(f(B))^2 - \beta. \quad (9.1)$$

We rewrite equation (9.1) as

$$\left(1+\frac{1}{\alpha}\right)(f(B))^2 = \alpha\left(B - \frac{f(B)}{\alpha}\right)^2 + \beta. \quad (9.2)$$

From equation (9.2), when

$$f(B) = \alpha B, \quad (9.3)$$

then $f(B)$ will attain its minimum value with

$$f(B^*) = \sqrt{\frac{\alpha\beta}{\alpha+1}}. \quad (9.4)$$

Consequently, we imply that

$$B^* = \frac{1}{\alpha} f(B^*) = \sqrt{\frac{\beta}{\alpha(\alpha+1)}}, \quad (9.5)$$

as predicted by Chang et al. [20]. Hence, we finish the open question.

Next, we will provide an alternative method to solve the minimum problem in equation (8.1). We assume two vectors U and V with

$$U = \left(\sqrt{\frac{h}{2}}Q, \sqrt{\frac{b}{2}}B\right), \quad (9.6)$$

and

$$V = \left(\sqrt{\frac{2}{h}}, \sqrt{\frac{2}{b}}\right). \quad (9.7)$$

From the Cauchy-Schwarz inequality, it yields

$$\langle U, V\rangle = Q + B \le \|U\|\|V\| = \sqrt{\frac{h}{2}Q^2 + \frac{b}{2}B^2}\sqrt{\frac{2}{h}+\frac{2}{b}}. \quad (9.8)$$

By equation (9.8), it implies that

$$\frac{1}{Q+B}\left(\frac{h}{2}Q^2 + \frac{b}{2}B^2\right) \ge \frac{bh}{2(b+h)}(Q+B), \quad (9.9)$$

and

$$\frac{1}{Q+B}\left(\frac{h}{2}Q^2 + \frac{b}{2}B^2\right) = \frac{bh}{2(b+h)}(Q+B), \quad (9.10)$$

holds only when U parallels to V. We combine equations (8.1) and (9.9) to derive that

$$C(Q,B) \ge \frac{DK}{Q+B} + \frac{bh}{2(b+h)}(Q+B) = \left(\sqrt{\frac{KD}{B+Q}} - \sqrt{(B+Q)\frac{hb}{2(h+b)}}\right)^2, +2\sqrt{\frac{hbKD}{2(b+h)}}. \quad (9.11)$$

From equation (9.11), we find that the minimum point satisfies

$$\frac{h}{2}Q = \frac{b}{2}B, \quad (9.12)$$

and the minimum value is $\sqrt{\frac{2bhDK}{b+h}}$. According to

$$(Q+B)^2 = \frac{2(b+h)DK}{bh}, \quad (9.13)$$

it implies the minimum point

$$Q^* = \sqrt{\frac{2bDK}{h(b+h)}}, \quad (9.14)$$

and

$$B^* = \sqrt{\frac{2hDK}{b(b+h)}}. \quad (9.15)$$

Our results in equation (9.15) are the same in Grubbström and Erdem [22] and Ronald et al. [24]. The above discussion points out that following our proposed new approach, then there is no complete square so the problem of which parameter should be completed the square first does not exist. It illustrates that with our new approach, the open question in Chang et al. [20] disappeared.

X. Application of our proposed method

In this paper, we not only solve the open question in Chang et al. [20] but also provide an alternative method to solve the minimum problem. Our method has the following advantages:

(a) We avoid the sophisticated decomposition in Grubbström and Erdem [22] and Cardenas-Barron [23].

(b) Our method used the complete square technique and Cauchy-Schwarz inequality, which are suitable for practitioners who are not familiar with calculus so that we might introduce inventory models to these practitioners.

(c) We avoid the complicated partition of the domain in Ronald et al. [24], so we provide an easy procedure to solve the minimum problem.

(d) Our method is simpler than Chang et al. [20]. Their method depends on which variable should be complete square first.

Owing to the above discussion, we may say that this paper not only presents a useful patchwork to further develop the inventory models without derivatives but also provides a new feature to apply Cauchy-Schwarz inequality to solve inventory models.

XI. A RELATED PROBLEM IN FUZZY ALGORITHM

We recall that Xu [34] tried to construct a consistent complete linguistic preference relation. However, we find that his algorithm contained unnecessary steps such that we will provide a revised version to construct a consistent complete linguistic preference relation.

From the Step 2 of the algorithm of Xu [34], he wanted to construct a consistent complete linguistic preference relation with

$$\overline{a}_{ij}^{(k)} = \overline{a}_{it}^{(k)} \oplus \overline{a}_{tj}^{(k)}, \tag{11.1}$$

for $1 \le i, j, t \le n$ and $1 \le k \le m$. By Step 3 of the algorithm of Xu [34], using the linguistic weighted averaging operator to obtain the collective complete linguistic preference relation

$$\overline{a}_{ij} = \omega_1 \overline{a}_{ij}^{(1)} \oplus \omega_2 \overline{a}_{ij}^{(2)} \oplus \cdots \oplus \omega_n \overline{a}_{ij}^{(n)}. \tag{11.2}$$

From Step 4 of the algorithm of Xu [34], it yields that using the linguistic averaging operator to fuse the preference degree

$$\overline{a}_i = \frac{1}{n}\overline{a}_{i1} \oplus \frac{1}{n}\overline{a}_{i2} \oplus \cdots \oplus \frac{1}{n}\overline{a}_{in}, \tag{11.3}$$

for all i. For the future computation, we use b_i to denote that $\overline{a}_i = s_{b_i}$, c_{ij} to denote that $\overline{a}_{ij} = s_{c_{ij}}$ and $d_{ij}^{(k)}$ to denote that $\overline{a}_{ij}^{(k)} = s_{d_{ij}^{(k)}}$. From the linguistic averaging operator, it follows that

$$b_i = \frac{1}{n}\sum_{j=1}^{n} c_{ij}. \tag{11.4}$$

Recalling the linguistic weighted averaging operator, it implies that

$$c_{ij} = \sum_{k=1}^{m} \omega_k d_{ij}^{(k)}. \tag{11.5}$$

For an incomplete linguistic preference relation, we assume that only the i_k th row and the i_k th column are offered by the k th decision maker where the diagonal are denoted as s_0. By the Step 2 of the algorithm in Xu [34] to create a consistent complete linguistic preference relation, then it derives that $\overline{a}_{st}^{(k)} = \overline{a}_{si_k}^{(k)} \oplus \overline{a}_{i_k t}^{(k)}$, such that

$$d_{st}^{(k)} = d_{si_k}^{(k)} \oplus d_{i_k t}^{(k)}. \tag{11.6}$$

Using Equation (11.6), we rewrite Equations (11.5) and (11.4) as

$$c_{st} = \sum_{k=1}^{m} \omega_k \left(d_{si_k}^{(k)} + d_{i_k t}^{(k)} \right), \tag{11.7}$$

and

$$b_s = \frac{1}{n}\sum_{j=1}^{n} c_{sj} = \frac{1}{n}\sum_{k=1}^{m} \omega_k \sum_{j=1}^{n} \left(d_{si_k}^{(k)} + d_{i_k j}^{(k)} \right)$$

$$= \Delta + \sum_{k=1}^{m} \omega_k d_{si_k}^{(k)}, \tag{11.8}$$

where

$$\Delta = \frac{1}{n}\sum_{k=1}^{m} \omega_k \sum_{j=1}^{n} d_{i_k j}^{(k)}, \tag{11.9}$$

is a constant that is independent of the index s.

Since $\overline{a}_i = s_{b_i}$, the ranking of alternatives, x_s for $1 \le s \le n$, is decided by the rank of b_s for $1 \le s \le n$. Hence, the ranking can be decided by $\sum_{k=1}^{m} \omega_k d_{si_k}^{(k)}$. We summarize our results in the next theorem.

Theorem 1. If for the k th decision maker, we know the linguistic preference relation for the i_k th row and the i_k th column, then the ranking of alternatives is decided by the algorithm of Xu [34] can be simplify to compare $\sum_{k=1}^{m} \omega_k d_{si_k}^{(k)}$.

In Xu [34], he prepared the incomplete linguistic preference relations for three decision makers such that for the first decision maker, the preference relations for the $1st$ row and the $1st$ column are given. For the second decision maker, the preference relations for the $4th$ row and the $4th$ column are provided. For the third decision maker, the preference relations for the above diagonal and the below diagonal are presented, i.e., $a_{ij}^{(3)}$ with $|i-j| = 1$ are offered.

Now, we consider how to derive one column, for example the first column, for the third decision maker. From the below diagonal preference relations, we find that

$$a_{i1}^{(3)} = a_{i,i-1}^{(3)} \oplus a_{i-1,i-2}^{(3)} \oplus \cdots \oplus a_{2,1}^{(3)}. \tag{11.10}$$

From Equation (9.10), it follows that

$$a_{i1}^{(3)} = a_{i,i-1}^{(3)} \oplus a_{i-1,1}^{(3)}. \tag{11.11}$$

Equation (11.11) means that if we derive the first column by the following order: $a_{3,1}^{(3)}$, $a_{4,1}^{(3)},\cdots,a_{n,1}^{(3)}$, then we have an easy to obtain the first column preference relations.

The essential computation to decide the ranking for alternatives x_i for $i=1,2,...,8$ is listed below:

$$b_i = 0.5a_{i1}^{(1)} + 0.3a_{i4}^{(2)} + 0.2\sum_{k=2}^{i} a_{k,k-1}^{(3)}, \qquad (11.12)$$

Consequently, we compute

$$\begin{aligned} &0.5(0,2,1,3,-1,0,2,-1) \\ &+0.3(-2,-1,-2,0,-3,-4,-2,-3) \\ &+0.2(0,2,2-2=0,0+1=1,1-3=-2, \\ &-2+1=-1,-1+1=0,0-1=-1) \\ &=(-0.6,1.1,-0.1,1.7,-1.8,-1.4,0.4,-1.6). \qquad (11.13) \end{aligned}$$

If we compare our computation with the results of Xu [34], then it reveals that

(I) Our computation is very simple that dramatically reduce the risk of computation mistake.

(II) Our computation is a shift of the result in Xu [34] such that both methods will imply the same rank.

(III) There is no need to construct the consistent complete linguistic preference relations. Hence, to use the linguistic weighted averaging operator to generate the collective complete linguistic preference relation is unnecessary. Moreover, to use the linguistic averaging operator to derive the preference degrees is redundant.

Based on the above discussion, we conclude that we have studied the incomplete linguistic preference relations to improve the algorithm of Xu [34]. Consequently, We provide an easy algorithm to decide the ranking for alternatives.

XII. Possible Directions for Future Research

There are several recently published papers that are worthy to mention so we list them in the following. There are four papers: Pappalardo et al. [35], Patil et al. [36], Zhu et al. [37], and Tan et al. [38] that discussed problems with computer sciences. We provide a short reviewing for them. Pappalardo et al. [35] talked about the modeling the longitudinal flight dynamics of a fixed-wing aircraft. Patil et al. [36] focused on deploying convolutional neural networks for the detection of osteoporosis of lumbar spine X-ray Images. Zhu et al. [37] studied power system by Archimedes optimization algorithm to locate the optimal power flow. For acoustic event classification, Tan et al. [38] applied pre-trained Dense Net-121 with multilayer perceptron.

There are four papers: Wichapa and Sodsoon [39], Zhang et al. [40], Raghu and Prameela [41], and Atatalab, and Najafabadi [42], that are studied for systems in engineering and their applications that are important directions for future study. Wichapa and Sodsoon [39] developed a relative closeness coefficient system according to the distance of virtual DMUs cross-efficiency approach to classify Thai economic growth. For electricity, heat and gas for virtual power plants, Zhang et al. [40] constructed economic and optimal dispatch system to consider low carbon targets. According to sparse Bayesian learning through covariance, Raghu and Prameela [41] developed direction of arrival estimation through employing intra-block correlations. Based on estimated delay probabilities, Atatalab and Najafabadi [42] considered estimations of IBNR and RBNS liabilities with a zero-inflated Gamma mixture system.

There are four papers: Yen [43, 44], Yang and Chen [45], and Wang and Chen [46] which are examined theoretical development for models in operation research and their applications in several research fields to show the current research trend. Yen [43] tried to solve inventory systems by his new algebraic approach. Yen [44] studied inventory systems with compounding to amend Çalışkan [47, 48] and pointed out the algebraic method proposed by Çalışkan [49] is a complex transformation of calculus that contains a severe problem to assume the existence of the optimal solution. Yang and Chen [45] pointed out questionable results in Yen [44], Osler [50] and Caliskan [51, 52] and then provided some improvements. Wang and Chen [46] examined the consistent test for local stability intervals in analytic hierarchy process.

XIII. Conclusion

In their conclusion, they warn researchers that some contradictory pairwise comparison matrices may pass the consistency test such that information is accepted. They suggested their contradictory test to delete those contradictory pairwise comparison matrices.

From the above discussion, we may say that

(a) They did not organize a sounded definition of their contradictory pairwise comparison matrices.

(b) They have problem for the definition of a_{ij} in the pairwise comparison matrix.

(c) By their example, we point out contradictory is a local property that will not decisively change the priority vector.

(d) They misused a local property to imply global property.

Hence, we advise researchers overlook their contradictory test to avoid unnecessary confusion demonstrated by our example.

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