Fixed Point Theorem for Orthogonal (varphi, psi)-(Lambda, delta, Upsilon)-Admissible Multivalued Contractive Mapping in Orthogonal Metric Spaces

Gunasekaran Nallaselli and Arul Joseph Gnanaprakasam*

***Abstract*—In the current research, we represent a novel class of multivalued contractive mappings that are cyclic orthogonal $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible. In the framework of O-complete metric spaces, we establish the fixed point results for these new cyclic orthogonal $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible contractive mappings.**

***Index Terms*—cyclic (φ,ψ)-admissible mapping, cyclic orthogonal (φ,ψ)-admissible mapping, cyclic $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible multivalued mapping, cyclic orthogonal $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible multivalued mapping, fixed point, orthogonal metric space.**

I. Introduction

MANY years ago, various fixed point findings were obtained in the context of metric spaces. If (X,d) is a complete metric space (abbreviated CMS) and $f : X \to X$ is a contraction mapping $(i.e., d(f(x), f(y)) \leq \alpha d(x,y),\ \forall\ x,y \in X$, where $0 \leq \alpha < 1)$, then f has a unique fixed point (abbreviated UFP). First, Kirk et al. [8] introduced the concept of cyclic contraction in the fixed point theory. There has been a lot of research done on the fixed points of multi-valued functions. A point x is said to be a fixed point of a single-valued mapping f (multi-valued mapping F) if $f(x) = x(x \in F(x))$. Nadler [1] examined the convergence of a sequence of the Banach contraction multivalued fixed point results of a convergent of multivalued contraction mappings of a CMS X into the nonempty CL(X) in 1969. In 2014, Ali et al. [2] introduced the concept of (α,ψ,ξ)-contractive multivalued mappings and extended the notion of $\alpha-\psi$-contractive mappings to closed valued multi-functions, as well as providing fixed-point theorems for (α,ψ,ξ)-contractive multivalued mappings in CMS's. Alizadeh et al. [3] introduced the concept of cyclic $(\alpha,\beta)-(\psi,\phi)$-contractive mappings, and cyclic rational weak $\alpha-\beta-\psi$-contraction mappings. In the situation of CMS's, they demonstrated some new fixed point results for such mappings. Hussain et al. [4] developed some fixed point theorems for multi and single-valued mappings via $\alpha-\psi$-contractive requirements in CMS in 2014. Samet et al. [5] developed the ideas of $\alpha-\psi$-contractive and α-admissible mappings in CMS's in 2012 and established different fixed point theorems for such mappings. Others have achieved significant results in this prominent field recently, more details see ([6], [7], [9], [10], [11]).

Gordji et al. [12] invented the concept of orthogonal sets and metric spaces in 2017. They also established the existence and uniqueness of fixed points for mappings on a generalized orthogonal metric space (shortly, OMS). Following that, several authors proved many existing fixed point theorems in various metric spaces (for example, [13] - [21]).

In this paper, we combine the ideas of cyclic $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible multivalued mapping(shortly, A.M.M.) and orthogonal concept of metric space and prove a fixed point theorem in these cyclic orthogonal $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible multivalued contraction mappings.

Manuscript received April 15, 2023; revised September 4, 2023.

Gunasekaran Nallaselli is a Research Scholar in Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603203, India. (e-mail: gn4255@srmist.edu.in).

Arul Joseph Gnanaprakasam is an Assistant Professor in Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur - 603203, India. (corresponding author e-mail: aruljosg@srmist.edu.in).

II. Preliminaries

Several results in the present context is listed below. Throughout this paper, we denote $\mathbb{N}$ and $\mathcal{R}^+$ by the set of all positive integers and real numbers, $\mathcal{R}$ by $(-\infty,+\infty)$ and $\mathcal{R}_0^+$ by $[0,\infty)$.

Definition 1. *[5] Let $\Im : \pounds \to \pounds$ and $\varphi : \pounds \times \pounds \to \mathcal{R}_0^+$ be functions. $\Im$ is called φ-admissible when $\beta,\zeta \in \pounds$ such that (s.t.) $\varphi(\beta,\zeta) \geq 1 \implies \varphi(\Im\beta,\Im\zeta) \geq 1$.*

Definition 2. *[3] Let $\mathfrak{e} : \pounds \to CL(\pounds)$ and $\varphi,\psi : \pounds \to \mathcal{R}^+$ be two functions. $\Im$ is said to be a cyclic (φ,ψ)-admissible mapping if*

(1) *$\varphi(\beta) \geq 1$ for some $\beta \in \pounds \implies \psi(\Im\beta) \geq 1$,*
(2) *$\psi(\beta) \geq 1$ for some $\beta \in \pounds \implies \varphi(\Im\beta) \geq 1$.*

Definition 3. *[3] Let $(\pounds,\partial)$ be a CMS and $\Im : \pounds \to \pounds$ be a cyclic (φ,ψ)-admissible mapping. We say that $\Im$ is a cyclic $(\varphi,\psi)-(\Lambda,\Upsilon)$-contractive mapping if for all $\beta,\zeta \in \pounds$,*

$$\varphi(\beta)\psi(\zeta) \geq 1 \implies \Lambda(\partial(\Im\beta,\Im\zeta)) \leq \Lambda(\partial(\beta,\zeta)) - \Upsilon(\partial(\beta,\zeta)),$$

where $\Lambda : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ is increasing and continuous function and $\Upsilon : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ is a lower semi-continuous function with $\Upsilon(\iota) = 0 \implies \iota = 0$.

Theorem 1. *[3] Let $(\pounds,\partial)$ be a CMS and $\Im : \pounds \to \pounds$ be a $(\varphi,\psi)-(\Lambda,\Upsilon)$-admissible mapping. Assume that the following axioms hold:*

(1) *there exists $\beta_0 \in \pounds$ s.t. $\varphi(\beta_0) \geq 1$ and $\psi(\beta_0) \geq 1$,*
(2) *$\Im$ is continuous, or*

(3) *if* $\{\beta_\varepsilon\}$ *is a sequence in* $\pounds$ *s.t.* $\beta_\varepsilon \to \beta$ *and* $\psi(\beta_\varepsilon) \geq 1$, $\forall\ \varepsilon \in \mathbb{N}$, *then* $\psi(\beta) \geq 1$,

then $\Im$ *has a fixed point. Moreover, if* $\varphi(\beta) \geq 1$ *and* $\psi(\zeta) \geq 1$, $\forall\ \beta, \zeta\ \in \mathcal{F}(\Im)$, *then* $\Im$ *has a UFP.*

Definition 4. *[2] The family* Δ *of all functions* $\delta : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ *satisfies the properties:*

(1) δ *is continuous;*
(2) δ *is nondereasing on* $\mathcal{R}^+$*;*
(3) $\delta(0) = 0$ *and* $\delta(\iota) > 0$, $\forall\ \iota \in (0, \infty)$*;*
(4) δ *is sub additive.*

Lemma II.1. *[2] Let* $(\pounds, \partial)$ *be a metric space, let* $\delta \in \Delta$ *and* $\Im \in CL(\pounds)$. *Suppose there exists* $\beta \in \pounds$ *s.t.* $\delta(\partial(\beta, \Im)) > 0$. *Then, there exists* $\zeta \in \Im$ *s.t.*

$$\delta(\partial(\beta, \zeta)) < \varrho\delta(\partial(\beta, \Im)),$$

where $\varrho > 1$.

Definition 5. *[12] Let* $\pounds \neq \emptyset$ *and define a binary relation* $\perp \subseteq \pounds \times \pounds$ *if* $\perp$ *satisfy:*

$$\exists\ \beta_0 \in \pounds,\ (\forall\ \beta \in \pounds, \beta\perp\beta_0) \quad or \quad (\forall\beta \in \pounds, \beta_0\perp\beta),$$

then, the pair $(\pounds, \perp)$ *is known as orthogonal set (briefly O-set).*

Example 1. *[12] Let* $\pounds = [0, 1)$. *Suppose* $\beta\perp\zeta$ *if* $\beta \leq \zeta$. $(\pounds, \perp)$ *is an O-set.*

Example 2. *[12] Let* $(\pounds, \partial)$ *be a metric space and* $\Im : \pounds \to \pounds$ *be a Picard operator, i.e.,* $\Im$ *has a UFP* $\beta^* \in \pounds$ *and* $\lim\limits_{\varepsilon\to\infty} \Im^\varepsilon(\beta) = \beta^*$, $\forall\ \zeta \in \pounds$. *We define the binary relation* $\perp$ *on* $\pounds$ *by* $\zeta\perp\beta$ *if*

$$\lim_{\varepsilon\to\infty} \partial(\beta, \Im^\varepsilon(\zeta)) = 0.$$

Then, $(\pounds, \perp)$ *is an O-set.*

Example 3. *Suppose that* $\mathcal{M}(\varepsilon)$ *is the set of all* $\varepsilon\times\varepsilon$ *matrices and* $\mathcal{Q}$ *is an invertible matrix. Define the relation* $\perp$ *on* $\mathcal{M}(\varepsilon)$ *by* $\mathcal{K}\perp\mathcal{E} \iff \exists\ \pounds \in \mathcal{M}(\varepsilon) : \mathcal{K}\pounds = \mathcal{E}$. *It is easy to seen that* $\mathcal{Q}\perp\mathcal{E}$, $\forall\ \mathcal{E} \in \mathcal{M}(\varepsilon)$.

Definition 6. *[12] Let* $(\pounds, \perp)$ *be an O-set. A sequence* $\{\beta_\varepsilon\}$ *is called an orthogonal sequence (briefly, O-sequence) if*

$$(\ \forall\ \varepsilon \in \mathbb{N}, \beta_\varepsilon\perp\beta_{\varepsilon+1}) \quad or \quad (\ \forall\ \varepsilon \in \mathbb{N}, \beta_{\varepsilon+1}\perp\beta_\varepsilon).$$

Definition 7. *[12] Let* $(\pounds, \perp, \partial)$ *be an OMS. Then, a mapping* $\Im : \pounds \to \pounds$ *is said to be orthogonally continuous (or* $\perp$*-continuous) in* $\beta \in \pounds$ *if for each O-sequence* $\{\beta_\varepsilon\}$ *in* $\pounds$ *with* $\beta_\varepsilon \to \beta$ *as* $n \to \infty$, *we have* $\Im(\beta_\varepsilon) \to \Im(\beta)$ *as* $\varepsilon \to \infty$. *Also,* $\Im$ *is said to be* $\perp$*-continuous on* $\pounds$ *if* $\Im$ *is* $\perp$*-continuous in each* $\beta \in \pounds$.

Example 4. *The continuity implies orthogonal continuity but the converse is not true. If* $\Im : \mathcal{R} \to \mathcal{R}$ *is defined by* $\Im(\beta) = [\beta]$, $\forall\ \beta \in \mathcal{R}$ *and the relation* $\perp \subset \mathcal{R}\times\mathcal{R}$ *is defined by*

$$\beta\perp\zeta \text{ if } \beta, \zeta \in \left(\mathrm{i} + \frac{1}{3}, \mathrm{i} + \frac{2}{3}\right), \mathrm{i} \in \mathcal{Z} \text{ or } \beta = 0.$$

Then, $\Im$ *is* $\perp$*-continuous while* $\Im$ *is discontinuous on* $\mathcal{R}$.

Example 5. *Let* $\pounds = \mathcal{R}$. *Suppose that* $\beta\perp\zeta$ *if and only if* $\beta = 0$ *or* $0 \neq \zeta \in \mathcal{Q}$. *It is easy to seen that* $(\pounds, \perp)$ *is an O-set. Define* $\Im : \pounds \to \pounds$ *by*

$$\Im(\beta) = \begin{cases} 1, & \text{if } \beta \in \mathcal{Q}, \\ 0, & \text{if } \beta \in \mathcal{Q}^c. \end{cases}$$

Therefore, $\Im$ *is* $\perp$*-continuous at all rational numbers.*

Definition 8. *[12] Let* $(\pounds, \perp, \partial)$ *be an OMS. Then,* $\pounds$ *is said to be orthogonal complete (briefly, O-complete) if every O-Cauchy sequence is convergent.*

Example 6. *The completeness of the metric space implies O-completeness, but the converse is not true. We know that* $\pounds = [0, 1)$ *with Euclidean metric* ∂ *is not a CMS. If we define the relation* $\perp \subset \pounds \times \pounds$ *by* $\beta\perp\zeta \iff \beta \leq \zeta \leq \frac{1}{2}$ *or* $\beta = 0$, *then* $(\pounds, \perp, \partial)$ *is an O-complete.*

Definition 9. *[12] Let* $(\pounds, \perp)$ *be an O-set. A mapping* $\Im : \pounds \to \pounds$ *is called* $\perp$*-preserving if* $\Im\beta\perp\Im\zeta$ *whenever* $\beta\perp\zeta$. *Also* $\Im : \pounds \to \pounds$ *is called weakly* $\perp$*-preserving if* $\Im(\beta)\perp\Im(\zeta)$ *or* $\Im(\zeta)\perp\Im(\beta)$ *whenever* $\beta\perp\zeta$.

Example 7. *Let* $\pounds = [0, 1)$ *and define a relation* $\perp \subset [0, 1) \times [0, 1)$ *by*

$$\beta\perp\zeta \text{ if } \beta\zeta \in \{\beta, \zeta\} \subset [0, 1).$$

Then, $\pounds = [0, 1)$ *is an O-set. Now, define a function* $\Im : \pounds \to CL(\pounds)$ *by*

$$\Im(\beta) = \begin{cases} [\frac{\beta}{15}, \frac{\beta+1}{7}], & \text{if } \beta \in \mathcal{Q} \cap \pounds, \\ \{0\}, & \text{if } \beta \in \mathcal{Q}^c \cap \pounds, \end{cases}$$

is a $\perp$*-preserving mapping.*

III. Main Results

Now, we introduce the definition of a cyclic orthogonal $(\varphi, \psi)-(\Lambda, \delta, \Upsilon)$(abbreviated C.O.$(\varphi, \psi)-(\Lambda, \delta, \Upsilon)$)-A.M.M and prove a fixed point theorem on O-CMS.

Definition 10. *Let* $\Im : \pounds \to \pounds$ *be a a self-mapping and a function* $\varphi : \pounds \times \pounds \to \mathcal{R}_0^+$. $\Im$ *is called orthogonal* φ*-admissible when if* $\beta, \zeta \in \pounds$ *with* $\beta\perp\zeta$ *s.t.* $\varphi(\beta, \zeta) \geq 1$ *then we have* $\varphi(\Im\beta, \Im\zeta) \geq 1$.

Definition 11. *Let* $\mathfrak{e} : \pounds \to CL(\pounds)$ *be a mapping and* $\varphi, \psi : \pounds \to \mathcal{R}^+$ *be two functions.* $\Im$ *is said to be a cyclic orthogonal* (φ, ψ)*-admissible mappingping if* $\forall\ \beta$ *with* $\beta\perp\beta$

(1) $\varphi(\beta) \geq 1$ *for some* $\beta \in \pounds \implies \psi(\Im\beta) \geq 1$,
(2) $\psi(\beta) \geq 1$ *for some* $\beta \in \pounds \implies \varphi(\Im\beta) \geq 1$.

Definition 12. *Let* $(\pounds, \partial)$ *be an O-CMS and* $\Im : \pounds \to \pounds$ *be a C.O.* (φ, ψ)*-admissible mapping. We say that* $\Im$ *is a C.O.* $(\varphi, \psi) - (\Lambda, \Upsilon)$*-contractive mapping if* $\forall\ \beta, \zeta \in \pounds$ *with* $\beta\perp\zeta$

$$\begin{aligned} &\partial(\Im\beta, \Im\zeta) > 0, \varphi(\beta)\psi(\zeta) \geq 1 \\ &\implies \Lambda(\partial(\Im\beta, \Im\zeta)) \leq \Lambda(\partial(\beta, \zeta)) - \Upsilon(\partial(\beta, \zeta)), \end{aligned}$$

where $\Lambda : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ *is a continuous and increasing function and* $\Upsilon : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ *is a lower semi-continuous function with* $\Upsilon(\iota) = 0 \implies \iota = 0$.

Definition 13. *Let* $(\pounds, \perp, \partial)$ *be an OMS and* $\Im : \pounds \to CL(\pounds)$ *by cyclic* (φ, ψ) *admissible mapping. We say that* $\Im$ *is a C.O.*$(\varphi, \psi) - (\Lambda, \delta, \Upsilon)$*-A.M.M of type A if* *there exists* $\varphi, \psi : \pounds \times \pounds \to \mathcal{R}_0^+, \Lambda \in \Xi, \delta \in \Delta$ *and* $\Upsilon \in \Pi$ *s.t.* $\forall\ \beta, \zeta \in \pounds$ *with* $\beta \perp \zeta$*:*

$$\begin{aligned}&\mathcal{H}(\Im\beta, \Im\zeta) > 0,\ \varphi(\beta)\psi(\zeta) \geq 1\\ &\implies \delta(\mathcal{H}(\Im\beta, \Im\zeta)) \leq \Lambda(\delta(\mathcal{M}(\beta,\zeta))) - \Upsilon(\mathcal{M}(\beta,\zeta)),\end{aligned} \tag{1}$$

where

$$\mathcal{M}(\beta,\zeta) = \max\Big\{\partial(\beta,\zeta), \partial(\beta,\Im\beta), \partial(\zeta,\Im\zeta), \frac{1}{2}[\partial(\beta,\Im\zeta) + \partial(\zeta,\Im\beta)]\Big\}.$$

Definition 14. *Let* $(\pounds, \perp, \partial)$ *be an OMS. The mapping* $\Im : \pounds \to CL(\pounds)$ *is said to be a C.O.* (φ, ψ)*- A.M.M of type B if there exists* $\varphi, \psi : \pounds \times \pounds \to \mathcal{R}_0^+, \Lambda \in \Xi, \delta \in \Delta$ *and* $\Upsilon \in \Pi$ *s.t.* $\forall\ \beta, \zeta \in \pounds$ *with* $\beta \perp \zeta$*:*

$$\begin{aligned}&\mathcal{H}(\Im\beta, \Im\zeta) > 0, \varphi(\beta)\psi(\zeta) \geq 1\\ &\implies \delta(\mathcal{H}(\Im\beta, \Im\zeta)) \leq \Lambda(\delta(\mathcal{P}(\beta,\zeta))) - \Upsilon(\mathcal{P}(\beta,\zeta))\end{aligned} \tag{2}$$

where

$$\mathcal{P}(\beta,\zeta) = \max\Big\{\partial(\beta,\zeta), \frac{[1+\partial(\beta,\Im\beta)]\partial(\zeta,\Im\zeta)}{\partial(\beta,\zeta)+1}\Big\}.$$

Theorem 2. *Let* $(\pounds, \perp, \partial)$ *be an orthogonal CMS and* $\Im : \pounds \to CL(\pounds)$ *by C.O.*$(\varphi, \psi) - (\Lambda, \delta, \Upsilon)$*-A.M.M of type A. Assume that the following postulations hold:*

1) *there exits* $\beta_0 \in \pounds$ *and* $\beta_1 \in \Im\beta_0$ *with* $\beta_0 \perp \beta_1$ *s.t.*
$$\begin{aligned}\varphi(\beta_0) \geq 1 &\implies \psi(\Im\beta_0) = \psi(\beta_1) \geq 1,\\ \psi(\beta_0) \geq 1 &\implies \varphi(\Im\beta_0) = \varphi(\beta_1) \geq 1,\end{aligned}$$
2) *if* $\{\beta_\varepsilon\}$ *is an O-sequence in* $\pounds$ *with* $\beta_\varepsilon \to \beta$ *as* $\beta \to \infty$ *and* $\psi(\beta_\varepsilon) \geq 1$, $\forall\ \varepsilon \in \mathbb{N}$*, then* $\psi(\beta) \geq 1$*,*
3) $\perp$*-continuous,*
4) $\perp$*-preserving,*

then $\Im$ *has a UFP.*

Proof: Since $(\pounds, \perp)$ is an O-set,

$$\exists\ \beta_0 \in \pounds\ (\forall\ \beta \in \pounds,\ \beta \perp \beta_0)\ \vee\ (\forall\ \beta \in \pounds,\ \beta_0 \perp \beta).$$

It follows that $\beta_0 \perp \Im(\beta_0)$ or $\Im(\beta_0) \perp \beta_0$.
Let

$$\beta_1 = \Im(\beta_0); \beta_2 = \Im(\beta_1); \ldots; \beta_{\varepsilon+1} = \Im(\beta_\varepsilon),\ \forall\ \varepsilon \in \mathbb{N}.$$

By starting from β_0 and $\beta_1 \in \Im\beta_0$ with $\beta_0 \perp \beta_1$ in axioms (1), we have

$$\begin{aligned}\varphi(\beta_0) \geq 1 &\implies \psi(\Im\beta_0) = \psi(\beta_1) \geq 1,\\ \psi(\beta_0) \geq 1 &\implies \varphi(\Im\beta_0) = \varphi(\beta_1) \geq 1.\end{aligned}$$

Therefore, $\varphi(\beta_0) \geq 1$ and $\psi(\beta_1) \geq 1$, equivalently, $\varphi(\beta_0)\psi(\beta_1) \geq 1$. If $\beta_0 = \beta_1$, we conclude that $\beta_1 \in \mathcal{F}(\Im)$ and so the proof is completed. Now, taking $\beta_0 \neq \beta_1$ and $\beta_1 \notin \Im\beta_1$. From (1), we have

$$\begin{aligned}0 &< \delta(\partial(\beta_1, \Im\beta_1))\\ &\leq \delta(\mathcal{H}(\Im\beta_0, \Im\beta_1))\\ &\leq \Lambda(\delta(\mathcal{M}(\beta_0,\beta_1))) - \Upsilon(\mathcal{M}(\beta_0,\beta_1)),\end{aligned} \tag{3}$$

where

$$\begin{aligned}\mathcal{M}(\beta_0,\beta_1) &= \max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_0,\Im\beta_0), \partial(\beta_1,\Im\beta_1), \frac{1}{2}[\partial(\beta_0,\Im\beta_1) + \partial(\beta_1,\Im\beta_0)]\Big\}\\ &= \max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1), \frac{1}{2}[\partial(\beta_0,\Im\beta_1)]\Big\}\\ &= \max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1), \frac{1}{2}[\partial(\beta_0,\beta_1) + \partial(\beta_1,\Im\beta_1)]\Big\}\\ &= \max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1)\Big\}.\end{aligned} \tag{4}$$

From (3) and (4) and by using the properties of Υ, we get

$$\begin{aligned}0 &< \delta(\partial(\beta_1,\Im\beta_1))\\ &\leq \Lambda\Big(\delta\Big(\max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1)\Big\}\Big)\Big)\\ &\quad - \Upsilon\Big(\max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1)\Big\}\Big).\end{aligned} \tag{5}$$

Assume that $\max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1)\Big\} = \partial(\beta_1,\Im\beta_1)$, then we obtain

$$\begin{aligned}0 < \delta(\partial(\beta_1,\Im\beta_1)) &\leq \Lambda(\delta(\partial(\beta_1,\Im\beta_1))) - \Upsilon(\partial(\beta_1,\Im\beta_1))\\ &< \Lambda(\delta(\partial(\beta_1,\Im\beta_1))),\end{aligned}$$

which is a contradiction. Thus

$$\max\Big\{\partial(\beta_0,\beta_1), \partial(\beta_1,\Im\beta_1)\Big\} = \partial(\beta_0,\beta_1).$$

From (5), we obtain

$$\begin{aligned}0 < \delta(\partial(\beta_1,\Im\beta_1)) &\leq \Lambda(\delta(\partial(\beta_0,\beta_1))) - \Upsilon(\partial(\beta_0,\beta_1))\\ &< \Lambda(\delta(\partial(\beta_0,\beta_1))).\end{aligned} \tag{6}$$

For $\varrho > 1$ by Lemma II.1, there exists $\beta_2 \in \Im\beta_1$ s.t.

$$0 < \delta(\partial(\beta_1,\beta_2)) < \varrho\delta(\partial(\beta_1,\Im\beta_1)). \tag{7}$$

From (6) and (7), we get

$$0 < \delta(\partial(\beta_1,\beta_2)) < \varrho\Lambda(\delta(\partial(\beta_0,\beta_1))). \tag{8}$$

By applying Λ in (8), we have

$$0 < \Lambda(\delta(\partial(\beta_1,\beta_2))) < \Lambda(\varrho\Lambda(\delta(\partial(\beta_0,\beta_1)))). \tag{9}$$

Set $\varrho_1 = \dfrac{\Lambda(\varrho\Lambda(\delta(\partial(\beta_0,\beta_1))))}{\Lambda(\delta(\partial(\beta_1,\beta_2)))}$.

Then $\varrho_1 \geq 1$. From the Definition 11, condition (1) and $\beta_2 \in \Im\beta_1$, we have

$$\begin{aligned}\varphi(\beta_1) \geq 1 &\implies \psi(\Im\beta_1) = \psi(\beta_2) \geq 1,\\ \psi(\beta_1) \geq 1 &\implies \varphi(\Im\beta_1) = \varphi(\beta_2) \geq 1.\end{aligned}$$

So, $\varphi(\beta_1) \geq 1$, and $\psi(\beta_2) \geq 1$. Equivalently, $\varphi(\beta_1)\psi(\beta_2) \geq 1$. If $\beta_2 \in \Im\beta_2$, then $\beta_2 \in \mathcal{F}(\Im)$. So, we assume that $\beta_2 \notin \Im\beta_2$. From (1), we conclude that

$$\begin{aligned}0 &< \delta(\partial(\beta_2,\Im\beta_2))\\ &\leq \delta(\mathcal{H}(\Im\beta_1,\Im\beta_2))\\ &\leq \Lambda(\delta(\mathcal{M}(\beta_1,\beta_2))) - \Upsilon(\mathcal{M}(\beta_1,\beta_2)),\end{aligned} \tag{10}$$

where

$$\begin{aligned}\mathcal{M}(\beta_1,\beta_2) &= \max\Big\{\partial(\beta_1,\beta_2),\partial(\beta_1,\Im\beta_1),\partial(\beta_2,\Im\beta_2),\\ &\qquad \frac{1}{2}[\partial(\beta_1,\Im\beta_2)+\partial(\beta_2,\Im\beta_1)]\Big\}\\ &= \max\Big\{\partial(\beta_1,\beta_2),\partial(\beta_1,\beta_2),\partial(\beta_2,\Im\beta_2),\\ &\qquad \frac{1}{2}\partial(\beta_1,\Im\beta_2)\Big\}\\ &= \max\Big\{\partial(\beta_1,\beta_2),\partial(\beta_2,\Im\beta_2),\\ &\qquad \frac{1}{2}[\partial(\beta_1,\beta_2)+\partial(\beta_2,\Im\beta_2)]\Big\}\\ &= \max\Big\{\partial(\beta_1,\beta_2),\partial(\beta_2,\Im\beta_2)\Big\}.\end{aligned}$$

If $\mathcal{M}(\beta_1,\beta_2)=\partial(\beta_2,\Im\beta_2)$ and by using properties of Υ, we have

$$\begin{aligned}0<\delta(\partial(\beta_2,\Im\beta_2)) &\le \Lambda(\delta(\partial(\beta_2,\Im\beta_2)))-\Upsilon(\partial(\beta_2,\Im\beta_2))\\ &< \Lambda(\delta(\partial(\beta_2,\Im\beta_2))),\end{aligned}$$

which is a contradiction. Thus, if $\mathcal{M}(\beta_1,\beta_2)=\partial(\beta_1,\beta_2)$, we get

$$\begin{aligned}0&<\delta(\partial(\beta_2,\Im\beta_2))\\ &\le \delta(\mathcal{H}(\Im\beta_1,\Im\beta_2))\\ &\le \Lambda(\delta(\partial(\beta_1,\beta_2)))-\Upsilon(\partial(\beta_1,\beta_2))\\ &< \Lambda(\delta(\partial(\beta_1,\beta_2))). \qquad (11)\end{aligned}$$

For $\varrho_1>1$ by Lemma II.1, then there exists $\beta_3\in\Im\beta_2$ s.t.

$$0<\delta(\partial(\beta_2,\beta_3))<\varrho_1\delta(\partial(\beta_2,\Im\beta_2)). \qquad (12)$$

From (11) and (12), we obtain

$$\begin{aligned}0<\partial(\beta_2,\beta_3) &< \varrho_1\Lambda(\delta(\partial(\beta_2,\Im\beta_2)))\\ &= \Lambda(\varrho\Lambda(\delta(\partial(\beta_0,\beta_1)))). \qquad (13)\end{aligned}$$

By applying Λ in (13), we have

$$0<\Lambda(\delta(\partial(\beta_2,\beta_3)))<\Lambda^2(\varrho\Lambda(\delta(\partial(\beta_0,\beta_1)))). \qquad (14)$$

By continuing this procedure and since $\Im$ is $\perp$-preserving, form the O-sequence $\{\beta_\varepsilon\}\in \pounds$ s.t. $\beta_{\varepsilon+1}\neq\beta_\varepsilon\in\Im\beta_\varepsilon$. Since $\Im$ is a C.O. (φ,ψ)-admissible mapping, we obtain

$$\varphi(\beta_\varepsilon)\ge 1 \textit{ and } \psi(\beta_\varepsilon)\ge 1,\ \forall\ \varepsilon\in\mathbb{N}.$$

This implies that

$$\varphi(\beta_\varepsilon)\psi(\beta_{\varepsilon+1})\ge 1,$$

and

$$0<\delta(\partial(\beta_\varepsilon,\beta_{\varepsilon+1}))\Lambda^{\varepsilon}(\varrho\Lambda(\delta(\partial(\beta_0,\beta_1)))),\ \forall\ \mathbb{N}\cup\{0\}.$$

Let $\mathfrak{o},\varepsilon\in\mathbb{N}$ s.t. $\mathfrak{o}>\varepsilon$. By the triangle inequality, we have

$$\begin{aligned}\delta(\partial(\beta_{\mathfrak{o}},\beta_\varepsilon)) &\le \sum_{\ell=\varepsilon}^{\mathfrak{o}-1}\delta(\partial(\beta_\ell,\beta_{\ell+1}))\\ &\le \sum_{\ell=\varepsilon}^{\mathfrak{o}-1}\Lambda^{\ell-1}(\varrho\Lambda(\delta(\partial(\beta_0,\beta_1)))).\end{aligned}$$

From the Λ properties, this implies that $\lim_{\varepsilon,\mathfrak{o}\to\infty}\delta(\partial(\beta_{\mathfrak{o}},\beta_\varepsilon))=0$ and from $\perp$-continuity of δ, we obtain $\lim_{\varepsilon,\mathfrak{o}\to\infty}\partial(\beta_{\mathfrak{o}},\beta_\varepsilon)=0$. Thus $\{\beta_\varepsilon\}$ is an O-Cauchy sequence in $(\pounds,\perp)$ s.t. $\beta_\varepsilon\to\beta$ as $\varepsilon\to\infty$, $\forall\ \ \varepsilon\in\mathbb{N}$.
For all $\varepsilon\in\mathbb{N}$, assume that axiom (2) hold. Hence $\varphi(\beta_\varepsilon)\psi(\zeta)\ge 1$. From (1), we have

$$\delta(\mathcal{H}(\Im\beta_\varepsilon,\Im\zeta))\le\Lambda(\delta(\mathcal{M}(\beta_\varepsilon,\zeta)))-\Upsilon(\mathcal{M}(\beta_\varepsilon,\zeta)), \qquad (15)$$

for all $\varepsilon\in\mathbb{N}$. Where

$$\begin{aligned}\max\Big\{&\partial(\beta_\varepsilon,\zeta),\partial(\Im\beta_\varepsilon,\beta_\varepsilon),\partial(\zeta,\Im\zeta),\\ &\frac{1}{2}[\partial(\beta_\varepsilon,\Im\zeta)+\partial(\zeta,\Im\beta_\varepsilon)]\Big\}.\end{aligned}$$

Assume that $\partial(\zeta,\Im\zeta)\neq 0$. Let $\epsilon=\frac{\partial(\zeta,\Im\zeta)}{2}$.
Since $\beta_\varepsilon\to\zeta$ as $\varepsilon\to\infty$, we can find $\varsigma_1\in\mathbb{N}$ s.t.

$$\partial(\zeta,\beta_\varepsilon)<\frac{\partial(\zeta,\Im\zeta)}{2},\ \forall\ \varepsilon\ge\varsigma_1. \qquad (16)$$

Also, we get

$$\begin{aligned}\partial(\beta_\varepsilon,\Im\zeta) &\le \partial(\beta_\varepsilon,\zeta)+\partial(\zeta,\Im\zeta)\\ &< \frac{\partial(\zeta,\Im\zeta)}{2}+\partial(\zeta,\Im\zeta)\\ &= \frac{3\partial(\zeta,\Im\zeta)}{2},\ \forall\ \varepsilon\ge\varsigma_2. \qquad (17)\end{aligned}$$

Furthermore, we obtain

$$\partial(\beta_\varepsilon,\Im\beta_\varepsilon)\le\partial(\beta_\varepsilon,\beta_{\varepsilon+1})<\frac{\partial(\zeta,\Im\zeta)}{2},\ \forall\ \varepsilon\ge\varsigma_3. \qquad (18)$$

Using (16) – (18), we have

$$\begin{aligned}\mathcal{M}(\beta_\varepsilon,\zeta) &= \max\Big\{\partial(\beta_\varepsilon,\zeta),\partial(\Im\beta_\varepsilon,\beta_\varepsilon),\partial(\zeta,\Im\zeta),\\ &\qquad \frac{1}{2}[\partial(\beta_\varepsilon,\Im\zeta)+\partial(\zeta,\Im\beta_\varepsilon)]\Big\}\\ &= \partial(\zeta,\Im\zeta),\ \forall\ \varepsilon\ge\varsigma=\{\varsigma_1,\varsigma_2,\varsigma_3\}. \qquad (19)\end{aligned}$$

For $\varepsilon\ge\varsigma$, from triangle inequality and equation (15) and the hypothesis of Υ, we obtain

$$\begin{aligned}\delta(\partial(\zeta,\Im\zeta)) &\le \delta(\partial(\zeta,\beta_{\varepsilon+1}))+\delta(\mathcal{H}(\Im\beta_\varepsilon,\Im\zeta))\\ &\le \delta(\partial(\zeta,\beta_{\varepsilon+1}))+\Lambda(\delta(\mathcal{M}(\beta_\varepsilon,\zeta)))\\ &\qquad -\Upsilon(\mathcal{M}(\beta_\varepsilon,\zeta))\\ &\le \delta(\partial(\zeta,\beta_{\varepsilon+1}))+\Lambda(\delta(\partial(\zeta,\Im\zeta)))\\ &\qquad -\Upsilon(\partial(\zeta,\Im\zeta))\\ &\le \delta(\partial(\zeta,\beta_{\varepsilon+1}))+\Lambda(\delta(\partial(\zeta,\Im\zeta))),\end{aligned}$$

taking $\varepsilon\to\infty$ in the above inequality, we get

$$\delta(\partial(\zeta,\Im\zeta))\le\Lambda(\delta(\partial(\zeta,\Im\zeta)))<\delta(\partial(\zeta,\Im\zeta)),$$

which is a contradiction. Thus, we have $\partial(\zeta,\Im\zeta)=0$, that is, $\zeta\in\Im\zeta$. Hence ζ is a fixed point of $\Im$.
To prove the uniqueness property of fixed point.
Let $\zeta^*\in\pounds$ be another fixed point of $\Im$. Then, we have $\Im^{\varepsilon}(\zeta^*)=\zeta^*$ and $\Im^{\varepsilon}(\zeta)=\zeta$, $\forall\ \varepsilon\in\mathbb{N}$. By the choice of β_0 in the first part of proof, we have

$$[\beta_0\perp\zeta \textit{ and } \beta_0\perp\zeta^*] \textit{ or } [\zeta\perp\beta_0 \textit{ and } \zeta^*\perp\beta_0].$$

Since $\Im$ is $\perp$-preserving, we have

$$[\Im^{\varepsilon}(\beta_0)\perp\Im^{\varepsilon}(\zeta) \textit{ and } \Im^{\varepsilon}(\beta_0)\perp\Im^{\varepsilon}(\zeta^*)],$$

or

$$[\Im^{\varepsilon}(\zeta)\perp\Im^{\varepsilon}(\beta_0) \text{ and } \Im^{\varepsilon}(\zeta^*)\perp\Im^{\varepsilon}(\beta_0)],\ \forall\ \varepsilon\in\mathbb{N}.$$

Therefore, from (15), we have

$$\begin{aligned}\delta(\partial(\zeta,\zeta^*)) &\le \delta(\mathcal{H}(\Im^{\varepsilon}(\zeta),\Im^{\varepsilon}(\zeta^*)))\\ &\le \Lambda(\delta(\mathcal{M}(\zeta,\zeta^*))) - \Upsilon(\mathcal{M}(\zeta,\zeta^*))\\ &\le \Lambda(\delta(\partial(\zeta,\zeta^*))) - \Upsilon(\partial(\zeta,\zeta^*))\\ &\le \Lambda(\delta(\partial(\zeta,\zeta^*)))\\ &< \delta(\partial(\zeta,\zeta^*)).\end{aligned}$$

Hence, $\delta(\partial(\zeta,\zeta^*)) \le \delta(\mathcal{H}(\Im^{\varepsilon}(\zeta),\Im^{\varepsilon}(\zeta^*))) < \delta(\partial(\zeta,\zeta^*))$, which is a contradiction, unless $\partial(\zeta,\zeta^*)=0 \implies \zeta=\zeta^*$. Therefore, $\Im$ has a UFP. ■

Corollary 1. *Let $(\pounds,\perp,\partial)$ be an orthogonal CMS and $\Im:\pounds\to CL(\pounds)$. There exists four functions $\varphi,\psi:\pounds\to\mathcal{R}_0^+,\Lambda\in\Xi,\delta\in\Delta$ and $\Upsilon\in\Pi$ s.t.*

$$\beta,\zeta\in\pounds \text{ with } \beta\perp\zeta, \mathcal{H}(\Im\beta,\Im\zeta)>0,$$

$$\varphi(\beta)\psi(\zeta)\delta(\mathcal{H}(\Im\beta,\Im\zeta)) \le \Lambda(\delta(\mathcal{M}(\beta,\zeta))) - \Upsilon(\mathcal{M}(\beta,\zeta)).$$

Assume that the following postulations hold:

1) $\exists\ \beta_0\in\pounds,\ \beta_1\in\Im\beta_0$ *s.t.*

$$\begin{aligned}\varphi(\beta_0)\ge 1 &\implies \psi(\Im\beta_0)=\psi(\beta_1)\ge 1,\\ \psi(\beta_0)\ge 1 &\implies \varphi(\Im\beta_0)=\varphi(\beta_1)\ge 1,\end{aligned}$$

2) *if $\{\beta_\varepsilon\}$ is an O-sequence in $\pounds$ with $\beta_\varepsilon\to\beta\in\pounds$ as $\varepsilon\to\infty$ and $\psi(\beta_\varepsilon)\ge 1$, $\forall\ \varepsilon\in\mathbb{N}$, then $\psi(\beta)\ge 1$,*
3) *$\perp$-continuous,*
4) *$\perp$-preserving,*

then $\Im$ has a UFP.

Proof: Let $\varphi(\beta)\psi(\zeta)\ge 1$ for every $\beta,\zeta\in\pounds$. Then by equation (4), we have:

$$\begin{aligned}\delta(\mathcal{H}(\Im\beta,\Im\zeta)) &\le \varphi(\beta)\psi(\zeta)\delta(\mathcal{H}(\Im\beta,\Im\zeta))\\ &\le \Lambda(\delta(\mathcal{M}(\beta,\zeta))) - \Upsilon(\mathcal{M}(\beta,\zeta)),\end{aligned}$$

this provides that $\Im$ C.O. $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-admissible multivalued mapping. Hence, So, by the proof of Theorem 2, we reach the required result. ■

If we let $\Lambda(\iota)=\delta(\iota)=\iota$ and $\Upsilon(\iota)=(1-\mathfrak{h})\iota$ in Theorem 2, we derive the following corollary.

Corollary 2. *Let $(\pounds,\perp,\partial)$ be an O-CMS and $\Im:\pounds\to CL(\pounds)$. There exists four functions $\varphi,\psi:\pounds\to\mathcal{R}_0^+,\Lambda\in\Xi,\delta\in\Delta$ and $\Upsilon\in\Pi$ s.t.*

$$\beta,\zeta\in\pounds \text{ with } \beta\perp\zeta, \mathcal{H}(\Im\beta,\Im\zeta)>0,$$

$$\varphi(\beta)\psi(\zeta)\ge 1 \implies \delta(\mathcal{H}(\Im\beta,\Im\zeta))\le\mathfrak{h}\mathcal{M}(\beta,\zeta),$$

for $\mathfrak{h}\in[0,1)$. Assume that the below axioms true:

1) $\exists\ \beta_0\in\pounds,\ \beta_1\in\Im\beta_0$ *s.t.*

$$\begin{aligned}\varphi(\beta_0)\ge 1 &\implies \psi(\Im\beta_0)=\psi(\beta_1)\ge 1,\\ \psi(\beta_0)\ge 1 &\implies \varphi(\Im\beta_0)=\varphi(\beta_1)\ge 1,\end{aligned}$$

2) *if $\{\beta_\varepsilon\}$ is an O-sequence in $\pounds$ with $\beta_\varepsilon\to\beta\in\pounds$ as $\varepsilon\to\infty$ and $\psi(\beta_\varepsilon)\ge 1$, $\forall\ \varepsilon\in\mathbb{N}$, then $\psi(\beta)\ge 1$,*
3) *$\perp$-continuous,*
4) *$\perp$-preserving,*

then $\Im$ has a UFP.

Theorem 3. *Let $(\pounds,\perp,\partial)$ be an orthogonal CMS and $\Im:\pounds\to CL(\pounds)$ be a C.O.$(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-A.M.M of type B. Suppose that the following assumptions hold:*

1) *for each $\beta_0\in\pounds$, $\beta_1\in\Im\beta_0$ s.t.*

$$\begin{aligned}\varphi(\beta_0)\ge 1 &\implies \psi(\Im\beta_0)=\psi(\beta_1)\ge 1,\\ \psi(\beta_0)\ge 1 &\implies \varphi(\Im\beta_0)=\varphi(\beta_1)\ge 1,\end{aligned}$$

2) *if $\{\beta_\varepsilon\}$ is an O-sequence in $\pounds$ with $\beta_\varepsilon\to\beta\in\pounds$ as $\varepsilon\to\infty$ and $\psi(\beta_\varepsilon)\ge 1$, $\forall\ \ \varepsilon\in\mathbb{N}$, then $\psi(\beta)\ge 1$,*
3) *$\perp$-continuous,*
4) *$\perp$-preserving,*

then $\Im$ has a UFP.

Proof: By similar way in Theorem 2, from β_0 and $\beta_1\in\Im\beta_0$ in condition (1), we have

$$\begin{aligned}\varphi(\beta_0)\ge 1 &\implies \psi(\Im\beta_0)=\psi(\beta_1)\ge 1,\\ \psi(\beta_0)\ge 1 &\implies \varphi(\Im\beta_0)=\varphi(\beta_1)\ge 1.\end{aligned}$$

Therefore, $\varphi(\beta_0)\ge 1$ and $\psi(\beta_1)\ge 1$, equivalently, $\varphi(\beta_0)\psi(\beta_1)\ge 1$. If $\beta_0=\beta_1$, we taking $\beta_1\in\mathcal{F}(\Im)$ and so the proof is obvious. Now, suppose that $\beta_0\ne\beta_1$ and $\beta_1\in\Im\beta_1$ implies $\partial(\beta_1,\Im\beta_1)>0$. From (1), we obtain

$$\begin{aligned}0 &< \delta(\partial(\beta_1,\Im\beta_1))\\ &\le \delta(\mathcal{H}(\Im\beta_0,\Im\beta_1))\\ &\le \Lambda(\delta(\mathcal{P}(\beta_0,\beta_1))) - \Upsilon(\mathcal{P}(\beta_0,\beta_1)),\end{aligned}\tag{20}$$

where

$$\begin{aligned}\mathcal{P}(\beta_0,\beta_1) &= \max\left\{\partial(\beta_0,\beta_1),\frac{[1+\partial(\beta_0,\Im\beta_0)\partial(\beta_1,\Im\beta_1)]}{\partial(\beta_0,\beta_1)+1}\right\}\\ &= \max\left\{\partial(\beta_0,\beta_1),\frac{[1+\partial(\beta_0,\beta_1)\partial(\beta_1,\Im\beta_1)]}{\partial(\beta_0,\beta_1)+1}\right\}\\ &= \max\left\{\partial(\beta_0,\beta_1),\partial(\beta_1,\Im\beta_1)\right\}.\end{aligned}$$

We will use the same procedure as in Theorem 2 to complete the proof after the above pause. ■

Definition 15. *Let $(\pounds,\perp,\partial)$ be an O-CMS and $\Im:\pounds\to CL(\pounds)$. $\Im$ is called an orthogonal $(\varphi,\psi-\Lambda,\delta,\Upsilon)$-Meir-Keeler-Khan multivalued mapping if there exists $\Lambda\in\Xi,\delta\in\Delta$ and $\Upsilon\in\Pi$ and $\varphi,\psi:[0,\infty)\to\mathcal{R}_0^+$ s.t.*

$$\begin{aligned}&\mathcal{H}(\Im\beta,\Im\zeta)>0,[\varphi(\beta)\psi(\zeta)\ge 1 \implies\\ &\delta(\mathcal{H}(\Im\beta,\Im\zeta))\le\Lambda(\delta(\mathcal{N}(\beta,\zeta)))-\Upsilon(\mathcal{N}(\beta,\zeta))],\end{aligned}\tag{21}$$

where

$$\mathcal{N}(\beta,\zeta)=\frac{\partial(\beta,\Im\beta)\partial(\beta,\Im\zeta)+\partial(\zeta,\Im\zeta)\partial(\zeta,\Im\beta)}{\partial(\beta,\Im\zeta)+\partial(\zeta,\Im\beta)},$$

$\forall\ \beta,\zeta\in\pounds$ with $\beta\perp\zeta$.

Now, we will state our results in this section.

Theorem 4. *Let $\Im:\pounds\to CL(\pounds)$ be a C.O. $(\varphi,\psi)-(\Lambda,\delta,\Upsilon)$-Meir-Keeler-Khan multivalued mapping on OMS $(\pounds,\perp,\partial)$. Assume that the following axioms hold:*

(1) *there exists $\beta_0\in\pounds$ and $\beta_1\in\Im\beta_0$ s.t.*

$$\begin{aligned}\varphi(\beta_0)\ge 1 &\implies \psi(\Im\beta_0)=\beta(\beta_1)\ge 1,\\ \psi(\beta_0)\ge 1 &\implies \varphi(\Im\beta_0)=\varphi(\beta_1)\ge 1,\end{aligned}$$

(2) $\perp$-*continuous,*
(3) $\perp$-*preserving,*
then $\Im$ *has a fixed point.*

Proof: Since $(\pounds, \perp)$ is an O-set,

$$\exists\ \beta_0 \in \pounds\ (\forall\ \beta \in \pounds,\ \beta \perp \beta_0)\ \vee\ (\forall\ \beta \in \pounds,\ \beta_0 \perp \beta).$$

It follows that $\beta_0 \perp \Im(\beta_0)$ or $\Im(\beta_0) \perp \beta_0$.
Let

$$\beta_1 = \Im(\beta_0); \beta_2 = \Im(\beta_1); \ldots; \beta_{\varepsilon+1} = \Im(\beta_\varepsilon),\ \forall\ \varepsilon \in \mathbb{N}.$$

By starting from β_0 and $\beta_1 \in \Im\beta_0$ with $\beta_0 \perp \beta_1$ in axioms (1), we have

$$\varphi(\beta_0) \geq 1 \implies \psi(\Im\beta_0) = \psi(\beta_1) \geq 1,$$
$$\psi(\beta_0) \geq 1 \implies \varphi(\Im\beta_0) = \varphi(\beta_1) \geq 1.$$

Therefore, $\varphi(\beta_0) \geq 1$ and $\psi(\beta_1) \geq 1$, equivalently, $\varphi(\beta_0)\psi(\beta_1) \geq 1$. If $\beta_0 = \beta_1$, we conclude that $\beta_1 \in \mathcal{F}(\Im)$ and so the proof is completed. Now, taking $\beta_0 \neq \beta_1$ and $\beta_1 \notin \Im\beta_1$. From (21), we have $\beta_0 \in \pounds$ and $\beta_1 \in \Im\beta_0$ s.t.

$$\begin{aligned} 0 < \partial(\beta_1, \Im\beta_1) &\leq \delta(\mathcal{H}(\Im\beta_0, \Im\beta_1)) \\ &\leq \Lambda(\delta(\mathcal{N}(\beta_0, \beta_1))) - \Upsilon(\mathcal{N}(\beta_0, \beta_1)), \end{aligned} \tag{22}$$

where

$$\begin{aligned} &\mathcal{N}(\beta_0, \beta_1) \\ &= \frac{\partial(\beta_0, \Im\beta_0)\partial(\beta_0, \Im\beta_1) + \partial(\beta_1, \Im\beta_1)\partial(\beta_1, \Im\beta_0)}{\partial(\beta_0, \Im\beta_1) + \partial(\beta_1, \Im\beta_0)} \\ &= \partial(\beta_0, \beta_1). \end{aligned} \tag{23}$$

From (22) and (23) and using the properties of Υ, we get

$$\begin{aligned} 0 &< \delta(\partial(\beta_1, \Im\beta_1)) \\ &\leq \Lambda(\delta(\partial(\beta_0, \beta_1))) - \Upsilon(\partial(\beta_0, \beta_1)) \\ &< \Lambda(\delta(\partial(\beta_0, \beta_1))). \end{aligned} \tag{24}$$

For $\sigma > 1$, by Lemma II.1, there exists $\beta_2 \in \Im\beta_1$ s.t.

$$0 < \delta(\partial(\beta_1, \beta_2)) < \sigma\delta(\partial(\beta_1, \Im\beta_1)). \tag{25}$$

From (24) and (25), we get

$$0 < \delta(\partial(\beta_1, \beta_2)) < \Lambda(\sigma\Lambda(\delta(\partial(\beta_0, \beta_1)))). \tag{26}$$

Since $\Im$ is a cyclic (φ, ψ)-admissible mapping, from condition (1) and $\beta_2 \in \Im\beta_2$, we have

$$\varphi(\beta_1) \geq 1 \implies \psi(\Im\beta_1) = \psi(\beta_2) \geq 1,$$
$$\psi(\beta_1) \geq 1 \implies \varphi(\Im\beta_1) = \varphi(\beta_2) \geq 1.$$

So, $\varphi(\beta_1) \geq 1$ and $\psi(\beta_2) \geq 1$.
Equivalently, $\varphi(\beta_1)\psi(\beta_2) \geq 1$. If $\beta_2 \in \Im\beta_2$, then $\beta_2 \in \mathcal{F}(\Im)$. So, we assume that $\beta_2 \notin \Im\beta_2$, that is $\partial(\beta_2, \Im\beta_2) > 0$. From (21), we deduce

$$\begin{aligned} 0 < \delta(\partial(\beta_2, \Im\beta_2)) &\leq \delta(\mathcal{H}(\Im\beta_1, \Im\beta_2)) \\ &\leq \Lambda(\delta(\mathcal{N}(\beta_1, \beta_2))) - \Upsilon(\mathcal{N}(\beta_1, \beta_2)), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &\mathcal{N}(\beta_1, \beta_2) \\ &= \frac{\partial(\beta_1, \Im\beta_1)\partial(\beta_1, \Im\beta_2) + \partial(\beta_2, \Im\beta_2)\partial(\beta_2, \Im\beta_1)}{\partial(\beta_1, \Im\beta_2) + \partial(\beta_2, \Im\beta_1)} \\ &= \partial(\beta_1, \beta_2). \end{aligned} \tag{28}$$

Using properties of Υ, we have

$$\begin{aligned} 0 < \delta(\partial(\beta_2, \Im\beta_2)) &\leq \delta(\mathcal{H}(\Im\beta_1, \Im\beta_2)) \\ &< \Lambda(\delta(\partial(\beta_1, \beta_2))). \end{aligned} \tag{29}$$

For $\sigma_1 > 1$ by Lemma II.1, there exists $\beta_3 \in \Im\beta_2$ s.t.

$$0 < \delta(\partial(\beta_2, \beta_3)) < \sigma_1\delta(\partial(\beta_2, \Im\beta_2)). \tag{30}$$

From (29) and (30), we obtain

$$0 < \delta(\partial(\beta_1, \beta_2)) < \Lambda^2(\sigma\Lambda(\delta(\partial(\beta_0, \beta_1)))). \tag{31}$$

By continuing in this way, we construct the O-sequence $\{\beta_\varepsilon\} \subset \pounds$ s.t. $\beta_{\varepsilon+1} \neq \beta_\varepsilon \in \Im\beta_\varepsilon$, again, since $\Im$ is a C.O. (φ, ψ)-admissible mapping, we have

$$\varphi(\beta_\varepsilon) \geq 1 \text{ and } \psi(\beta_\varepsilon) \geq 1,\ \forall\ \varepsilon \in \mathbb{N}.$$

This implies that

$$\varphi(\beta_\varepsilon)\psi(\beta_{\varepsilon+1}) \geq 1,$$

$$0 < \delta(\partial(\beta_\varepsilon, \beta_{\varepsilon+1})) < \Lambda^\varepsilon(\varrho\Lambda(\delta(\partial(\beta_0, \beta_1)))),\ \forall\ \mathbb{N} \cup \{0\}. \tag{32}$$

Let $\mathfrak{o}, \varepsilon \in \mathbb{N}$ s.t. $\mathfrak{o} > \varepsilon$. By the triangle inequality, we get

$$\begin{aligned} \delta(\partial(\beta_\mathfrak{o}, \beta_\varepsilon)) &\leq \sum_{\ell=\varepsilon}^{\mathfrak{o}-1} \delta(\partial(\beta_\ell, \beta_{\ell+1})) \\ &\leq \sum_{\ell=\varepsilon}^{\mathfrak{o}-1} \Lambda^{\ell-1}(\varrho\Lambda(\delta(\partial(\beta_0, \beta_1)))). \end{aligned} \tag{33}$$

Since $\Lambda \in \Xi$ and δ is $\perp$-continuous, we have

$$\lim_{\varepsilon, \mathfrak{o} \to \infty} \partial(\beta_\mathfrak{o}, \beta_\varepsilon) = 0.$$

Thus, $\{\beta_\varepsilon\}$ is O-Cauchy sequence in $(\pounds, \perp, \partial)$. By the O-completeness of $(\pounds, \perp, \partial)$, there exists $\beta^* \in \pounds$ s.t. $\beta_\varepsilon \to \beta^*$ as $\varepsilon \to \infty$. Since $\Im$ is $\perp$-continuous, we get

$$\partial(\beta^*, \Im\beta^*) = \lim_{\varepsilon \to \infty} \partial(\beta_{\varepsilon+1}, \Im\beta^*) \leq \lim_{\varepsilon \to \infty} \mathcal{H}(\Im\beta_\varepsilon, \Im\beta^*) = 0.$$

Therefore, we have $\beta^* \in \Im\beta^*$.
To prove the uniqueness property of fixed point. Let $\zeta^* \in \pounds$ be another fixed point of $\Im$. Then, we have $\Im^\varepsilon(\zeta^*) = \zeta^*$ and $\Im^\varepsilon(\zeta) = \zeta,\ \forall\ \ \varepsilon \in \mathbb{N}$. By the choice of β_0 in the first part of proof, we have

$$[\beta_0 \perp \zeta \textit{ and } \beta_0 \perp \zeta^*] \textit{ or } [\zeta \perp \beta_0 \textit{ and } \zeta^* \perp \beta_0].$$

Since $\Im$ is $\perp$-preserving, we have

$$[\Im^\varepsilon(\beta_0) \perp \Im^\varepsilon(\zeta) \textit{ and } \Im^\varepsilon(\beta_0) \perp \Im^\varepsilon(\zeta^*)],$$

or

$$[\Im^\varepsilon(\zeta) \perp \Im^\varepsilon(\beta_0) \textit{ and } \Im^\varepsilon(\zeta^*) \perp \Im^\varepsilon(\beta_0)],\ \forall\ \varepsilon \in \mathbb{N}.$$

Therefore, from (15), we have

$$\begin{aligned} \delta(\partial(\zeta, \zeta^*)) &\leq \delta(\mathcal{H}(\Im^\varepsilon(\zeta), \Im^\varepsilon(\zeta^*))) \\ &\leq \Lambda(\delta(\mathcal{M}(\zeta, \zeta^*))) - \Upsilon(\mathcal{M}(\zeta, \zeta^*)) \\ &\leq \Lambda(\delta(\partial(\zeta, \zeta^*))) - \Upsilon(\partial(\zeta, \zeta^*)) \\ &\leq \Lambda(\delta(\partial(\zeta, \zeta^*))) \\ &< \delta(\partial(\zeta, \zeta^*)). \end{aligned}$$

Hence, $\delta(\partial(\zeta,\zeta^*)) \leq \delta(\mathcal{H}(\Im^{\varepsilon}(\zeta),\Im^{\varepsilon}(\zeta^*))) < \delta(\partial(\zeta,\zeta^*))$, which is a contradiction, unless $\partial(\zeta,\zeta^*) = 0 \implies \zeta = \zeta^*$. Therefore, $\Im$ has a UFP. ■

Example 8. *Let $\pounds = \mathcal{R}_0^+$ and $\partial : \pounds \times \pounds \to \mathcal{R}_0^+$ be defined by $\partial(\beta,\zeta) = |\beta - \zeta|$ for all $\beta,\zeta \in \pounds$ with $\beta\perp\zeta$. Define a relation $\perp$ on $\pounds$ by*

$$\beta\perp\zeta \iff \beta\zeta \in \{\beta,\zeta\} \subseteq \pounds.$$

Thus, $(\pounds,\perp,\partial)$ is an OCMS.
Define $\Im : \pounds \to \pounds$ and $\varphi,\psi : \pounds \to \mathcal{R}_0^+$ by

$$\Im\beta = \begin{cases} \frac{\beta}{3}, & \text{if } \beta \in [0,1], \\ 3\beta, & \text{if } \beta \in (1,\infty). \end{cases}$$

$$\varphi(\beta) = \begin{cases} \frac{\beta+5}{2}, & \text{if } \beta \in [0,1], \\ 0, & \text{if } \beta \in (1,\infty). \end{cases}$$

$$\psi(\beta) = \begin{cases} \frac{\beta+8}{3}, & \text{if } \beta \in [0,1], \\ 0, & \text{if } \beta \in (1,\infty). \end{cases}$$

Now, we prove that the existence of fixed point of the Theorem 2 of $\Im$. Firstly, we want to show that $\Im$ is a C.O. (φ,ψ)-admissible mapping.
For $\beta,\zeta \in \pounds$, we have

$$\begin{aligned} \varphi(\beta) \geq 1 &\implies \beta \in [0,1] \\ &\implies \psi(\Im\beta) = \psi(\frac{\beta}{3}) = \frac{\beta+24}{9} \geq 1, \end{aligned}$$

and

$$\begin{aligned} \psi(\beta) \geq 1 &\implies \beta \in [0,1] \\ &\implies \varphi(\Im\beta) = \psi(\frac{\beta}{3}) = \frac{\beta+15}{6} \geq 1. \end{aligned}$$

Next, we prove that $\Im$ is a C.O. $(\varphi,\psi-\Lambda,\delta,\Upsilon)$-multivalued contractive mapping. Define functions $\Lambda,\Upsilon : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ by

$$\Lambda(\gamma) = \frac{8}{3}\gamma, \delta(\gamma) = \gamma \text{ and } \Upsilon(\gamma) = \frac{3}{11}\gamma, \forall\, \gamma \in \mathcal{R}_0^+.$$

If $\{\beta_\varepsilon\}$ is an O-sequence in $\pounds$ s.t.$\psi(\beta_\varepsilon) \geq 1$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to \infty$. So, $\beta_\varepsilon \in [0,1]$. Hence, i.e., $\psi(\beta) \geq 1$.
Let $\varphi(\beta)\psi(\beta) \geq 1$. Then $\beta,\zeta \in [0,1]$ and $\delta(\gamma) = \gamma$. Therefore, we have

$$\begin{aligned} \delta(\mathcal{H}(\Im\beta,\Im\zeta)) &= \frac{1}{3}|\beta-\zeta| \\ &\leq \frac{8}{11}|\beta-\zeta| \\ &= \frac{8}{11}\partial(\beta,\zeta) \\ &\leq \frac{8}{11}\mathcal{M}(\beta,\zeta) \\ &= \frac{8}{3}(\frac{3}{11}\mathcal{M}(\beta,\zeta)) - \Upsilon(\mathcal{M}(\beta,\zeta)) \\ &= \Lambda(\delta(\mathcal{M}(\beta,\zeta))) - \Upsilon(\mathcal{M}(\beta,\zeta)). \end{aligned}$$

So, all the axioms of Theorem 2 hold, which imply that $\Im$ has fixed point.

IV. Application to fractional differential equations

Let $\Delta = \{\mathfrak{w} \in \mathcal{C}_{0,1}, \mathfrak{w}(\mathfrak{c}) > 0\ \forall\ \mathfrak{c} \in [0,1]\}$.
Define an orthogonal relation $\perp$ on Δ as follows:

$$\mathfrak{q}\perp\varsigma \iff \mathfrak{q}(\mathfrak{c})\varsigma(\mathfrak{c}) \geq \mathfrak{q}(\mathfrak{c}) \text{ or } \mathfrak{q}(\mathfrak{c})\varsigma(\mathfrak{c}) \geq \varsigma(\mathfrak{c}),\ \forall\ \mathfrak{c} \in [0,1].$$

Let $\mathcal{C}_{0,1}$ be the space of continuous functions $\omega : [0,1] \to (-\infty,\infty)$. Define the metric $\partial : \mathcal{C}_{0,1} \times \mathcal{C}_{0,1} \to [0,\infty)$ by

$$\partial(\mathfrak{q},\varsigma) = ||\mathfrak{q}-\varsigma||_\infty = \max_{\mathfrak{c}\in[0,1]} |\mathfrak{q}(\mathfrak{c}) - \varsigma(\mathfrak{c})|,$$

$\forall\ \mathfrak{q},\varsigma \in \mathcal{C}_{0,1}$ with $\mathfrak{q}\perp\varsigma$. Then the space $(\mathcal{C}_{0,1},\perp,\partial)$ is an O-complete metric space. Let $\mathfrak{f} : \mathcal{C}_{0,1} \times \mathcal{C}_{0,1} \to [0,\infty)$ be a mapping defined by

$$\mathfrak{f}(\mathfrak{q},\varsigma) = \mathfrak{e}^{||\mathfrak{q}+\varsigma||_\infty},$$

for $\mathfrak{q},\varsigma \in \mathcal{C}_{0,1}$. Let $\mathcal{K}_1 : [0,1] \times (-\infty,\infty) \to (-\infty,\infty)$ be a $\perp$-continuous mapping. We will investigate the Caputo fractional differential equations

$$^{\mathcal{C}}\mathcal{D}^{\beta}\mathfrak{q}(\mathfrak{c}) = \mathcal{K}_1(\mathfrak{c},\mathfrak{q}(\mathfrak{c})) \tag{34}$$

with boundary conditions

$$\mathfrak{q}(0) = 0, \mathcal{I}\mathfrak{q}(1) = \mathfrak{q}^{'}(0).$$

Here $^{\mathcal{C}}\mathcal{D}^{\beta}$ denotes the CFD of order β defined by

$$^{\mathcal{C}}\mathcal{D}^{\beta}\mathcal{K}_1(\mathfrak{c}) = \frac{1}{\Gamma(\pi-\beta)}\int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\pi-\beta-1}\mathcal{K}_1^{\pi}(\eta)\partial\eta,$$

where $\pi - 1 < \beta < \pi$ and $\pi = [\beta]+1$, and $\mathcal{I}^{\beta}\mathcal{K}_1$ is given by

$$\mathcal{I}^{\beta}\mathcal{K}_1(\mathfrak{c}) = \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1}\mathcal{K}_1(\eta)\partial\eta, \text{ with } \beta > 0.$$

Then equation (34) can be modified to

$$\begin{aligned} \mathfrak{q}(\mathfrak{c}) &= \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1}\mathcal{K}_1(\eta,\mathfrak{q}(\eta))\partial\eta \\ &+ \frac{2\mathfrak{c}}{\Gamma(\beta)}\int_0^1\int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1}\mathcal{K}_1(\mathfrak{u},\mathfrak{q}(\mathfrak{u}))\partial\mathfrak{u}\partial\eta. \end{aligned}$$

Now, we show that $\mathbb{R}$ is $\perp$-preserving. For each $\mathfrak{q},\varsigma \in \mathcal{C}_{0,1}$ with $\mathfrak{q}\perp\varsigma$ and $\mathfrak{c} \in [0,1]$, we have

$$\begin{aligned} \mathbb{R}(\mathfrak{q}(\mathfrak{c})) &= \frac{1}{\Gamma(\beta)}\int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1}\mathcal{K}_1(\eta,\mathfrak{q}(\eta))\partial\eta \\ &+ \frac{2\mathfrak{c}}{\Gamma(\beta)}\int_0^1\int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1}\mathcal{K}_1(\mathfrak{u},\mathfrak{q}(\mathfrak{u}))\partial\mathfrak{u}\partial\eta \geq 1. \end{aligned}$$

Accordingly, we have $[\mathbb{R}(\mathfrak{q}(\mathfrak{c}))][\mathbb{R}(\varsigma(\mathfrak{c}))] \geq \mathbb{R}(\mathfrak{q}(\mathfrak{c}))$, and thus $\mathbb{R}(\mathfrak{q}(\mathfrak{c}))\perp\mathbb{R}(\varsigma(\mathfrak{c}))$. Then, $\mathbb{R}$ is $\perp$-preserving.

Theorem IV.1. *Equation* (34) *admits a solution in $\mathcal{C}_{0,1}$ provided that:*

(I) $\exists$ $\partial(\mathfrak{q},\varsigma) > 0$ *such that for all* $\mathfrak{q},\varsigma \in \mathcal{C}_{0,1}$ *with* $\mathfrak{q}\perp\varsigma$*, we have*

$$\begin{aligned} &\mathcal{K}_1(\eta,\mathfrak{q}(\eta)) - \mathcal{K}_1(\eta,\varsigma(\eta)) \\ &\qquad\leq \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}\Gamma(\beta+1)}{4\delta}|\mathfrak{q}(\eta)-\varsigma(\eta)| \\ &\qquad(\delta = \min\{\mathfrak{f}(\mathfrak{q},\varsigma)|\mathfrak{q},\varsigma \in \mathcal{C}_{0,1}\}), \end{aligned}$$

(II) $\exists\ \mathfrak{q}_0 \in \mathcal{C}_{0,1}$ *such that for all* $\mathfrak{c} \in [0,1]$, *we have*

$$\mathfrak{q}_0(\mathfrak{c}) \le \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \mathcal{K}_1(\eta, \mathfrak{q}_0(\eta)) \partial\eta + \frac{2\mathfrak{c}}{\Gamma(\beta)} \int_0^1 \int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1} \mathcal{K}_1(\mathfrak{u}, \mathfrak{q}_0(\mathfrak{u})) \partial\mathfrak{u}\partial\eta.$$

Proof: According to the newly introduced notations, we define the mapping $\mathbb{R} : \mathcal{C}_{0,1} \to \mathcal{C}_{0,1}$ by

$$\mathbb{R}(\mathfrak{q}(\mathfrak{c})) = \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \mathcal{K}_1(\eta, \mathfrak{q}(\eta)) \partial\eta + \frac{2\mathfrak{c}}{\Gamma(\beta)} \int_0^1 \int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1} \mathcal{K}_1(\mathfrak{u}, \mathfrak{q}(\mathfrak{u})) \partial\mathfrak{u}\partial\eta.$$

By (II) $\exists\ \mathfrak{q}_0 \in \mathcal{C}_{0,1}$ such that $\mathfrak{q}_\pi = \mathbb{R}^\pi(\mathfrak{q}_0)$. The $\perp$-continuity of the mapping $\mathcal{K}_1$ leads to the $\perp$-continuity of the mapping $\mathbb{R}$ on $\mathcal{C}_{0,1}$. It is easy to verify the assumptions of Theorem 2. Let us verify the contractive conditions of Theorem 2.

$$\begin{aligned}
&|\mathbb{R}(\mathfrak{q}(\mathfrak{c})) - \mathbb{R}(\varsigma(\mathfrak{c}))| \\
&= \Bigg| \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \mathcal{K}_1(\eta, \mathfrak{q}(\eta)) \partial\eta \\
&\quad + \frac{2\mathfrak{c}}{\Gamma(\beta)} \int_0^1 \int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1} \mathcal{K}_1(\mathfrak{u}, \mathfrak{q}(\mathfrak{u})) \partial\mathfrak{u}\partial\eta \\
&\quad - \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \mathcal{K}_1(\eta, \varsigma(\eta)) \partial\eta \\
&\quad - \frac{2\mathfrak{c}}{\Gamma(\beta)} \int_0^1 \int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1} \mathcal{K}_1(\mathfrak{u}, \varsigma(\mathfrak{u})) \partial\mathfrak{u}\partial\eta \Bigg| \\
&\le \Bigg| \bigg(\frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \mathcal{K}_1(\eta, \mathfrak{q}(\eta)) \\
&\quad - \frac{1}{\Gamma(\beta)} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \mathcal{K}_1(\eta, \varsigma(\eta)) \bigg) \partial\eta \Bigg| \\
&\quad + \Bigg| \int_0^1 \int_0^{\eta} \bigg(\frac{2}{\Gamma(\beta)} (\eta-\mathfrak{u})^{\beta-1} \mathcal{K}_1(\mathfrak{u}, \mathfrak{q}(\mathfrak{u})) \\
&\quad - \frac{2}{\Gamma(\beta)} (\eta-\mathfrak{u})^{\beta-1} \mathcal{K}_1(\mathfrak{u}, \varsigma(\mathfrak{u})) \bigg) \partial\mathfrak{u}\partial\eta \Bigg| \\
&\le \frac{1}{\Gamma(\beta)} \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}\Gamma(\beta+1)}{4\delta} \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} (\mathfrak{q}(\eta) - \varsigma(\eta)) \partial\eta \\
&\quad + \frac{2}{\Gamma(\beta)} \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}\Gamma(\beta+1)}{4\delta} \\
&\quad \int_0^1 \int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1} (\varsigma(\eta) - \mathfrak{q}(\eta)) \partial\mathfrak{u}\partial\eta \\
&\le \frac{1}{\Gamma(\beta)} \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}\Gamma(\beta+1)}{4\delta} \partial(\mathfrak{q},\varsigma) \int_0^{\mathfrak{c}} (\mathfrak{c}-\eta)^{\beta-1} \partial\eta \\
&\quad + \frac{2}{\Gamma(\beta)} \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}\Gamma(\beta)\Gamma(\beta+1)}{4\delta\Gamma(\mathfrak{s})\Gamma(\beta+1)} \partial(\varsigma,\mathfrak{q}) \\
&\quad \int_0^1 \int_0^{\eta} (\eta-\mathfrak{u})^{\beta-1} \partial\mathfrak{u}\partial\eta \\
&\le \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}\Gamma(\beta)\Gamma(\beta+1)}{4\delta\Gamma(\beta)\Gamma(\beta+1)} \partial(\mathfrak{q},\varsigma) \\
&\quad + 2\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)} \mathcal{B}(\beta+1,1) \frac{\Gamma(\beta)\Gamma(\beta+1)}{4\delta\Gamma(\beta)\Gamma(\beta+1)} \partial(\mathfrak{q},\varsigma) \\
&\le \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}}{4\delta} \partial(\mathfrak{q},\varsigma) + \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}}{2\delta} \partial(\mathfrak{q},\varsigma) \\
&< \frac{\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)}}{\delta} \partial(\mathfrak{q},\varsigma).
\end{aligned}$$

Define the mapping $\Lambda(\delta(\partial(\mathfrak{q},\varsigma))) = ln(\partial(\mathfrak{q},\varsigma))$ and $\Upsilon(\partial(\mathfrak{q},\varsigma)) = ln(\mathfrak{e}^{-\partial(\mathfrak{q},\varsigma)})$ for $\mathfrak{q},\varsigma \in \mathcal{C}_{0,1}$. Then the last inequality can be written as

$$\delta(\partial(\mathbb{R}(\mathfrak{q}), \mathbb{R}(\varsigma))) \le \Lambda(\delta(\partial(\mathfrak{q},\varsigma))) - \Upsilon(\partial(\mathfrak{q},\varsigma)).$$

By Theorem 2, the self-mapping $\mathbb{R}$ admits a fixed point, and hence equation (34) has a solution. ■

V. Conclusion

In this paper, we proved fixed point theorems on O-complete metric space using C.O. $(\varphi,\psi) - (\Lambda,\delta,\Upsilon)$-admissible multivalued mapping. Furthermore, we presented example to strengthen our main results. Also, we provided an application to the fractional differential equations.

References

[1] J. Nadler, "Multivalued contraction mappings", *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475-488, 1969.

[2] M. U. Ali, T. Kamran, and E. Karapinar, "(α,ψ,ξ)-contractive multivalued mappings", *Fixed Point Theory and Applications*, vol. 2014, no. 7, pp. 1-8, 2014.

[3] S. Alizadeh, F. Moradlou, and P. Salimi, "Some fixed point results for $(\alpha,\beta)-(\psi,\phi)$-contractive mappings", *Filomat*, vol. 28, no. 3, pp. 635-647, 2014.

[4] N. Hussain, J. Ahmad, and A. Azam, "Generalized fixed point theorems for multivalued (α,ψ)-contractive mappings", *Journal of Inequalities and Applications*, vol. 2014, no. 348, pp. 1-15, 2014.

[5] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for a (α,ϕ)-contractive type mappings", *Nonlinear Analysis*, vol. 75, no. 4, pp. 2154-2165, 2012.

[6] G. Nallaselli, A. J. Gnanaprakasam, G. Mani, and O. Ege, "Solving integral equations via admissible contraction mappings", *Filomat*, vol. 36, no. 14, pp. 4947-4961, 2022.

[7] H. Alsamir, M. S. Noorani, and W. Shatanawi, "On new fixed point theorems for three types of $(\alpha,\beta)-(\psi,\delta,\phi)$-multivalued contractive mappings in metric spaces", *Cogent Mathematics*, vol. 3, no. 1, pp. 1-13, 2016.

[8] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, "Fixed points for mappings satisfying cyclical contractive conditions", *Fixed Point Theory*, vol. 4, no. 1, pp. 79-89, 2003.

[9] M. Pcurar and I. A. Rus, "Fixed point theory for cyclic ϕ-contractions", *Nonlinear Analysis*, vol. 72, no. 3-4, pp. 1181-1187, 2010.

[10] W. Shatanawi and S. Manro, "Fixed point results for cyclic (ψ,ϕ,A,B)-contraction in partial metric space", *Fixed Point Theory and Applications*, vol. 2012, no. 165, pp. 1-13, 2012.

[11] A. Rabaiah, A. Tallafha, and W. Shatanawi, "Common fixed point results for mappings under nonlinear contraction of cyclic form in b-metric spaces", *Nonlinear Functional Analysis and Applications*, vol. 26, no. 2, pp. 289-301, 2021.

[12] M. E. Gordji, M. Ramezani, M. De La Sen, and Y. J. Cho, "On orthogonal sets and Banach fixed point theorem", *Fixed Point Theory*, vol. 18, no. 2, pp. 569-578, 2017.

[13] M. Eshaghi Gordji and H. Habibi, "Fixed point theory in generalized orthogonal metric space", *Journal of Linear and Topological Algebra*, vol. 6, no. 03, pp. 251-260, 2017.

[14] A. J. Gnanaprakasam, G. Nallaselli, Haq AU, G. Mani, I. A. Baloch, and K. Nonlaopon, "Common fixed-points technique for the existence of a solution to fractional integro-differential equations via orthogonal Branciari metric spaces", *Symmetry*, vol. 14, no. 9, pp. 1-23, 2022.

[15] C. Haitao and Shoujin Li, "A rapid iterative algorithm for solving split variational inclusion problems and fixed point problems", *IAENG International Journal of Applied Mathematics*, vol. 47, no. 3, pp. 248-254, 2017.

[16] P. Senthil Kumar and G. Arul Joseph, "Solution of the volterra integral equation in orthogonal partial ordered metric spaces", *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, pp. 613-621, 2023.

[17] D. Menaha, G. Arul Joseph, M. Gunaseelan, R. Rajagopalan, Khizar Hyatt Khan, Ola Ashour A. Abdelnaby, and S. Radenovic´ "Fixed point theorem on an orthogonal extended interpolative $\psi\mathcal{F}$-contraction", *AIMS Mathematics*, vol. 8, no. 7, pp. 16151-16164, 2023.

[18] N. Gunasekaran, A. S. Baazeem, G. Arul Joseph, M. Gunaseelan, Khalil Javed, E. Ameer, and N. Mlaiki, “Fixed point theorems via orthogonal convex contraction in orthogonal b-metric spaces and applications”, *Axioms*, vol. 12, no. 2, pp. 1-17, 2023.

[19] N. Gunasekaran, G. Arul Joseph, M. Gunaseelan, and E. Ozgur, “Solving integral equations via admissible contraction mappings”, *Filomat*, vol. 36, no. 14, pp. 4947-4961, 2022.

[20] M. Amin Abdellaoui, Z. Dahmani, and N. Bedjaoui, “Applications of fixed point theorems for coupled systems of fractional integro-differential equations involving convergent series”, *IAENG International Journal of Applied Mathematics*, vol. 45, no. 4, pp. 273-278, 2015.

[21] M. Arshad, A. Azam, and P. Vetro, “Common fixed point of generalized contractive type mappings in cone metric spaces”, *IAENG International Journal of Applied Mathematics*, vol. 41, no. 3, pp. 246-251, 2011.