Some Identities Involving Chebyshev Polynomial of Third Kind, Lucas Numbers and Fourth Kind, Fibonacci Numbers

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Abstract - In this article, we demonstrate certain identities involving both the third-fourth kinds of the Chebyshev polynomials taking into account the Fibonacci and Lucas numbers. This study brings to light some significant results and defines the relationship between these polynomials. We have used mathematical induction to establish the relationship between third-fourth kinds of Chebyshev’s polynomials, Fibonacci, and Lucas numbers. We also used the Binet formula and the second-order differential equation to establish their relationship. We also present some results for Fibonacci and Lucas numbers, particularly by using the second-order derivative of third-kind Chebyshev polynomials. We prove some results that connect the fourth-kind of Chebyshev polynomials with Fibonacci and Lucas numbers. These findings will pave the way for further exploration of the third-fourth kinds of Chebyshev polynomials. In addition, we look at the practical application of Chebyshev’s polynomials in approximation theory.

Index Terms- Chebyshev polynomials of third-fourth kinds; Fibonacci numbers; Lucas numbers.

I. INTRODUCTION
A. Chebyshev polynomials
As usual the first kind of Chebyshev polynomials are defined as:
\[ T_0(y) = 1, T_1(y) = y, \]
For \( h \geq 2,3, ... \)
\[ T_h(y) = 2yT_{h-1}(y) - T_{h-2}(y). \] (1)
The second kind of Chebyshev polynomials are defined as:
\[ U_0(y) = 1, U_1(y) = 2y, \]
For \( h \geq 2,3, ... \)
\[ U_h(y) = 2yU_{h-1}(y) - U_{h-2}(y). \] (2)
The third kind of Chebyshev polynomials are defined as:
\[ V_0(y) = 1, V_1(y) = 2y - 1, \]
For \( h \geq 2,3, ... \)
\[ V_h(y) = 2yV_{h-1}(y) - V_{h-2}(y). \] (3)
The fourth kind of Chebyshev polynomials are defined as:
\[ W_0(y) = 1, W_1(y) = 2y + 1, \]
\[ W_h(y) = 2yW_{h-1}(y) - W_{h-2}(y). \] (4) For \( y = \cos \theta \) the Chebyshev polynomials (all four kinds) can also be structured as (trigonometric way):
\[ T_h(y) = \cos(h \theta), \]
\[ U_h(y) = \frac{\sin((h+1)\theta)}{\sin \theta}, \]
\[ V_h(y) = \frac{\cos(h+1)\theta}{\cos \theta}, \]
\[ W_h(y) = \frac{\sin(h+2)\theta}{\sin 2 \theta}. \] (5)
B. Fibonacci Numbers
Leonardo of Pisa, an Italian mathematician, is renowned for his work on Fibonacci numbers, which are a sequence of numbers defined by the following equation:
\[ F_h = F_{h-1} + F_{h-2}, \text{ with } F_0 = 0 \text{ and } F_1 = 1. \] (5)
C. Lucas Numbers
Lucas numbers named the mathematician Francois Edouard Anatole Lucas. Lucas’s numbers are closely related to Fibonacci numbers. The recurrence relation of both Lucas and Fibonacci sequences is the same, although the initial values differ. Lucas’s sequence of numbers is defined by:
\[ L_h = L_{h-1} + L_{h-2}, \text{ with } L_0 = 2 \text{ and } L_1 = 1. \] (6)

This paper comprises three primary sections. The initial section provides an introduction and definition of Chebyshev polynomials, which comprehensively cover all four kinds, including the Fibonacci and Lucas numbers. The second section encompasses the research conducted and work done on deriving six theorems and six corollaries related to Chebyshev polynomials, Fibonacci, and Lucas numbers. In the last sections, we have discussed the significance of this work and delved into the vast range of applications that Chebyshev polynomials have to offer.

In the 19th century, Pafnuty Lvovich Chebyshev introduced the idea of Chebyshev polynomials. The analysis of the initial and subsequent Chebyshev polynomials was conducted by Mason and Wenpeng Zhang, who also presented several interesting identities [6, 19, 21]. The basic definitions of Chebyshev polynomials have been taken from a very notable scholarly work, the classical orthogonal polynomials [1]. Jonny Griffiths [5] presented many results which connected all four kinds of Chebyshev polynomials. Kamal Aghigh et al. [8], M.R. Eslahchi et al. [12], and Taekyun Kim et al. [15, 16, 17] gave many identities attributed to both the third-fourth kinds of
Chebyshev polynomials. Wenpeng Zhang [18] gave the foundational idea to solve the summation of recurrence relations and also studied some identities related to Fibonacci sequences. Chebyshev polynomials are widely studied by researchers and defined in various forms like recurrence relations and trigonometric formulae, etc. Fibonacci and Lucas numbers share a close relationship with Chebyshev polynomials. These recursive relationships are employed in counting. Zhang [19] found a number of properties related to the Chebyshev polynomials' derivatives and showed the relationship between them as follows:

Let $h, q$ be integers with $h \geq 0, q \geq 1$, we have:

$$
\sum_{b_1 + b_2 + \cdots + b_{q+1} = i}^{q+1} U_{b_q}(y) = \frac{1}{2^q q!} U_{h+q}^{(q)}(y).
$$

Sanjay Harne et al. [13] found identities related to the Chebyshev polynomials, Lucas and Fibonacci numbers at certain variables with their derivatives as follows:

$$
\sum_{b_1 + b_2 + \cdots + b_{q+1} = i}^{q+1} \frac{q+1}{2^q q!} U_{h+q}^{(q)}(y) = \frac{h!}{2^q q!} \left( \frac{1}{2} \right)^h U_{h+q}^{(q)}(y), \\
$$

$$
\sum_{b_1 + b_2 + \cdots + b_{q+1} = i}^{q+1} \frac{q-2}{2^q q!} U_{h+q}^{(q)}(y) = \frac{h!}{2^q q!} \left( \frac{1}{2} \right)^h U_{h+q}^{(q)}(y).
$$

Kamal et al. [8] examined the Chebyshev polynomials of both third-fourth kinds with their applications and obtained highly advantageous outcomes. Li et al. [9] used the Chebyshev polynomials matrices to obtain numerical solutions with examples for BVP of the Caputo fractional derivative integro-D.E. of fourth-order. T. Korkiatsakul et al. [14] looked at a new way of using the Chebyshev operating matrix to figure out how to solve a problem with integrals of non-linear Caputo fractions in a static beam. Frontczak, and Goy [3] obtained Chebyshev-Fibonacci polynomial relations using generating functions. Shoukralla [2] obtained a numerical solution to the first kind of Fredholm integral equation using the matrix form of the second-kind Chebyshev polynomials. Zhang [20] gave results linked to the Lucas and Fibonacci numbers. Yang Li [22] determined the connection between derivatives of the Chebyshev, and Fibonacci polynomials. In addition, Yang Li [23] analysed the both first and second kinds of Chebyshev polynomials and applied the elementary method to find the relationship between both kinds of Chebyshev polynomials connected to Fibonacci polynomials and finally found some results related to the Fibonacci and the Lucas numbers. L. Zhang and W. Zhan [10, 19] used mathematical induction, to solve the problem of Chebyshev polynomials sums of powers and obtained many other properties related to Chebyshev polynomials. T. Kim et al. [16] studied sums of finite products of first kind Chebyshev, and Lucas’s polynomials represented each one in terms of all kinds of Chebyshev polynomials. Y. Zhang and Z. Chen [24] found identity related to the Chebyshev polynomials of the second-kind by using combinatorial.

M. Arya and V. Verma [11] found a special type of representation of the Chebyshev polynomials. Han and Lv [4] studied the Chebyshev polynomials to explore some new identities. Zhang and Han [21] provided identities for reciprocal sums of the Chebyshev polynomials through mathematical induction and the properties of symmetrical polynomial sequences. Kim et al. [17] obtained the sums of Chebyshev polynomials’ finite products of two different types and each one of them represented a linear combination of all types. Robert et al. [3] studied the first and second kinds of Chebyshev polynomials as well as the polynomials of the Fibonacci sequence and gave new connections and combinatorial identities by using generating functions. Jugal Kishore and VIPIN VERMA [7] used a computational method to give identities concerning product’s finite sums of the Lucas, Fibonacci, and complex Fibonacci numbers. T. Kim et al. [15] studied the sum related to finite products of both third-fourth kinds of Chebyshev polynomials and have obtained the following relation among the fourth kind of the Chebyshev polynomials and their derivatives as follows:

$$
\sum_{i=0}^{h} \sum_{b_1 + b_2 + \cdots + b_{q+1} = i}^{l} (-1)^{h-i} C_{q+1}^{q} U_{b_1} W_{b_2} W_{b_3} \cdots W_{b_{q+1}}(y) = \frac{1}{2^q q!} W_{h+q}^{(q)}(y),
$$

here $h, q$ be integers with $h \geq 0, q \geq 1$.

For obtaining the relationship between the third-kind of the Chebyshev polynomials and Lucas numbers, we also used the result obtained by Kim et. al. [15] and W. Zhang [19]:

$$
\sum_{i=0}^{h} \sum_{b_1 + b_2 + \cdots + b_{q+1} = i}^{l} \left( C_{q+1}^{q} V_{b_1} V_{b_2} \cdots V_{b_{q+1}}(y) = \frac{1}{2^q q!} V_{h+q}^{(q)}(y),
$$

inside sum covers all non-negative integers $b_1, b_2 \cdots b_{q+1}$ with $b_1 + b_2 + \cdots + b_{q+1} = i$, $h, q$ be integers with $h \geq 0, q \geq 1$.

$V_{h}^{(q)}(y), W_{h}^{(q)}(y)$ denotes the $q$th derivative of $V_{h}(y)$ , $W_{h}(y)$.

II. SOME SIMPLE LEMMAS

To prove our primary outcome, we require numerous lemmas. For $h \geq 0$, we have these identities:

$$
W_h(161) = \frac{1}{8} F_{6(2h+1)}, \\
W_h(47) = \frac{1}{3} F_{6(2h+1)}, \\
W_h(123) = \frac{1}{11} L_{5(2h+1)}, \\
V_h(47) = u^{-1} L_{4(2h+1)}, \\
V_h(161) = u^{-1} L_{6(2h+1)}, \\
V_h(-9) = u^{-1} L_{3(2h+1)}.
$$

Lemma 1: The following identity holds true for all $h \geq 0$:

$$
W_h(161) = \frac{1}{8} F_{6(2h+1)}. \\
$$

Proof: To prove lemma 1, take $y = 161, u = \sqrt{\frac{1+\sqrt{2}}{2}}$.

Utilizing the identity.

$$
U_h(u) = \frac{1}{8} F_{6(h+1)}, \\
U_{2h}(u) = \frac{1}{8} F_{6(2h+1)}. \\
$$

We also utilizing,

$$
U_{2h}(u) = W_h(y), \\
$$

to get
\[ W_h(161) = \frac{1}{2} F_6(2h+1). \]

This demonstrates lemma 1.

**Lemma 2:** The following identity holds true for all \( h \geq 0: \)

\[ W_h \left( \frac{47}{2} \right) = \frac{1}{3} F_4(2h+1). \]

**Proof:** To prove lemma 2, take \( y = \frac{47}{2}, u = \sqrt{\frac{1+y}{2}} \).

We utilize the identity,

\[ "U_h(u) = \frac{1}{3} F_4(h+1)," \]

\[ \Rightarrow U_{2h}(u) = \frac{1}{3} F_4(2h+1). \]

Also utilizing,

\[ "U_{2h}(u) = W_h(y)," \]

\[ W_h \left( \frac{47}{2} \right) = \frac{1}{3} F_4(2h+1). \]

This demonstrates lemma 2.

**Lemma 3:** The following identity holds true for all \( h \geq 0: \)

\[ W_h \left( \frac{123}{2} \right) = \frac{1}{11} L_5(2h+1). \]

**Proof:** To prove lemma 3, take \( y = \frac{123}{2}, u = \sqrt{\frac{1+y}{2}} \).

Utilizing the identity,

\[ "U_h(u) = \frac{1}{11} L_5(h+1)," \]

\[ \Rightarrow "U_{2h}(u) = \frac{1}{11} L_5(2h+1)." \]

We also utilizing,

\[ "U_{2h}(u) = W_h(y)," \]

\[ W_h \left( \frac{123}{2} \right) = \frac{1}{11} L_5(2h+1). \]

This demonstrates lemma 3.

**Lemma 4:** The following identity holds true for all \( h \geq 0: \)

\[ V_h \left( \frac{47}{2} \right) = u^{-1} \frac{1}{2} L_4(2h+1). \]

**Proof:** To prove lemma 4, take \( y = \frac{47}{2}, u = \sqrt{\frac{1+y}{2}} \).

Using the identity,

\[ "T_h(u) = \frac{1}{2} L_4(h)," \]

\[ \Rightarrow "T_{2h+1}(u) = \frac{1}{2} L_4(2h+1)." \]

We also use,

\[ "V_h(y) = u^{-1} T_{2h+1}(u)," \]

\[ V_h \left( \frac{47}{2} \right) = u^{-1} \frac{1}{2} L_4(2h+1). \]

This demonstrates lemma 4.

**Lemma 5:** The following identity holds true for all \( h \geq 0: \)

\[ V_h(161) = u^{-1} \frac{1}{2} L_6(2h+1). \]

**Proof:** To prove lemma 5, take \( y = 161, u = \sqrt{\frac{1+y}{2}} \).

We utilizing the identity,

\[ "T_h(u) = \frac{1}{2} L_6(h)," \]

\[ \Rightarrow "T_{2h+1}(u) = \frac{1}{2} L_6(2h+1)." \]

We also using,

\[ "V_h(y) = u^{-1} T_{2h+1}(u)," \]

\[ V_h(161) = u^{-1} \frac{1}{2} L_6(2h+1). \]

This demonstrates lemma 5.

**Lemma 6:** The following identity holds true for all \( h \geq 0: \)

\[ V_h(-9) = u^{-1} \frac{(-i)^{2h+1}}{2} L_3(2h+1). \]

**Proof:** To prove lemma 6, take \( y = -9, u = \sqrt{\frac{1+y}{2}} \).

We utilizing the identity,

\[ "T_h(-2i) = \frac{(-i)^h}{2} L_3(h)," \]

\[ \Rightarrow T_{2h+1}(-2i) = \frac{(-i)^{2h+1}}{2} L_3(2h+1). \]

We also utilizing,

\[ "V_h(y) = u^{-1} T_{2h+1}(u)," \]

\[ V_h(-9) = u^{-1} \frac{(-i)^{2h+1}}{2} L_3(2h+1). \]

This demonstrates lemma 6.

**III. MAIN RESULTS**

**A. Relation between fourth-kind of the Chebyshev polynomials, Fibonacci numbers, and Lucas numbers.**

Theorem 1 and theorem 2 present the relation between Fibonacci numbers and fourth-kind Chebyshev polynomials. While theorem 3 explains the relationships between Lucas numbers and fourth-kind Chebyshev polynomials.

**Theorem 1.** Let \( h, q \) be integers with \( h \geq 0, q \geq 1, F_h \) be the \( h^{th} \) Fibonacci number, we have:

\[ \sum_{i=0}^{q-1} (-1)^{h-i} (q - 1 + h - i) F_{i+1}(2b_q + i) \]

\[ F_{6(2b_q + 1)} \cdots F_{6(2b_q + 1)} = \frac{2q+3}{q!} W_h(q) \]

inside sum covers all non-negative integers \( b_1, b_2 \ldots b_q+1 \) with \( b_1 + b_2 + \cdots + b_q + 1 = i \).

**Theorem 2.** Let \( h, q \) be integers with \( h \geq 0, q \geq 1, F_h \) be the \( h^{th} \) Fibonacci number, we have:

\[ \sum_{i=0}^{q-1} (-1)^{h-i} (q - 1 + h - i) F_{i+1}(2b_q + i) \]

\[ F_{6(2b_1 + 1)} \cdots F_{6(2b_1 + 1)} = \frac{3q+1}{2q!} W_h(q) \]

inside sum covers all non-negative integers \( b_1, b_2 \ldots b_q+1 \) with \( b_1 + b_2 + \cdots + b_q + 1 = i \).

**Theorem 3.** Let \( h, q \) be integers with \( h \geq 0, q \geq 1, L_h \) be the \( h^{th} \) Lucas number, then the following identity we can state:

\[ \sum_{i=0}^{q-1} (-1)^{h-i} (q - 1 + h - i) L_{i+1}(2b_q + i) \]

\[ L_5(2b_1 + 1) \cdots L_5(2b_q + 1) = \frac{123}{2q!} W_h(q) \]
inside sum covers all non-negative integers 
\[ b_1, b_2, \ldots, b_{q+1} \]
with 
\[ b_1 + b_2 + \cdots + b_{q+1} = i. \]

B. Relation between the third-kind Chebyshev polynomials, Lucas numbers.

Here, we have obtained three theorems. We present the relationship between Lucas numbers and third kind Chebyshev polynomials at certain points.

Theorem 4. Let \( h, q \) be integer with \( h \geq 0, q \geq 1, L_h \) be the \( h^{th} \) Lucas number, we have the following:

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \left( \binom{q-1}{q-1} \right) L_4(2b_1+1)L_4(2b_2+1) \ldots L_4(2b_{q+1}+1) = \frac{2}{q!}u^{q+1}V_{h+q}(q) \left( \frac{47}{2} \right),
\]

inside sum covers all non-negative integers \( b_1, b_2, \ldots, b_{q+1} \) with 
\[ b_1 + b_2 + \cdots + b_{q+1} = i, \quad u = \sqrt{\frac{1+y}{2}} \]

Theorem 5. Let \( h, q \) be integer with \( h \geq 0, q \geq 1, L_h \) be the \( h^{th} \) Lucas number, we have:

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \left( \binom{q-1}{q-1} \right) L_6(2b_1+1)L_6(2b_2+1) \ldots L_6(2b_{q+1}+1) = \frac{2}{q!}u^{q+1}V_{h+q}(q) \left( 161 \right),
\]

inside sum covers all non-negative integers \( b_1, b_2, \ldots, b_{q+1} \) with 
\[ b_1 + b_2 + \cdots + b_{q+1} = i, \quad u = \sqrt{\frac{1+y}{2}} \]

Theorem 6. Let \( h, q \) be integer with \( h \geq 0, q \geq 1, L_h \) be the \( h^{th} \) Lucas number, we have:

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+\cdots+b_{q+1}=i} \left( \binom{q-1}{q-1} \right) (-i)^{2b_1+1} L_3(2b_1+1)
\]
\[ + (-i)^{2b_2+1} L_3(2b_2+1) \ldots (-i)^{2b_{q+1}+1} L_3(2b_{q+1}+1) = \frac{2}{q!}u^{q+1}V_{h+q}(q) \left( -9 \right),
\]

inside sum covers all non-negative integers \( b_1, b_2, \ldots, b_{q+1} \) with 
\[ b_1 + b_2 + \cdots + b_{q+1} = i, \quad u = \sqrt{\frac{1+y}{2}} \]

We can draw the following six Corollary from the above six theorems:

Corollary 1. Let \( h, q \) be integers with \( h \geq 0, q \geq 1, \) the resulting output is as follows:

\[
\text{For } q = 2, \quad \sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( \binom{h-1}{h-1} \right) F_6(2b_1+1)F_6(2b_2+1) = \left[ \frac{264(2h+5)F_4(2h+3) - F_4(2h+7)}{207360(h+2)(h+3)F_6(2h+5)} + 190304F_6(2h+5) + 646(2h+5) \right] \left[ \frac{2}{(25920)^2} \right].
\]

Corollary 2. Let \( h, q \) be integers with \( h \geq 0, q \geq 1, \) the resulting output is as follows:

\[
\text{For } q = 2, \quad \sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( \binom{h-1}{h-1} \right) F_6(2b_1+1)F_6(2b_2+1) = \left[ \frac{264(2h+5)F_4(2h+3) - F_4(2h+7)}{207360(h+2)(h+3)F_6(2h+5)} + 190304F_6(2h+5) + 646(2h+5) \right] \left[ \frac{2}{(25920)^2} \right].
\]

Corollary 3. Let \( h, q \) be integers with \( h \geq 0, q \geq 1, \) the resulting output is as follows:

\[
\forall \quad q = 2, \quad \sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( \binom{h-1}{h-1} \right) L_5(2b_1+1)L_5(2b_2+1) = \left[ 5702(2h+5)\left[ L_5(2h+3) - L_5(2h+7) \right] + 937750L_5(2h+5) \right] \left[ \frac{20131375}{22} (h+2)(h+3)L_6(2h+5) \right].
\]

Corollary 4. Let \( h, q \) be integers with \( h \geq 0, q \geq 1, \) the resulting output is as follows:

\[
\forall \quad q = 2, \quad \sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( \binom{h-1}{h-1} \right) F_4(2b_1+1)F_4(2b_2+1) = \left[ 216(2h+5)F_4(2h+3) - F_4(2h+7) \right] \left[ \frac{10584F_4(2h+5)}{19845(h+2)(h+3)} \right] \left[ \frac{2}{(2205)^2} \right].
\]

Proof of the Theorems

Theorems and corollaries will be proved in this section.

Theorem 1: Proof. Let \( W_0(y) \) be defined by (4), then for any \( h, q \) be integers with \( h \geq 0, q \geq 1, \) by (7)
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) \prod_{k=0}^{q+i} \binom{q}{k} = \frac{1}{2^{2q+3}} W_{h+q}^{(q)}(y).
\]

From lemma 1, we have the identity between Chebyshev polynomials of fourth kind and Lucas numbers,
\[
W_h(161) = \frac{1}{8} F_6(2h+1).
\] (10)

From (10), we have
\[
W_{b_1}(161) = \frac{1}{8} F_6(2b_1+1),
\]
\[
W_{b_2}(161) = \frac{1}{8} F_6(2b_2+1),
\]
\[
\ldots
\]
\[
W_{b_{k+1}}(161) = \frac{1}{8} F_6(2b_{k+1}+1).
\]

Using above equations in (9), we get
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) F_6(2b_{i+1}) = \frac{2^{2q+3}}{q!} W_{h+q}^{(q)}(y).
\]

Hence theorem 1 is formulated in this manner.

**Theorem 2: Proof.** Let \( W_h(y) \) be defined by (4), then for any \( h, q \) be integers with \( h \geq 0, q \geq 1 \), by (7) yields
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) \prod_{k=0}^{q+i} W_{b_k}(y) = \frac{1}{2^{2q+3}} W_{h+q}^{(q)}(y).
\] (12)

From lemma 2, we have the identity between Chebyshev polynomials of fourth kind and Fibonacci numbers i.e.
\[
W_h \left( \frac{2}{3} \right) = \frac{1}{3} F_4(2h+1).
\] (13)

From (13), we have
\[
W_{b_1} \left( \frac{47}{2} \right) = \frac{1}{3} F_4(2b_1+1),
\]
\[
W_{b_2} \left( \frac{47}{2} \right) = \frac{1}{3} F_4(2b_2+1),
\]
\[
\ldots
\]
\[
W_{b_{k+1}} \left( \frac{47}{2} \right) = \frac{1}{3} F_4(2b_{k+1}+1).
\]

Using above equations in (12), we get
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) F_4(2b_{i+1}) = \frac{3^{q+i}}{2^{2q+3}} W_{h+q}^{(q)}(\frac{47}{2}).
\] (14)

Thus theorem 2 established through proof.

**Theorem 3: Proof.** Let \( W_h(y) \) be defined by (4), then for any \( h, q \) be integers with \( h \geq 0, q \geq 1 \), by using (7),
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) \prod_{k=0}^{q+i} W_{b_k}(y) = \frac{1}{2^{2q+3}} W_{h+q}^{(q)}(y).
\]

From lemma 3, we have the identity between Chebyshev polynomials of fourth kind and Lucas numbers, i.e.
\[
W_h \left( \frac{123}{2} \right) = \frac{1}{11} L_5(2h+1).
\] (16)

From (16), we have
\[
W_{b_1} \left( \frac{123}{2} \right) = \frac{1}{11} L_5(2b_1+1),
\]
\[
W_{b_2} \left( \frac{123}{2} \right) = \frac{1}{11} L_5(2b_2+1),
\]
\[
\ldots
\]
\[
W_{b_{k+1}} \left( \frac{123}{2} \right) = \frac{1}{11} L_5(2b_{k+1}+1).
\]

Using above equations in (15), we get
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) L_5(2b_{i+1}) = \frac{1}{2^{2q+3}} W_{h+q}^{(q)}(\frac{47}{2}).
\] (17)

Hence, proof is established for theorem 3.

**Theorem 4: Proof.** Let \( V_h(y) \) be defined by (3), then for any \( h, q \) be integers with \( h \geq 0, q \geq 1 \), by using (8)
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) \prod_{k=0}^{q+i} V_{b_k}(y) = \frac{1}{2^{2q+3}} V_{h+q}^{(q)}(y).
\] (18)

From lemma 4, we have the identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.
\[
V_h \left( \frac{47}{2} \right) = u^{-1} \frac{1}{2} L_4(2h+1).
\] (19)

From (19), we have
\[
V_{b_1} \left( \frac{47}{2} \right) = u^{-1} \frac{1}{2} L_4(2b_1+1),
\]
\[
V_{b_2} \left( \frac{47}{2} \right) = u^{-1} \frac{1}{2} L_4(2b_2+1),
\]
\[
\ldots
\]
\[
V_{b_{k+1}} \left( \frac{47}{2} \right) = u^{-1} \frac{1}{2} L_4(2b_{k+1}+1).
\]

Using above equations in (18), we get
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) L_4(2b_{i+1}) = \frac{1}{2^{2q+3}} V_{h+q}^{(q)}(\frac{47}{2}).
\] (20)

Hence, theorem 4’s proof is now complete.

**Theorem 5: Proof.** Let \( V_h(y) \) be defined by (3), then for any \( h, q \) be integers with \( h \geq 0, q \geq 1 \), by (8)
\[
\sum_{i=0}^{n} a_i b_i \sum_{q=0}^{n} (-1)^{n-i} \left( q - 1 + h - i \right) \prod_{k=0}^{q+i} V_{b_k}(y) = \frac{1}{2^{2q+3}} V_{h+q}^{(q)}(y).
\] (21)

From lemma 5, we have the identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.
\[
V_h(161) = u^{-1} \frac{1}{2} L_6(2h+1).
\] (22)

From (22), we have
\[
V_{b_1}(161) = u^{-1} \frac{1}{2} F_6(2b_1+1).
Proof of Theorem 6: Proof. Let \( V_n(y) \) be defined by (3), then for any \( h, q \) be integers with \( h \geq 0, q \geq 1 \), by (8) yields

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+\ldots+b_q+1=i} \left( \frac{q-1+h-i}{q-1} \right) L_6(2b_1+1)L_6(2b_2+1) \cdots L_6(2b_q+1) = \frac{1}{2^q q!} V_{h+q}(q)(y).
\]  

(24)

From lemma 6, we have the identity between Chebyshev polynomials of third kind and Lucas numbers, i.e.

\[
V_h(-9) = u^{-1} \frac{(-i)^{2h+1}}{2} L_{3(2h+1)}.
\]  

(25)

From (25), we have

\[
V_{b_1}(-9) = u^{-1} \frac{(-i)^{2b_1+1}}{2} L_{3(2b_1+1)},
\]

\[
V_{b_2}(-9) = u^{-1} \frac{(-i)^{2b_2+1}}{2} L_{3(2b_2+1)},
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]

\[
V_{b_k}(-9) = u^{-1} \frac{(-i)^{2b_k+1}}{2} L_{3(2b_k+1)}.
\]

Using above equation in (24), we get

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+\ldots+b_q+1=i} \left( \frac{q-1+h-i}{q-1} \right) L_6(2b_1+1)L_6(2b_2+1) \cdots L_6(2b_q+1) = \frac{1}{2^q q!} V_{h+q}(q)(-9).
\]  

(26)

All of our theorem now has through proof.

Proof of the Corollaries

Proof of Corollary 1: We know that \( W_n(y) \) satisfies the differential equations:

\[
(1-y^2)W_n'(y) = \frac{1}{2} \left( h + \frac{1}{2} \right) \left( W_{n-1}(y) - W_{n+1}(y) \right) + \frac{1}{2} \left( 1 + y \right) W_n(y).
\]  

(27)

and

\[
(1-y^2)W_n''(y) = (1 + 2y)W_n'(y) - h(n+1)W_n(y).
\]  

(28)

Putting \( q = 2 \) in (11), by (5), (27), (28),

We get

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( -1 \right)^{h-i} \left( 1 + h - i \right) F_6(2b_1+1)F_6(2b_2+1)F_6(2b_3+1) = \frac{1}{2} \left( h + \frac{1}{2} \right) \left( V_{h-1}(y) - V_{h+1}(y) \right) + \frac{1}{2} \left( 1 - y \right) V_h(y).
\]  

(29)

(1 - \( y^2 \))\( W_n''(y) = -(1 - 2y)W_n'(y) - h(n+1)W_n(y). \)

(30)

Put \( q = 2 \) in (20), and by (6), (29), (30),

We get

\[
\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( 1 + h - i \right) L_4(2b_1+1)L_4(2b_2+1)L_4(2b_3+1)
\]  

(31)
Proof of Corollary 5: 
Put \( q = 2 \) in (23), and by (6), (29), (30), we get 
\[
\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( \frac{1}{h+1} \right) L_6(2b_1+1) L_6(2b_2+1) L_6(2b_3+1) = \\
\left( \frac{221}{8} \right) u \left[ (2h+5) \left( L_6(2h+5)^{-5} + 12840 L_6(2h+5) \right) + 12960 (h+2)(h+3) L_6(2h+5) \right].
\]

Proof of Corollary 6: 
Put \( q = 2 \) in (26), and by (6), (29), (30), we get 
\[
\sum_{i=0}^{h} \sum_{b_1+b_2+b_3=i} \left( \frac{1}{h+1} \right) (-i)^{2b_1+1} L_3(2b_1+1) \left( (-i)^{2b_2+1} L_3(2b_2+1) + (-i)^{2b_3+1} L_3(2b_3+1) \right) = \\
u^{-1} \left[ -19 (2h+5) \left( (-i)^{2h+3} L_3(2h+3) - (-i)^{2h+7} L_3(2h+7) \right) \right] + 95 \left( -i \right)^{2h+5} L_3(2h+5) + 800 (h+2)(h+3) (-i)^{2h+5} L_3(2h+5). 
\]

IV. APPLICATIONS 
Chebyshev polynomials are widely used in various fields like mathematics, computer science, engineering and physics etcetera. Chebyshev polynomials fall under the category of recurrence relations and are extensively used to improve the advanced techniques for counting. These polynomials are used to study the integer function that allows us to establish new relationships among other polynomials. Chebyshev polynomials are of significant importance when it comes to solving other polynomials for obtaining novel trigonometric identities, finding the solutions to second-order differential equations, interpolating large data and in approximation theory.

With the help of Chebyshev polynomials, approximate numerical solutions can be obtained for differential and integral equations. Chebyshev polynomials play a key role in signal processing, primarily in the design of filters known as Chebyshev filters. They are in high demand in the field of computer graphics to generate a variety of shapes, surfaces, and curves.

V. PRATICAL APPLICATIONS 
Express \( y^4 - 4y^3 - 2y^2 + 3y - 1 \) in terms of third-kind of the Chebyshev polynomials.

SOL: The first four third-kind Chebyshev polynomials are:
\[
V_0(y) = 1, \quad V_1(y) = 2y - 1, \quad V_2(y) = 4y^2 - 2y - 1, \quad V_3(y) = 8y^3 - 4y^2 - 4y + 1, \quad V_4(y) = 16y^4 - 8y^3 - 12y^2 + 4y + 1,
\]

From above equations, we get 
\[
y = \frac{1}{2} \left[ V_0(y) + V_1(y) \right], \quad y^2 = \frac{1}{4} \left[ 2V_0(y) + V_1(y) + V_2(y) \right],
\]
\[
y^3 = \frac{1}{8} \left[ 3V_0(y) + 3V_1(y) + V_2(y) + V_3(y) \right], \quad y^4 = \frac{1}{16} \left[ 6V_0(y) - 4V_1(y) + 4V_2(y) + V_3(y) + V_4(y) \right].
\]

Putting above values in \( y^4 - 4y^3 - 2y^2 + 3y - 1 \), we get,
\[
\frac{1}{16} \left[ 6V_0(y) - 4V_1(y) + 4V_2(y) + V_3(y) + V_4(y) \right] - \frac{1}{2} \left[ 3V_0(y) + 3V_1(y) + V_2(y) + V_3 \right] - \frac{1}{2} \left[ 2V_0(y) + V_1(y) + V_2(y) \right] + \frac{3}{2} \left[ V_0(y) + V_1(y) - V_0(y) \right].
\]

Hence,
\[
- \frac{13}{8} V_0(y) - \frac{3}{4} V_1(y) - \frac{3}{4} V_2(y) - \frac{7}{16} V_3(y) + \frac{1}{16} V_4(y).
\]

VI. CONCLUSION 
In relation to both the third-fourth kinds of Chebyshev polynomials with Fibonacci numbers and Lucas numbers, we have discovered six theorems and six corollaries. The first two theorems present the relationship between fourth-kind Chebyshev polynomials and Fibonacci numbers. Theorem 3 clearly demonstrates the strong correlation between the Lucas numbers and fourth-kind Chebyshev polynomials, emphasizing the significance of the relationship. Theorems 4, 5 and 6 provided the link between Lucas numbers and third-kind Chebyshev polynomials at certain points. The six corollaries are merely particular cases associated with our six theorems. These findings are useful for understanding the properties and identities of Chebyshev polynomials with Fibonacci and Lucas numbers.

VII. SIGNIFICANCE OF THE WORK
Here, we illustrate the significance of the present work in the following elements:

Obtaining connections between fourth-kind of the Chebyshev polynomials and Fibonacci numbers with some variables: \( x = 161 \cdot \frac{47}{2} \).

Obtaining connections between fourth-kind of the Chebyshev polynomials and Lucas numbers at a particular variable: \( x = \frac{123}{2} \).

Obtaining connections between third-kind of the Chebyshev polynomials and Lucas numbers with some variables:

Figure 1: Flow Chart of Chebyshev Polynomial Application
It is worth mentioning here that the above-achieved results and analysis are fruitful. Some of their presumed uses are given below:

- These results strengthen the correlation of Chebyshev polynomials to Fibonacci and Lucas.
- They are also beneficial in studying problems connected to calculating general summations.
- They help study integer sequences.
- These polynomials are fruitful in solving convolution sum problems.
- These polynomials can be used to solve differential equations, whether they are linear or non-linear.
- To acquire numerical answers to differential equations, whether linear or nonlinear.
- The connections between the Chebyshev polynomials, Fibonacci, and Lucas numbers are highly helpful in obtaining the identities related to them.
- The Chebyshev polynomials are fruitful in approximation theory.

REFERENCES


