Impact of Swapping Migration And Allee Effect in Single-species Model

Ninghua Chen, Lijuan Chen, Junyan Xu, and Chunchen Wu

Abstract—In this paper, we propose two kinds of singlespecies models with swapping migration and Allee effect. We aim to discuss how the Allee effect and swapping migration influence the above models' dynamical behaviours. The existence and stability of equilibria are addressed under weak Allee effect and strong Allee effect, respectively. Also we investigate the occurrence of saddle-node bifurcation which doesn't exist in the single-species Logistic model. The corresponding dynamical behaviors are richer and more interesting than those in the traditional logistic model with dispersal. We find that both Allee effect and dispersal, especially the capacity of respective patches can make the species "unstable".

Index Terms—Allee effect, swapping migration, extinction, stability, bifurcation.

I. INTRODUCTION

A large number of researchers have investigated the dynamical behaviors for single-species model with logistic growth. The asymptotic stability, permanence, extinction and existence of positive periodic solutions have been considered during the last decades. Also, since the population can spread over the homogeneous/heterogeneous patchy landscape, there may be interaction between the patches. Migration between different patches in ecosystem has become a focus of intense research in order to keep the ecological balance ([1] -[4]). An increasing number of scholars have extensively studied the Logistic model with dispersal and many interesting results have been obtained ([5] -[9]). For example, in [6], the authors analyzed source-sink systems with asymmetric dispersal between two patches. By using dynamical system theory and applying the graphical analysis method, the authors provided complete analysis on the system and demonstrated a mechanism by which the dispersal asymmetry can lead to either an increased total size of the species in two patches, a decreased total size with persistence in the patches, or even extinction in both patches. Subsequently, in [9], the authors investigated a predator-prey system in which the predator moves between two patches. One patch is a source, where the predator and prey can persist, while the other is a sink where the predator cannot survive. The authors

Manuscript received March 10, 2023; revised August 22, 2023.

This work was partically supported by the Young and Middle-aged Teachers Education Research Project of Fujian Provincial Department of Education (JAT191340).

N. Chen is an Associate Professor of the College of Information Science, Fujian Preschool Education College, Fuzhou, Fujian, 350007 China (Corresponding author to provide e-mail: 5655341@qq.com).

L. Chen is an Associate Professor of the School of Mathematics and Statistics, Fuzhou University, Fuzhou, Fujian, 350108 China (e-mail: chenlijuan@fzu.edu.cn).

J. Xu is an Associate Professor of the School of Mathematics and Statistics, Fuzhou University, Fuzhou, Fujian, 350108 China (e-mail: xu-junyan159@163.com).

C. Wu is an Associate Professor of the Department of Date Science Statistic, Fuzhou University Zhicheng College, Fuzhou, Fujian, 350002 China (237621488@qq.com).

showed whether or not the dispersal is beneficial to the predator's total abundance at equilibrium. We notice that for the following system:

$$\begin{cases}
\frac{dx_1}{dt} = -x_1 + D_2 x_2 - D_1 x_1, \\
\frac{dx_2}{dt} = x_2 (1 - x_2) + D_1 x_1 - D_2 x_2,
\end{cases}$$
(1)

it is not difficult to obtain the corresponding asymptotic stability as follows.

Theorem A System (1) has a trial equilibrium which is a saddle and the unique positive equilibrium is globally asymptotically stable.

From the above, the species are always permanent due to the logistic growth for the species in the second patch. The migration term is usually expressed by linear function of densities. Nevertheless, as was pointed out in [3] and [4], a kind of nonlinear migration, i.e., the swapping migration is more reasonable. In other words, the species can move from one patch to the other only when there are empty cells in the destination. The authors in [3] and [4] took into account the capacities of patches when considering the metapopulation dynamic model on two patches. Let $E_i(t) = S_i - X_i(t)$ be the density of empty cell in patch i at time t, where S_i, X_i denote the the total cell number and the densities of the species in patch i at time t, respectively. As a result, the migration term in patch i can be presented as $\overline{E}_i X_i - \overline{E}_i X_j$ which is a nonlinear function of densities, where $i \neq j$ and i, j = 1, 2. It is found that the population dynamics are largely affected by nonlinear migration. However, we notice that the species in each patch in [3] and [4] decay exponentially.

Actually, due to complication in finding mates, predation, inbreeding depression, environment conditioning, etc., Allee [10] pointed out that Allee effect may lead to lower birth rates and higher death rates. Allee effects are mainly classified into two categories: strong and weak. Recently, an increasing number of scholars have investigated the ecological model with Allee effect. For example, Conway and Smoller [11], Bazykin [12] discussed the following single-species model with strong Allee effect:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 - \frac{x}{K}\right)(x - m)x,\tag{2}$$

where 0 < m < K. K is the carrying capacity and m reflects the Allee effect. Also, a population with weak Allee effect can be stated as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1-x)\frac{x}{\beta+x}.$$
(3)

In the case of strong Allee effect, there exists a threshold population level and the species become extinct when the Allee effect constant is below this threshold. The other case of weak Allee effect, the growth rate may reduce and remain positive at lower population density. For details on Allee effect, one can refer to [13]-[21]. In [16], the authors proposed a single species logistic model with feedback control and additive Allee effect in the growth of species. The result shows that not only the Allee effect but also feedback control can make the system become "unstable". In [17], the authors considered a regime switching model for species subject to additive Allee effect. They concluded that Allee effect may lead to the extinction of species.

However, it comes to our attention that, to this day, still no scholars have investigated the ecological model with Allee effect and migration. Motivated by the above, one can naturally propose the following interesting question: How about the dynamic behaviours when the species suffer from Allee effect and swapping migration? Is the species still permanent? Is the Allee effect and dispersal possible to make the species become extinct? Different with the logistic growth in [6] and [9], in this paper, we will discuss the following two systems with Allee effect and swapping migration:

$$\begin{cases} \frac{dx_1}{dt} = -x_1 + D(\bar{E}_1 x_2 - \bar{E}_2 x_1), \\ \frac{dx_2}{dt} = x_2(S_2 - x_2)(x_2 - m) + D(\bar{E}_2 x_1 - \bar{E}_1 x_2), \end{cases}$$
(4)

and

$$\begin{cases} \frac{dx_1}{dt} = -x_1 + D(\bar{E}_1 x_2 - \bar{E}_2 x_1), \\ \frac{dx_2}{dt} = x_2(S_2 - x_2)\frac{x_2}{\beta + x_2} + D(\bar{E}_2 x_1 - \bar{E}_1 x_2). \end{cases}$$
(5)

In the above, $\overline{E}_i = S_i - x_i$ and S_i , x_i denote the total cell number and the densities of the species in patch *i* at time *t*, respectively. And *D* is the dispersal constant.

The main contributions of this paper are summarized as follows. (i) The existence and stability for the systems (4) and (5) under strong Allee effect and weak Allee effect are comprehensively discussed, respectively. On one hand, System (4) and (5) in this paper may have one or two positive equilibria which is different from that in system (1). Besides, the unique positive equilibrium in system (1) is globally asymptotically stable which means that the species is always permanent. However, for system (4) and (5), in order to decide whether the positive equilibrium is stable or not, one should compare the size of S_1 , S_2 and the Allee constant m,β . In other words, dispersal, the capacity of different patches and Allee effect can make the corresponding system "unstable". (ii) The bifurcation phenomenon under different Allee effect cases is also presented. Comparing with (1), system (4) and (5) are experiencing more interesting dynamics such as the saddle-node bifurcation when incorporating the Allee effect and dispersal. (iii) Our results will be useful to understand populations on heterogeneous landscapes and then to take possible measures in order to protect desired species.

The rest of this paper is as follows. In section 2 and 3, we investigate the dynamic behaviours for system (4) and (5), respectively. We also illuminate the main results. The effect of dispersal and Allee effect for the above two kinds of

systems are both exhibited in Section 4. Finally, we present a conclusion to end up this paper in Section 5.

II. DYNAMICS OF SYSTEM (4)

It is easy to see that system (4) always admits the trivial equilibrium $E_0(0,0)$. In the sequel, we will illustrate the existence of the positive equilibria. We consider the positive solution of the following equation:

$$x_2^2 - (S_2 + m)x_2 + mS_2 + \frac{DS_1}{1 + DS_2} = 0.$$
 (6)

Denote the discriminant of equation (6) by

$$\Delta(m) = (S_2 - m)^2 - \frac{4DS_1}{1 + DS_2}$$

Let

$$m^* = S_2 - 2\sqrt{\frac{DS_1}{1 + DS_2}}$$

be the unique root of $\triangle(m) = 0$. If $m < m^*$, then $\triangle(m) > 0$. So it infers that (4) has two positive equilibria, i.e., $E_1(x_{11}, x_{12})$, $E_2(x_{21}, x_{22})$, where

$$x_{1i} = \frac{DS_1}{1 + DS_2} x_{2i}, \ i = 1, 2,$$
$$x_{21} = \frac{S_2 + m - \sqrt{\Delta(m)}}{2}, \ x_{22} = \frac{S_2 + m + \sqrt{\Delta(m)}}{2}.$$

On the other hand, if $m > m^*$, then $\triangle(m) < 0$. So (4) has no positive equilibria. Also, system (4) has a unique positive equilibrium $E_3(x_{13}, x_{23})$ when $m = m^*$ which implies that $\triangle(m) = 0$, where

$$x_{23} = \frac{S_2 + m}{2}, \quad x_{13} = \frac{DS_1}{1 + DS_2} x_{23}$$

A statement concerning the existence of the positive equilibria of system (4) is listed below.

Theorem 2.1.

(i) If $4DS_1 < (1 + DS_2)S_2^2$ and

(a) if $m < m^*$, then system (4) has two positive equilibria $E_1(x_{11}, x_{12})$ and $E_2(x_{21}, x_{22})$,

(b) if $m = m^*$, then system (4) has a unique positive equilibrium $E_3(x_{13}, x_{23})$,

(c) if $m > m^*$, then system (4) has no positive equilibria. (ii) If $4DS_1 \ge (1+DS_2)S_2^2$, then system (4) has no positive equilibria.

Consequently, we will discuss the stability of the above equilibria. Let's start with the stability of the trivial equilibrium E_0 . In fact, the characteristic equation of J_{E_0} is

$$\lambda^2 + (1 + DS_2 + DS_1 + mS_2)\lambda + (1 + DS_2)mS_2 + DS_1 = 0.$$

Thus $E_0(0,0)$ is a hyperbolic stable node.

Secondly, for the two positive equilibria $E_i(x_{1i}, x_{2i})$, i = 1, 2, the corresponding Jacobian matrix is

$$J_{E_i} = \begin{bmatrix} -1 - DS_2 & DS_1 \\ DS_2 & -3x_{2i}^2 + 2(S_2 + m)x_{2i} - mS_2 - DS_1 \\ \end{bmatrix}$$
(7)

Thus the determinant and the trace of J_{E_i} are denoted by

$$Det[J_{E_i}] = (1 + DS_2)(2x_{2i} - (S_2 + m))x_{2i},$$

$$Tr[J_{E_i}] = -2x_{2i}^2 + (S_2 + m)x_{2i} - 1 - DS_2 - \frac{D^2S_1S_2}{1 + DS_2}.$$

Furthermore, we have

$$Det[J_{E_1}] = -(1 + DS_2)x_{21}\sqrt{\triangle(m)} < 0,$$
$$Det[J_{E_2}] = (1 + DS_2)x_{22}\sqrt{\triangle(m)} > 0.$$

Therefore, E_1 is always a saddle. At the same time,

 $Tr[J_{E_2}] < -x_{22}(2x_{22} - (S_2 + m)) = -x_{22}\sqrt{\Delta(m)} < 0.$

Thus E_2 is a hyperbolic stable node.

In short, the stability of the equilibria can be concluded as follows.

Theorem 2.2. For system (4), $E_0(0,0)$ is always a hyperbolic stable node. If $4DS_1 < (1 + DS_2)S_2^2$ and

(a) if $m < m^*$, then E_1 is a saddle and E_2 is a hyperbolic stable node,

(b) if $m = m^*$, then E_3 is an attracting saddle-node.

Proof. Here we only need to prove that E_3 is an attracting saddle-node when $m = m^*$. Let $X_1 = x - x_{13}$, $X_2 = x_2 - x_{23}$, then we transform system (4) to the following system:

$$\begin{cases} \dot{X}_1 = -(1+DS_2)X_1 + DS_1X_2, \\ \dot{X}_2 = DS_2X_1 - \frac{D^2S_1S_2}{1+DS_2}DS_1X_2 - \frac{S_2+m}{2}X_2^2 \\ -X_2^3 + P(X_2), \end{cases}$$
(8)

where $P(X_2)$ is the power serry with term X_2^i satisfying $i \ge 4$. The Jacobian matrix of the above system at E_3 is

$$J_{E_3} = \begin{bmatrix} -(1+DS_2) & DS_1 \\ DS_2 & -\frac{D^2 S_1 S_2}{1+DS_2} \end{bmatrix}.$$
 (9)

So the eigenvalues of J_{E_3} are

$$\lambda_1 = 0, \lambda_2 = -\left(1 + DS_2 + \frac{D^2 S_1 S_2}{1 + DS_2}\right).$$

Let

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} DS_1 & 1 + DS_2 \\ 1 + DS_2 & -DS_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad (10)$$

system (8) can be rewritten as

$$\begin{cases} \dot{Y}_1 = c_{20}Y_1^2 + c_{11}Y_1Y_2 + c_{02}Y_2^2 + P_1(Y_1, Y_2), \\ \dot{Y}_2 = d_{01}Y_2 + d_{20}Y_1^2 + d_{11}Y_1Y_2 + d_{02}Y_2^2 + P_2(Y_1, Y_2). \end{cases}$$
(11)

Here $P_1(Y_1,Y_2)$, $P_2(Y_1,Y_2)$ denote the power series with term $Y_1^i Y_2^j$ satisfying $i+j \geq 3$ and

$$c_{20} = -\frac{(S_2 + m)(1 + D_1)D_2^2}{2((1 + D_1)^2 + D_1D_2)},$$

$$c_{11} = -\frac{(S_2 + m)(1 + D_1)^2D_2}{(1 + D_1)^2 + D_1D_2},$$

$$c_{02} = -\frac{(S_2 + m)(1 + D_1)^3}{2((1 + D_1)^2 + D_1D_2)},$$

$$d_{01} = -\frac{(1 + D_1)^2 + D_1D_2}{1 + D_1},$$

$$d_{20} = \frac{(S_2 + m)D_2^2}{2((1 + D_1)^2 + D_1D_2))},$$

$$d_{11} = \frac{(S_2 + m)(1 + D_1)D_2}{(1 + D_1)^2 + D_1D_2},$$

$$d_{02} = \frac{(S_2 + m)(1 + D_1)^2}{2((1 + D_1)^2 + D_1D_2)},$$

Also, $D_1 = DS_2, D_2 = DS_1$.

Let a new time variable $\tau = -\frac{(1+D_1)^2 + D_1D_2}{1+D_1}t$, we obtain

$$\begin{cases} \dot{Y}_1 = e_{20}Y_1^2 + e_{11}Y_1Y_2 + e_{02}Y_2^2 + Q_1(Y_1, Y_2), \\ \dot{Y}_2 = Y_2 + f_{20}Y_1^2 + f_{11}Y_1Y_2 + f_{02}Y_2^2 + Q_2(Y_1, Y_2). \end{cases}$$
(12)

Note that we still retain t to denote τ for notational simplicity. And

$$e_{ij} = -\frac{(1+D_1)c_{ij}}{(1+D_1)^2 + D_1D_2},$$

 c_{ij} , $d_{ij}(i + j = 2)$ are the same as those in (11). When $m = m^*$, e_{20} can be simplified as $e_{20} = \frac{S_2 + m}{2} > 0$. Considering the new time variable τ and using the Theorem 7.1 in [22], E_3 is an attracting saddle-node. This completes the proof.

From the above, we notice that the species will be extinct when $(1 + DS_2)S_2^2 \le 4DS_1$. In other words, dispersal may lead to extinction when the capacity of the second patch is low. Moreover, when $(1+DS_2)S_2^2 > 4DS_1$, the species may be permanent or extinct. Whether the species can survive or not depends on the size of m. The smaller m is, the more likely the population is permanent.

Next, we can prove the existence of saddle-node bifurcation when $m = m_{SN}$, where

$$m_{SN} = 1 - 2\sqrt{\frac{DS_1}{1 + DS_2}}.$$

Theorem 2.3. System (4) undergoes the saddle-node bifurcation when $m = m_{SN}$.

Proof. We will use Sotomayor's theorem in [23] to verify the transversality condition and obtain the existence of the saddle-node bifurcation when $m = m_{SN}$. The Jacobian matrix at E_3 is given by (9). Since $Det[J_{E_3}] = 0$, J_{E_3} has a unique zero eigenvalue denoted by λ_1 .

For the matrices $J(E_3)$ and $J(E_3)^T$, V and W are the eigenvectors corresponding to the eigenvalue λ_1 as follows.

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} DS_1 \\ 1 + DS_2 \end{pmatrix},$$
$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} DS_2 \\ 1 + DS_2 \end{pmatrix}.$$

Moreover,

$$F_m(E_3; m_{SN}) = \begin{pmatrix} \frac{\partial F_1}{\partial m} \\ \frac{\partial F_2}{\partial m} \end{pmatrix}_{(E_3; m_{SN})} = \begin{pmatrix} 0 \\ \frac{S_2^2 - m^2}{4} \end{pmatrix}$$

where

$$F_1 = -x_1 + D(\bar{E}_1 x_2 - \bar{E}_2 x_1)$$

and

$$F_2 = x_2(S_2 - x_2)(x_2 - m) + D(\bar{E}_2x_1 - \bar{E}_1x_2)$$

Volume 53, Issue 4: December 2023

It's not difficult to obtain that V and W satisfy the transversality condition

$$W^{T}F_{m}(E_{3};m_{SN}) = \frac{(1+D^{2}S_{2}^{2})(S_{2}^{2}-m^{2})}{4} \neq 0,$$

$$W^{T}[D^{2}F(E_{3};m_{SN})(V,V)] = -(1+DS_{2})^{2}(S_{2}+m) \neq 0$$

The above implies the existence of the saddle-node bifurcation occurs at E_3 when $m = m_{SN}$. Thus the proof is completed.

The corresponding illustrative phase portrait in Theorem 2.1-2.3 is shown in Figure 1. Here we choose D = 0.5, $S_1 = 0.6$, $S_2 = 1$, m = 0.1 in case (a), D = 0.8, $S_1 = 0.16$, $S_2 = 1$, m = 0.2 in case (b) and D = 0.5, $S_1 = 0.6$, $S_2 = 1$, m = 0.4 in case (c). In case (a), system (4) has a trivial equilibrium E_0 and two positive equilibria E_1 , E_2 . Also, E_0 and E_2 is stable nodes while E_1 is a saddle. In case (b), system (4) has a trivial equilibrium E_0 which is a stable node and a unique positive equilibrium E_3 which is an attracting saddle-node. In case (c), system (4) has a trivial equilibrium E_0 which is globally asymptotically stable.

III. DYNAMICS OF SYSTEM (5)

System (5) always has the trivial equilibrium $E_0(0,0)$. The existence of the positive equilibria is determined by the following equation:

$$(1+DS_2)x_2^2 - ((1+DS_2)S_2 - DS_1)x_2 + \beta DS_1 = 0.$$
(13)

It shows that the above equation has no positive equilibrium when $(1 + DS_2)S_2 \le DS_1$. In the sequel, we will analyze the case $(1 + DS_2)S_2 > DS_1$. It is easy to obtain that the discriminant of the above equation is

$$\Delta(\beta) = ((1 + DS_2)S_2 - DS_1)^2 - 4\beta DS_1(1 + DS_2).$$

2

Let

$$\beta^* = \frac{((1+DS_2)S_2 - DS_1)}{4DS_1(1+DS_2)}$$

which satisfies that $\triangle(\beta^*) = 0$. In this case, system (5) has a unique equilibrium $\overline{E}_3(\overline{x}_{13}, \overline{x}_{23})$ with

$$\bar{x}_{23} = \frac{(1+DS_2)S_2 - DS_1}{2(1+DS_2)}, \quad \bar{x}_{13} = \frac{DS_1}{1+DS_2}\bar{x}_{23}.$$

If $\beta < \beta^*$, then $\triangle(\beta) > 0$. So system (5) has two positive equilibria, i.e., $\bar{E}_1(\bar{x}_{11}, \bar{x}_{21})$, $\bar{E}_2(\bar{x}_{12}, \bar{x}_{22})$ with

$$\bar{x}_{1i} = \frac{DS_1}{1 + DS_2} \bar{x}_{2i}, i = 1, 2,$$
$$\bar{x}_{21} = \frac{(1 + DS_2)S_2 - DS_1 - \sqrt{\Delta(\beta)}}{2(1 + DS_2)}$$

and

$$\bar{x}_{22} = \frac{(1+DS_2)S_2 - DS_1 + \sqrt{\Delta(\beta)}}{2(1+DS_2)}.$$

Whereas, if $\beta > \beta^*$, then $\triangle(\beta) < 0$. So system (5) has no positive equilibria. Thus, we can get the resluts regarding the existence of the positive equilibria as follows.

Theorem 3.1.

(i) If DS₁ < (1 + DS₂)S₂ and
(a) if β < β*, then system (5) has two distinct positive

equilibria $\bar{E}_1(\bar{x}_{11}, \bar{x}_{21})$ and $\bar{E}_2(\bar{x}_{12}, \bar{x}_{22})$,

+ 0.3 x₂ - 0.5 x $x_2 (1 - x_2) (x_2 - 0.1) + 0.5 x_1 - 0.3 x_2$ 0.8 0. 0.0 0.5 ×[∾] 0.4 0.3 0.3 0. 0.5 0.1 0.2 0.3 0.4 0.6









(c) $m > m^*$

Fig. 1. The dynamical behaviors of system (4) when $4DS_1 < (1 + DS_2)S_2^2$.

(b) if $\beta = \beta^*$, then system (5) has a unique positive equilibrium $\bar{E}_3(\bar{x}_{13}, \bar{x}_{23})$,

(c) if $\beta > \beta^*$, then system (5) has no positive equilibria. (ii) If $DS_1 \ge (1 + DS_2)S_2$, then system (5) has no positive equilibria.

Next, we will further investigate the stability of the above equilibria. For the trivial equilibrium \bar{E}_0 , the character equation of $J_{\bar{E}_0}$ is

$$\lambda^2 + (1 + DS_1 + DS_2)\lambda + DS_1 = 0.$$

So $\overline{E}_0(0,0)$ is a hyperbolic stable node.

For the two positive equilibria $\bar{E}_i(\bar{x}_{1i}, \bar{x}_{2i}), i = 1, 2$, the Jacobian matrix is

$$J_{\bar{E}_i} = \begin{bmatrix} -1 - DS_2 & DS_1 \\ DS_2 & \frac{\Delta_1}{(1 + DS_2)(\beta + \bar{x}_{2i})} \end{bmatrix}, \quad (14)$$

where

$$\Delta_1 = \beta DS_1(2 - DS_2) - (S_2(1 + DS_2) - DS_1 + D^2S_1S_2)x_{2i}$$

The determinant and the trace of the Jacobian matrix are given by

$$Det[J_{\bar{E}_i}] = \frac{1}{\beta + x_{2i}} [(S_2(1 + DS_2) - DS_1)x_{2i} - 2\beta DS_1],$$
$$Tr[J_{\bar{E}_i}] = -\frac{\Delta_2}{(1 + DS_2)(\beta + x_{2i})},$$

where

$$\Delta_2 = ((1+D_1)^2 + S_2(1+D_1) - DS_1 + D^2S_1S_2)x_{2i} + \beta((1+DS_2)^2 - 2DS_1 + D^2S_1S_2).$$

So we have

$$Det[J_{\bar{E}_1}] = \frac{\sqrt{\Delta(\beta)}}{2(1+DS_2)(\beta+x_{21})}(\sqrt{\Delta(\beta)} - (S_2(1+DS_2)-DS_1)) < 0,$$
$$Det[J_{\bar{E}_2}] = \frac{\sqrt{\Delta(\beta)}}{2(1+DS_2)(\beta+x_{22})}(\sqrt{\Delta(\beta)} + (S_2(1+DS_2)-DS_1)) > 0.$$

Thus E_1 is always a saddle. Notice that

$$Tr[J_{\bar{E}_2}] \le -\frac{ riangle_3}{4(1+DS_2)^2 DS_1(\beta+x_{22})},$$

where

$$\Delta_3 = ((1+DS_2)^2 + D^2 S_1 S_2) S_2 (1+DS_2)$$
$$+ D^3 S_2 S_1^2 + D S_1 (1+DS_2)^2 > 0.$$

We have $Tr[J_{\bar{E}_2}] < 0$ when $\beta < \beta^*$. So E_2 is always a stable node. Thus, the stability of the equilibria is shown as follows.

Theorem 3.2. $\overline{E}_0(0,0)$ is always a hyperbolic stable node. If $DS_1 < (1 + DS_2)S_2$ and

(1) if $\beta < \beta^*$, then \overline{E}_1 is a saddle and E_2 is a hyperbolic stable node,

(2) if $\beta = \beta^*$, then \overline{E}_3 is an attracting saddle-node.

Proof. We will prove that \overline{E}_3 is an attracting saddle-node when $\beta = \beta^*$. Let $X_1 = x - \overline{x}_{13}$, $X_2 = x_2 - \overline{x}_{23}$, we transform system (5) to the following system:

$$\begin{cases} \dot{X}_1 = -(1+DS_2)X_1 + DS_1X_2, \\ \dot{X}_2 = DS_2X_1 - \frac{D^2S_1S_2}{1+DS_2}X_2 + a_0X_2^2 + \bar{P}(X_2), \end{cases}$$
(15)

where $\bar{P}(X_2)$ denotes the power sery with term X_2^i satisfying $i \ge 3$ and

$$a_0 = \frac{2}{(\beta S_2 + \bar{x}_{23})^2} (\beta - (3\beta + S_2 - \frac{DS_1}{1 + DS_2})\bar{x}_{23}) < 0.$$

The Jacobian matrix of system (15) at \overline{E}_3 is the same as that in Theorem 2.2. The eigenvalues of $J_{\overline{E}_3}$ are

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{D^2 S_1 S_2}{1 + D S_2}.$$

Under the same linear transformation as (10), system (15) can be rewritten as

$$\begin{cases} \dot{Y}_{1} = \bar{c}_{20}Y_{1}^{2} + \bar{c}_{11}Y_{1}Y_{2} + \bar{c}_{02}Y_{2}^{2} + \bar{P}_{1}(Y_{1}, Y_{2}), \\ \dot{Y}_{2} = \bar{d}_{01}Y_{2} + \bar{d}_{02}Y_{1}^{2} + \bar{d}_{11}Y_{1}Y_{2} + \bar{d}_{20}Y_{2}^{2} + \bar{P}_{2}(Y_{1}, Y_{2}), \\ \vdots \qquad \vdots \qquad \vdots \qquad (16)$$

where $P_1(Y_1, Y_2)$, $P_2(Y_1, Y_2)$ denote the power series with term $Y_1^i Y_2^j$ satisfying $i + j \ge 3$ and

$$\bar{c}_{20} = \frac{a_0(1+D_1)^3}{(1+D_1)^2 + D_1D_2}, \quad \bar{c}_{02} = \frac{a_0(1+D_1)D_1^2}{(1+D_1)^2 + D_1D_2},$$
$$\bar{c}_{11} = -\frac{2a_0D_1(1+D_1)^2}{(1+D_1)^2 + D_1D_2}, \quad \bar{d}_{01} = -\frac{(1+D_1)^2 + D_1D_2}{1+D_1},$$
$$\bar{d}_{20} = -\frac{a_0D_2(1+D_1)^2}{(1+D_1)^2 + D_1D_2}, \quad \bar{d}_{02} = -\frac{a_0D_1D_2}{(1+D_1)^2 + D_1D_2},$$

 $\bar{d}_{11} = \frac{2a_0D_1D_2(1+D_1)}{(1+D_1)^2 + D_1D_2}.$ And $D_1 = DS_2, D_2 = DS_1.$ Introducing a new time variable

$$\tau = -\frac{(1+D_1)^2 + D_1 D_2}{1+D_1}t,$$

we have the following system:

$$\begin{cases} \dot{Y}_1 = \bar{e}_{20}Y_1^2 + \bar{e}_{11}Y_1Y_2 + \bar{e}_{02}Y_2^2 + \bar{Q}_1(Y_1, Y_2), \\ \dot{Y}_2 = Y_2 + \bar{f}_{02}Y_1^2 + \bar{f}_{11}Y_1Y_2 + \bar{f}_{20}Y_2^2 + \bar{Q}_2(Y_1, Y_2), \\ \end{cases}$$
(17)

where

$$\bar{e}_{ij} = -\frac{(1+D_1)\bar{c}_{ij}}{(1+D_1)^2 + D_1D_2},$$

and \bar{c}_{ij} , $\bar{d}_{ij}(i + j = 2)$ are the same as those in (16). Moreover, \bar{e}_{20} can be simplified as

$$\bar{e}_{20} = -\frac{a_0(1+D_1)^4}{((1+D_1)^2+D_1D_2)^2} > 0$$

Thus we can deduce that \overline{E}_3 is an attracting saddle-node. This completes the proof.

For system (3), the unique positive equilibrium x = 1 is globally asymptotically stable, which indicates that the species is always permanent. Nevertheless, when dispersal is incorporated, extinction may appear. That is, dispersal is not beneficial for the species in the second patch. In detail, for system (5), the species will extinguish if $(1 + DS_2)S_2 \ge$

 DS_1 . On the contrary, if $(1 + DS_2)S_2 < DS_1$, the species may survive. Lower β is propitious to the persistence of the population.

Similar to Section 2, we will prove the existence of the saddle-node bifurcation for system (5).

Theorem 3.3. System (5) undergoes the saddle-node bifurcation when $\beta = \beta_{SN}$.

Proof. The Jacobian matrix at \overline{E}_3 is given by

$$J_{\bar{E}_3} = \begin{bmatrix} -(1+DS_2) & DS_1 \\ DS_2 & -\frac{D^2S_1S_2}{1+DS_2} \end{bmatrix}.$$

Obviously $Det[J_{\bar{E}_3}] = 0$, then $J_{\bar{E}_3}$ has a unique zero eigenvalue, named λ_1 .

One can find that V and W in Theorem 2.3 are two eigenvectors corresponding to the eigenvalue λ_1 for the matrices $J(\bar{E}_3)$ and $J(\bar{E}_3)^T$. Moreover,

$$F_{\beta}(\bar{E}_3;\beta_{SN}) = \left(\begin{array}{c} 0\\ \frac{-4D^3S_1^2S_2}{(S_2(1+DS_2)+DS_1)(S_2(1+DS_2)-DS_1)} \end{array}\right).$$

Here

$$F_1 = -x_1 + D(\bar{E}_1 x_2 - \bar{E}_2 x_1)$$

and

$$F_2 = x_2(S_2 - x_2)\frac{x_2}{x_2 + \beta} + D(\bar{E}_2x_1 - \bar{E}_1x_2).$$

We can derive that

$$W^{T}F_{\beta}(E_{3};\beta_{SN}) = \frac{-4D^{3}S_{2}(1+DS_{2})S_{1}^{2}}{(S_{2}(1+DS_{2})+DS_{1})(S_{2}(1+DS_{2})-DS_{1})} \neq 0,$$

$$W^{T}[D^{2}F(E_{3};\beta_{SN})(V,V)] = a_{0}(1+DS_{2})^{2} \neq 0,$$

where a_0 is the same as that in Theorem 3.2. The above deduces that the saddle-node bifurcation occurs at \bar{E}_3 when $\beta = \beta_{SN}$. Thus we complete the proof.

The corresponding illustrative phase portrait in Theorem 3.1-3.3 is shown in Figure 2. Here we take D = 0.9, $S_1 = \frac{5}{9}$, $S_2 = 1$, $\beta = 0.5$ in case (a), D = 0.4, $S_1 = 1.75$, $S_2 = 1$, $\beta = 0.125$ in case (b) and D = 0.9, $S_1 = \frac{5}{9}$, $S_2 = 1$, $\beta = 0.5$ in case (c). In case (a), system (5) has three equilibria, i.e., \bar{E}_0 , \bar{E}_1 , \bar{E}_2 . And \bar{E}_0 , \bar{E}_2 is stable nodes while \bar{E}_1 is a saddle. In case (b), system (5) has a stable node \bar{E}_0 and an attracting saddle-node \bar{E}_3 . For system (5), the species in both patches will die out which is presented in case (c).

IV. EFFECT OF ALLEE EFFECT AND DISPERSAL FOR SYSTEM (4) AND (5)

In this section, by analyzing the possible positive equilibrium, we focus on the impact of Allee effect and dispersal on the total population abundances. For system (4) and (5), we denote the total population abundances by T_1 and T_2 , where

$$T_1 = x_{12} + x_{22} = \frac{(1 + D(S_1 + S_2))(S_2 + m + \sqrt{\Delta(m)})}{2(1 + DS_2)},$$

$$T_2 = \bar{x}_{12} + \bar{x}_{22}$$

$$= \frac{(1 + D(S_1 + S_2))((1 + DS_2)S_2 - DS_1) + \sqrt{\Delta(\beta)}}{2(1 + DS_2)^2}.$$











(c) $\beta > \beta^*$

Fig. 2. The Dynamical behaviours of system (5) when $(1 + DS_2)S_2 > DS_1$.

Volume 53, Issue 4: December 2023

A. Effect of Allee effect

Through complicated analysis, we have

and

$$\frac{dT_2}{d\beta} = -\frac{(1 + DS_1 + DS_2)DS_1}{(1 + DS_2)\sqrt{\Delta(\beta)}} < 0.$$

 $\frac{dT_1}{dm} = \frac{1 + DS_1 + DS_2}{2(1 + DS_2)^2} (1 - \frac{S_2 - m}{\sqrt{\Delta(m)}}) < 0$

So we can conclude that not only weak Allee effect but also strong Allee effect can lead to lower total population density than without Allee effect.

B. Effect of dispersal

Due to the complexity of direct analyzing the impact of dispersal on the total population T_1 and T_2 , for system (4) and (5), we mainly pay attention to the special case $m = m^*$ and $\beta = \beta^*$, respectively.

For system (4), one can calculate that

$$\frac{dT_1}{dD} = \frac{S_1}{2(1+DS_2)^2 \sqrt{\frac{DS_1}{1+DS_2}}} \left[-3\frac{DS_1}{1+DS_2} + 2S_2 \sqrt{\frac{DS_1}{1+DS_2}} - 1\right]$$
(18)

Let

$$u = \sqrt{\frac{DS_1}{1 + DS_2}}, \ f(u) = -3u^2 + 2S_2u - 1, \ \Delta = 4(S_2^2 - 3).$$

Next, we can examine the impact of dispersal on the species concretely. If $S_2 \leq \sqrt{3}$, then $\Delta \leq 0$. Thus dispersal can lead to lower total population density. If $S_2 > \sqrt{3}$, then $\Delta > 0$. So the equation f(u) = 0 exists two positive solutions, i.e.,

$$u_1 = \frac{S_2 - \sqrt{S_2^2 - 3}}{3}$$
 and $u_2 = \frac{S_2 + \sqrt{S_2^2 - 3}}{3}$.

Furthermore, if

$$\sqrt{\frac{DS_1}{1+DS_2}} < u_1 \text{ or } \sqrt{\frac{DS_1}{1+DS_2}} > u_2,$$

one has $\frac{dT_1}{dD} < 0$. In addition, if

$$u_1 < \sqrt{\frac{DS_1}{1 + DS_2}} < u_2$$

it results in $\frac{dT_1}{dD} > 0$. Thus, the corresponding analysis is gained as follows.

Theorem 4.1. For system (4), assume that S_1 and S_2 are fixed.

(1) Let $S_1 \leq S_2^3$. We have

$$\lim_{D \to +\infty} T_1 = \frac{(S_1 + S_2)(S_2^3 - S_1)}{S_1(S_2^2 + \sqrt{S_1S_2})}$$

and

(a) if $S_2 \leq \sqrt{3}$, then T_1 decreases when D increases.,

(b) if $S_2 > \sqrt{3}$, then there exist D_0 and $D_1 > D_0$ such that T_1 reaches the minimum at D_0 and reaches the maximum at D_1 , where D_0 , D_1 satisfy that

 $\sqrt{\frac{D_0 S_1}{1 + D_0 S_2}} = \frac{S_2 - \sqrt{S_2^2 - 3}}{3}$

$$\sqrt{\frac{D_1 S_1}{1 + D_1 S_2}} = \frac{S_2 + \sqrt{S_2^2 - 3}}{3}$$



Fig. 3. Impact of dispersal for system (4)

Volume 53, Issue 4: December 2023

and

(2) Let $S_1 > S_2^3$ and $D_2 = \frac{S_2^2}{S_1 - S_2^3}$,

(a) if $S_2 \leq \sqrt{3}$, then T_1 decreases when D increases, and if $D \geq D_2$, then $T_1 = 0$,

(b) if $S_2 > \sqrt{3}$, then T_1 reaches the minimum at D_0 and reaches the maximum at D_1 , where D_0 , D_1 is the same as those in case (1). And if $D \ge D_2$, then $T_1 = 0$.

In terms of system (4), as shown in Figure 3, the influence of dispersal on the possible total population indensity is depicted. We choose $S_1 = 0.5, S_2 = 1$ (case (a)), $S_1 =$ $0.5, S_2 = 1$ (case (b), (c)), $S_1 = 0.5, S_2 = 1$ (case (d)) and $S_1 = 0.5, S_2 = 1$ (case (e), (f)). In case (a) and (c), one can find that if $S_2 \leq \sqrt{3}$, T_1 decreases when D increases. The underlying reason is that the intrinsic growth rate of the first patch is negative. When the capacity of the species in the second patch is low, i.e., $S_2 \leq \sqrt{3}$, dispersal may lead to lower total population intensity (see case (a)) and even go to extinction in both patches (see case (d)). However, when the capacity of the species in the second patch is high, i.e., $S_2 > \sqrt{3}$, the total population density T_1 is nonmonotonically decreasing. Actually, T_1 has two extreme points , i.e., D_0 and $D_1(>D_0)$. If $0 < D < D_0$ or $D > D_1$, T_1 decreases when D increases. If $D_0 < D < D_1$, T_1 is increasing when D is increasing (see case (b), (c), (e), (f)). Especially, when D is sufficiently large, the total population intensity T_1 remains at a steady positive level (see case (c)) or go to extinction in both patches (see case (f)). That is to say, dispersal may lead to the extinction or the survival of the species in both patches.

For system (5), notice that

$$\frac{dT_2}{dD} = \frac{S_1[S_2 - 1 + (S_2(S_2 - 1) - 2S_1)D]}{2(1 + DS_2)^3}.$$

Firstly, if $0 < S_2 \leq 1$, dispersal is unfavorable due to $\frac{dT_2}{dD} < 0$. Namely, T_2 is always less than $T_2(0)$ which is the total population density without dispersal. Secondly, if $S_2 > 1$ and $S_2(S_2 - 1) \geq 2S_1$, we recieve that $\frac{dT_2}{dD} > 0$ and then $T_2 > T_2(0)$. Thirdly, if $0 < S_2(S_2 - 1) < 2S_1$, it follows that

$$\frac{dT_2}{dD} > 0 \text{ for } D \in \left(0, \frac{S_2 - 1}{2S_1 - S_2(S_2 - 1)}\right)$$

and

$$\frac{dT_2}{dD} < 0 \text{ for } D \in \Big(\frac{S_2 - 1}{2S_1 - S_2(S_2 - 1)}, +\infty\Big).$$

It implies that T_2 reaches its maximum at

$$\bar{D}_0 = \frac{S_2 - 1}{2S_1 - S_2(S_2 - 1)}.$$

Denote $\bar{T}_2 = \frac{(S_1 + S_2)(S_2^2 - S_1)}{2S_2^2}$, $\bar{D}_1 = \frac{S_2}{S_1 - S_2^2}$. To sum up, we can arrive at the effect of dispersal as follows.

Theorem 4.2. For system (5), assume that S_1 and S_2 are fixed.

(1) Let $S_2 \leq 1$, then T_2 decreases when D increases. If $S_2^2 \geq S_1$, then $\lim_{D \to +\infty} T_2 = \overline{T}_2$. If $S_2^2 < S_1$, then $T_2 = 0$

when $D \geq \overline{D}_1$.

(2) Let $S_2 > 1$ and $S_2(S_2 - 1) \ge 2S_1$ (it follows that $S_2^2 > S_1$), then T_2 increases when D increases. Also, $\lim_{D \to +\infty} T_2 = \overline{T}_2$.

(3) Let $S_2 > 1$ and $S_2(S_2 - 1) < 2S_1$, then T_2 reaches the maximum at \overline{D}_0 . Furthermore, if $S_2^2 \ge S_1$, $\lim_{D \to +\infty} T_2 = \overline{T}_2$. If $S_2^2 < S_1$, then $T_2 = 0$ when $D > \overline{D}_1$.

In Figure 4, by numerical simulation, we show how dispersal affects the possible total population intensity for system (5). Similarly, we select $S_1 = 0.2, S_2 = 0.5$ (case (a)), $S_1 = 0.5, S_2 = 0.5$ (case(b)), $S_1 = 0.5, S_2 = 2$ (case (c)), $S_1 = 2, S_2 = 2$ (case (d)) and $S_1 = 5, S_2 = 2$ (case (e)). In case (a) and (b), one can find that T_2 decreases when D increases. When the capacity of the species in the second patch is low, i.e., $S_2 \leq 1$, dispersal may lead to lower total population intensity (see case (a)) or go to extinction in both patches (see case (b)). However, when the capacity of the species in the second patch is high, i.e., $S_2 > 1, S_2(S_2 - 1) \ge 2S_1$ the total population density T_2 increases when D increases(see case (c)). Also, if $S_2 > 1$ and $S_2(S_2-1) < 2S_1$, T_2 achieve the maximum at \overline{D}_1 . In fact, if $0 < D < \overline{D}_1$, T_2 is increasing when D is increasing. If $D > \overline{D}_1$, T_2 is decreasing when D is increasing (see case (d), (e)). Particularly, when D is sufficiently large, the total population intensity T_2 remains at a steady positive level (see case (d)) and even go to extinction in both patches (see case (e)). This indicates that dispersal may result in the extinction or the survival of the species in both patches.

Through the above analysis, for the single-species system with strong Allee effect, dispersal is always beneficial under certain condition. Also, under certain condition, intermediate migration intensity is favorable, but extremely small or extremely large intensity is unfavorable. All the above results will be useful in understanding population survival and planning possible ways to maintain the diversity of the population.

V. CONCLUSION

In this paper, we have investigated how Allee effect and swapping migration have an influence on the single-species models. We have achieved the dynamic behaviours under strong Allee effect and weak Allee effect, respectively. The above two models always have a trivial equilibrium which is locally asymptotically stable. For the case of strong Allee effect, only when $4DS_1 < S_2^2(1 + DS_2)$, $m < m^*$, system (4) has two positive equilibrium. For the case of weak Allee effect, the condition for the existence of two positive equilibrium is $DS_1 < S_2(1 + DS_2), \ \beta < \beta^*$. One of the above two positive equilibrium is local asymptotically stable which shows that the species can be permanent. In other words, for system (4), if the capacity constants S_1 and S_2 satisfy $4DS_1 \ge S_2^2(1+DS_2)$ or $4DS_1 < S_2^2(1+DS_2), m > 0$ m^* , the trivial equilibrium is globally asymptotically stable which implies that the species in both patches are extinct. However, for system (5), when $DS_1 \ge S_2(1 + DS_2)$ or $DS_1 < S_2(1 + DS_2), \ \beta > \beta^*,$ the species become extinguished. From the above, we conclude that whether the population lasts or not is strictly determined by the capacity constants S_1 , S_2 , the dispersal intensity and the Allee effect constant m, β . In detail, the bigger the capacity constant in



(e) $S_2 > 1, S_2^2 < S_1$

Fig. 4. Impact of dispersal for system (5)

the second patch and the smaller the Allee effect constant m, β , the less likely the species be extinct.

In brief, the obtained results, i.e., Theorem 2.1-2.3 and Theorem 3.1-3.3 have revealed that both Allee effect and the capacity have a vital effect on the dynamical behaviours such as the species' permanence, extinction and stability. Especially, notice that $S_2(1 + DS_2) \leq DS_1$ implies $S_2^2(1 + DS_2)$ $DS_2 \leq 4DS_1$ for sufficiently small capacity S_2 . The above shows that the species under strong Allee effect is more possible to become extinct than the case under weak Allee effect. As a result, in order to retain the permanence of the population, we should try to adjust the Allee effect and dispersal parameters simultaneously. In addition, it follows from Theorem 4.1 and Theorem 4.2 that when the capacity of the second patch is low, i.e., $S_2 \leq \sqrt{3}$ (strong Allee effect case) or $S_2 \leq 1$ (weak Allee effect case), dispersal is not beneficial to the population survival, or even lead to the extinction in both patches. The potential reason is that the species in patch 1 is exponentially decaying. For the strong Allee effect case, if $S_2 \geq \sqrt{3}$, the possible total population size T_1 firstly decreases, then increases and lastly decreases when dispersal intensity D increases. For the weak Allee effect case, if $S_2 > 1$ and $S_2(S_2 - 1) \ge 2S_1$, the possible total population size T_2 increases when dispersal intensity D increases and the species can survive in stationary state. For the weak Allee effect case, if $S_2 > 1$ and $S_2(S_2 - 1) < 2S_1$, when dispersal intensity D increases, the possible total population size T_2 firstly increases and then decreases. Finally, the species can stabilize in stationary state. Based on the above discussion, we come to a conclusion that dispersal under weak Allee effect is more advantageous to the species survival than that under strong Allee effect.

REFERENCES

- A.-D. Isam, X.H. Tang, Y. Yuan, "A prey-predator model with migrations and delays," Discrete and Continuous Dynamical Systems Series B, vol. 21, no. 3, pp 737-761, 2016.
- [2] S.K. Sasmal, D. Ghosh, "Effect of dispersal in two-patch prey-predator system with positive density dependence growth of preys," Biosystems, vol. 151, pp 8-20, 2017.
- [3] H. Yokoi, K.-i. Tainaka, K. Sato, "Metapopulation model for a preypredator system: Nonlinear migration due to the finite capacities of patches," Journal of Theoretical Biology, vol. 477, pp 24-35, 2019.
- [4] H. Yokoi, K.-i. Tainaka, N. Nakagiri, K. Sato, "Self-oranized habitat segregation in an ambush-predator system: Nonlinear migration of prey between two patches with finite capacities," Ecological Informatics, vol. 55, 101022, 2020.
- [5] R. Arditi, C. Lobry, T. Sari, "Asymmetric dispersal in the multi-patch logistic equation," Theoretical Population Biology, vol. 120, pp 11-15, 2018.
- [6] H. Wu, Y.S. Wang, Y.F. Li, D.L. DeAngelis, "Dispersal asymmetry in a two-patch system with source-sink populations," Theoretical Population Biology, vol. 131, pp 54-65, 2020.
- [7] A. S. Purnomo, I. Darti, A. Suryanto, W. M. Kusumawinahyu, "Fear effect on a modified Leslie-Gower predator-prey model with disease transmission in prey population," Engineering Letters, vol. 31, no. 2, pp 764-773, 2023.
- [8] J. Sooknum, N. Pochai, "A mathematical model for the evaluation of airborne infection risk for bus passengers," IAENG International Journal of Computer Science, vol.50, no. 1, pp 14-22, 2023.
- [9] R. Huang, Y.S. Wang, H. Wu, "Population abundance in predator-prey systems with predator's dispersal between two patches," Theoretical Population Biology, vol. 135, pp 1-8, 2020.
- [10] W.C. Allee, "Animal Aggregations: A study in General Sociology," University of Chicago Press, Chicago, 1931.
- [11] E.D. Conway, J.A. Smoller, "Global analysis of a system of predatorprey equations," SIAM Journal on Applied Mathematics, vol. 46, no. 4, pp 630-642, 1986.

- [12] A.D. Bazykin, "Nonlinear Dynamics of Interacting Populations," World Scientific, Singapore, 1998.
- [13] N. Min, M.X. Wang, "Hopf bifurcation and steady-state bifurcation for a Leslie-Gower prey-predator model with strong Allee effect in prey," Discrete Continuous Dynamical Systems: Series A, vol. 39, no. 2, pp 1071-1099, 2019.
- [14] T.T. Yu, Y. Tian, H.J. Guo, X.Y. Song, "Dynamical analysis of an integrated pest management predator-prey model with weak Allee effect," Journal of Biological Dynamics, vol. 13, no. 1, pp 218-244, 2019.
- [15] J.M. Zhang, L.J. Zhang, Y.Z. Bai, "Stability and bifurcation analysis on a predator-prey system with the weak Allee effect," Mathematics, vol. 7, no. 5, pp 432, 2019.
- [16] Y.Y. Lv, L.J. Chen, F.D. Chen, "Stability and bifurcation in a single species logistic model with additive Allee effect and feedback control," Advances in Difference Equations, vol. 2020, no. 1, pp 1-15, 2020.
- [17] J.Y. Xu, T.H. Zhang, M.A. Han, "Regime switching model for species subject to environmental noises and additive Allee effect," Physica A: Statistical Mechanics and Its Applications, vol. 527, 121300, 2020.
- [18] J. Jiang, Y.L. Song, P. Yu, "Delay-induced triple-zero bifurcation in a delayed Leslie-type predator-prey model with additive Allee effect," International Journal of Bifurcation and Chaos, vol. 26, no. 7, 1650117, 2016.
- [19] Y.L. Cai, C.D. Zhao, W.M. Wang, J.F. Wang, "Dynamics of a Leslie-Gower predator-prey model with additive Allee effect," Applied Mathematical Modelling, vol. 39, no. 7, pp 2092-2106, 2015.
 [20] T.T. Liu, L.J. Chen, F.D. Chen, Z. Li, "Stability analysis of a Leslie-
- [20] T.T. Liu, L.J. Chen, F.D. Chen, Z. Li, "Stability analysis of a Leslie-Gower model with strong Allee effect on prey and fear effect on predator," International Journal of Bifurcation and Chaos, vol. 32, no. 6, 2250086, 2022.
- [21] T.T. Liu, L.J. Chen, F.D. Chen, Z. Li, "Dynamics of a Leslie-Gower model with weak Allee effect on prey and fear effect on predator," International Journal of Bifurcation and Chaos, vol. 33, no. 1, 2350008, 2023.
- [22] Z.F. Zhang, T.R. Ding, W.Z. Huang, Z.X. Dong, "Qualitative Theory of Differential Equations," Science Press, Beijing(in Chinese), English edition: Translations of Mathematical Monographs, Vol. 101, American Mathematical Society, Providence, Rhode Island, 1992.
- [23] L. Perko, "Differential Equations and Dynamical Systems," third ed., in: Texts in Applied Mathematics, Vol. 7, SpringerVerlag, New York, 2001.