On a Fractional Model in Magneto-Elastic Interactions

Mohamed EL IDRISSI, El-Hassan ESSOUFI and Idriss ELLAHIANI

Abstract—**The aim of this research paper is to study the global existence of weak solution of magneto elastic interactions, which is modeled by a three-dimensional mathematical framework. A fractional generalization of the harmonic map heat flow is coupled to an evolution equation for displacement to characterize the model. Faedo-Galerkin method with some commutator is used to estimate and prove global existence of weak solutions for the proposed model.**

Key words: Ferromagnets, fractional derivative, Landau-Lifshitz equation, elasticity, weak solution, commutator estimates.

AMS subject classifications: 78A25, 35Q60, 35B40, 35K55, 65M12

I. Introduction

There is an interconnection among magnetic and elastic properties of ferromagnetic materials. The different couplings among these properties are known as magnetoelastic effects. This latter can be classified into two main classes, direct effects and their inverse effects. The crucial direct effect is magnetostriction that basically represents the phenomenon whereby a ferromagnetic sample deforms thanks to magnetic interactions that could be either within the sample itself (spontaneous magnetostriction) or the result of an external magnetic field (forced magnetostriction).This magnetostriction leads to a state of constraints(even in the absence of any external stress) which is responsible for the rearrangement of the fields and other phenomena as well. The magnetization in the magnetoelastic model results from non-mechanical external forces and is not influenced by the material's mechanical state. The aim of this research is to propose a model for the theoretical study of the interaction between the elastic and magnetic processes. The magnetoelastic dynamics in $\Theta = (0,T) \times \mathbf{D}$ ($\mathbf{D}$ is a bounded open set of $\mathbb{R}^d$, $d \geqslant 1$ and $\partial\mathbf{D}$ its boundary) are described by the nonlinear parabolic hyperbolic coupled system (see [22]):

$$\mathbf{Z}_t = \nu \mathbf{Z} \times \mathbf{H}_{\text{eff}} - \mu \mathbf{Z} \times (\mathbf{Z} \times \mathbf{H}_{\text{eff}}). \tag{1}$$

$$\rho\, \mathbf{u}_{tt} - \operatorname{div}\Big(\mathcal{S}(\mathbf{u}) + \frac{1}{2}\mathcal{L}(\mathbf{Z})\Big) = 0. \tag{2}$$

The equation (1) is the equation of Landau-Lifshitz and the equation (2) represent the displacement's evolution. The unknown $\mathbf{Z}$ is the magnetization vector,which is a map from $\mathbf{D}$ to S^2 (the unit sphere of $\mathbb{R}^3$) and $\mathbf{Z}_t$ is its derivative with respect to time. Let $\times$ denotes the vector cross product in $\mathbb{R}^3$. In addition, we designate by $m_i, i = 1,2,3$ the $\mathbf{Z}$ components. And $\nu \in \mathbb{R}$, and $\rho > 0$ are two physical constants and μ is a positive parameter called the Gilbert damping parameter. the effective field is denoted by $\mathbf{H}_{\text{eff}}$, and in this work we have

$$\mathbf{H}_{\text{eff}} = -\Lambda^{2\alpha}\mathbf{Z} - \boldsymbol{\ell}(\mathbf{Z},\mathbf{u}) \tag{3}$$

Where $\Lambda = (-\Delta)^{\frac{1}{2}}$ indicates the Laplacian's square root. It can be defined trough Fourier transformation [20],[24]. Throughout this work, for repeated indices, we use the Einstein summation convention, and we are focused on the case $\alpha \in (1, \frac{3}{2})$.

The vector $\boldsymbol{\ell}(\mathbf{Z},\mathbf{u})$ and the tensors $\mathcal{L}(\mathbf{Z})$, $\mathcal{S}(\mathbf{u})$ are given by these following equations

$$\ell_i = \zeta_{ijkl} Z_j \epsilon_{kl}(\mathbf{u}), \qquad i = 1,2,3.$$

$$\mathcal{L}_{kl} = \zeta_{ijkl} Z_i Z_j \ \text{ and } \ S_{kl} = \sigma_{ijkl}\epsilon_{ij}(\mathbf{u}).$$

Here

$$\begin{cases} \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \\ \text{refer to the components of the linearized strain tensor } \epsilon, \\ \zeta_{ijkl} = \zeta_1 \delta_{ijkl} + \zeta_2 \delta_{ij}\delta_{kl} + \zeta_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \\ \sigma_{ijkl} = \tau_1(\delta_{ijkl} - \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \tau_2 \delta_{ij}\delta_{kl} \\ \text{with} \begin{cases} \delta_{ijkl} = 1 \text{ if } i = j = k = l, \\ \delta_{ijkl} = 0 \ \text{ otherwise.} \end{cases} \end{cases}$$

where $\sigma = (\sigma_{ijkl})$ is the elasticity tensor. It is assumed to satisfy these two following assumptions :

$$\sigma_{ijkl} = \sigma_{klij} = \sigma_{jikl} \qquad \text{(symmetry property)}$$

and

$$(\sigma_{ijkl}\epsilon_{ij}\epsilon_{kl}) \geq \beta \sum |\epsilon_{ji}|^2 \tag{4}$$

holds for some$\beta \geq 0$.

In the case where the effective field $\mathbf{H}_{\text{eff}}$ is reduced to $-\Lambda^{2\alpha}\mathbf{Z}$, the equation (1) will be rewritten as

$$\mathbf{Z}_t = -\nu \mathbf{Z} \times \Lambda^{2\alpha}\mathbf{Z} + \mu \mathbf{Z} \times (\mathbf{Z} \times \Lambda^{2\alpha}\mathbf{Z}). \tag{5}$$

Note that when $\alpha = 1$, equation (5) conforms to the standard Landau-Lifshitz equation. This equation is proposed in 1935 by Landau and Lifshitz, and it has been widely studied in the past decades (we quote some references here [1], [2], [10], [23]). Also, when $\nu = 0$ and $\alpha = 1$, the equation (5) is called the heat flow of harmonic maps to the unit sphere which also was widely studied in the past decades [16]. The authors in [17], have studied the equation (5) for $\nu = 0$ (generalization

Manuscript submitted May 11, 2023; revised October 20, 2023

Mohamed EL IDRISSI is a Ph.D. student at Hassan First University of Settat, FST, MISI Lab, Settat, Morocco (e-mail: medelidrissi2020@gmail.com)

El-Hassan ESSOUFI is a professor at Hassan First University of Settat, FST, MISI Lab, Settat, Morocco (e-mail: ehessoufi@gmail.com)

Idriss ELLAHIANI is a professor at Hassan First University of Settat, FST, MISI Lab, Settat, Morocco (e-mail: ellahiani.i@gmail.com)

of the harmonic map heat flow to the fractional order) under these two conditions $\alpha \in (0,1)$ and $\alpha > \frac{d}{4}$. In the general case, several works have been carried out on the fractional Landau-Lifshitz equation we refer, for instance to [11] that "Guo and Zeng" have proven the existence of weak solutions for equation of a simplified model for thin-film micromagnetics under periodical boundary condition. The paper [18] have considered the fractional Landau-Lifshitz equation's well-posedness without Gilbert damping appears to be main purpose. Vanishing viscosity method proves the global existence of weak solutions. For magneto-elasticity coupling we refer, for instance in [22] where the authors managed to establish the weak solution's existence basing their study on a three-dimensional case. For a reduced model, in [4], the authors demonstrate both the existence and uniqueness of the solutions and in [3], tackled the problem of a one dimensional penalty and studied the gradient flow of the associated type Ginzburg-Landau functional. A classical solution, whose existence and uniqueness are demonstrated, tends asymptotically for subsequences towards a stationary point of the energy functional. In a recent work [7], the authors proved the global existence of weak solutions to a one-dimensional mathematical model describing magneto elastic interactions by using Faedo-Galerkin/Penalty method. In [8], and also based on the F.G.P method, the authors establish the existence of weak solutions for a magneto-viscoelastic model but without taking into account the external forces acting on the system. Our aim here is to study a fractional model arising in magneto elasticity, but this time in higher dimension more precisely when d = 3 and in the more general case in which mechanical external force act on the system.

Throughout the whole document, we put to use the subsequent notations. Let $\mathbf{D}$ be an open-bounded domain of $\mathbb{R}^3$, for $p \in \mathbb{N}^*$,$\mathbf{L}^p(\mathbf{D}) = (L^p(\mathbf{D}))^3$ and $\mathbf{H}^1(\mathbf{D}) = (H^1(\mathbf{D}))^3$ are the classical Hilbert spaces equipped with the usual norm denoted by $||.||_{\mathbf{L}^p(\mathbf{D})}$ and $||.||_{\mathbf{H}^1(\mathbf{D})}$ (generally, the $\mathbf{X}$ symbolizes the product functional spaces $(X)^3$). For all $s > 0$, $W^{s,p}$ designates the usual Sobolev space consisting of all g such that

$$\|g\|_{W^{s,p}} := \|\mathcal{F}^{-1}\big(1+|\cdot|^2\big)^{\frac{s}{2}}\big(\mathcal{F}g\big)(\cdot)\|_{L^p} < \infty$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier transform and its inverse. Having $\dot{W}^{s,p}$ denotes the corresponding homogeneous Sobolev space, when $p = 2$, $W^{s,p}$ conforms to the usual Sobolev space H^s and in this case we have

$$\|g\|_{\dot{H}^s} := \|\Lambda^s g\|_{L^2}$$

The following parts of this paper are structured as follows. Section 2 Introduces the model on which we are going to work, and it gives a preliminary result. Section 3 presents our main results which will be proving in section 5. In Section 4 we propose some lemmas which are beneficial to the work in the rest of the paper. Section 6 concludes the paper.

II. Model and preliminary results

The objective of this research is to prove the global existence of weak solutions for the magnetization vector in the spatial domain $\mathbf{D} = (0, 2\pi)^d$ with periodic boundary conditions. Consider $d = 3$ and assume that the gyromagnetic ratio $\nu \neq 0$ and $\mu > 0$. The $x = (x_1, x_2, x_3)$ denotes the generic point of $\mathbf{D}$. We consider the following system,

$$\begin{cases} \mathbf{Z}_t = \nu \mathbf{Z} \times \mathbf{H}_{\text{eff}} - \mu \mathbf{Z} \times (\mathbf{Z} \times \mathbf{H}_{\text{eff}}) \\ \rho\ \mathbf{u}_{tt} - \operatorname{div}\Big(\mathcal{S}(\mathbf{u}) + \dfrac{1}{2}\mathcal{L}(\mathbf{Z})\Big) = 0, \end{cases} \tag{6}$$

where $\mathbf{H}_{\text{eff}}$ is given by (3) and $\mathbf{h}$ is a given external force. For initial conditions let

$$\mathbf{Z}(\cdot, 0) = \mathbf{Z}_0, \ \ |\mathbf{Z}_0| = 1, \tag{7}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad \mathbf{u}_t(\cdot, 0) = \mathbf{u}_1, \text{ in } \ \mathbf{D},$$

where the boundary conditions for the displacement vector are

$$\mathbf{u} = 0 \quad \text{on} \quad \Sigma := \partial\mathbf{D} \times (0, T). \tag{8}$$

The major difficulty in the first equation of (6) is the double vector product. To overcome this difficulty, the equation (6) is equivalent to Gilbert equation

$$\mathbf{Z}_t = \frac{\nu^2 + \mu^2}{\nu}\mathbf{Z} \times \mathbf{H}_{\text{eff}} - \frac{\mu}{\nu}\mathbf{Z} \times \mathbf{Z}_t. \tag{9}$$

According to [4], we replace the first equation of (6) by the quasi linear parabolic equation (Ginzburg-Landau type equation).

$$\mathbf{Z}_t^{\varepsilon} - \eta \mathbf{Z} \times \mathbf{Z}_t + \gamma \Lambda^{2\alpha}\mathbf{Z}^{\varepsilon} + \gamma \boldsymbol{\ell}(\mathbf{Z}^{\varepsilon}, \mathbf{u}^{\varepsilon}) + \frac{|\mathbf{Z}^{\varepsilon}|^2 - 1}{\varepsilon}\mathbf{Z}^{\varepsilon} = 0. \tag{10}$$

where $\eta = \frac{\nu}{\mu}$, $\gamma = \frac{\nu^2+\mu^2}{\mu}$, the parameter ε is positive and $\mathbf{Z}^{\varepsilon} : \mathbf{D} \times \mathbb{R}^+ \to \mathbb{R}^3$. The ε-penalization in the equation (10) replaces the magnitude constraint $|\mathbf{Z}| = 1$.

III. Main results

The aim of the present section is to define the weak solution of the problem (6)-(7)-(8).

Definition III.1. *Given $\mathbf{Z}_0 \in \mathbf{H}^{\alpha}(\boldsymbol{D}), |\mathbf{Z}_0| = 1$ a.e., $\mathbf{u} \in \mathbf{H}_0^1(\boldsymbol{D})$, $\mathbf{u}_1 \in \mathbf{L}^2(\boldsymbol{D})$ and $\mathbf{h} \in \mathbf{L}^2(\Theta)$. The pair $(\mathbf{Z}, \mathbf{u})$ is said to be a weak solution of the problem (6)-(7)-(8) if:*

- *for all $T > 0$, $\mathbf{Z} \in L^{\infty}(0, T; \mathbf{H}^{\alpha}(\boldsymbol{D})), \mathbf{Z}_t \in L^2(0, T; \mathbf{L}^2(\boldsymbol{D}))$, $|\mathbf{Z}| = 1$ a.e., $\mathbf{u} \in L^2(0, T; \mathbf{H}_0^1(\boldsymbol{D}))$ and $\mathbf{u}_t \in L^2(0, T; \mathbf{L}^2(\boldsymbol{D}))$;*
- *for all $\boldsymbol{\varphi} \in \mathbf{C}^{\infty}(\overline{\Theta})$ and $\boldsymbol{\psi} \in \mathbf{H}_0^1(\Theta)$, we have:*

$$\int_{\Theta}(\mathbf{Z}_t \times \mathbf{Z}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t - \eta\int_{\Theta}\mathbf{Z}_t \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t$$
$$+\gamma\int_{\Theta}\Lambda^{\alpha}\mathbf{Z} \cdot \Lambda^{\alpha}(\mathbf{Z} \times \boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t$$
$$+\gamma\int_{\Theta}(\boldsymbol{\ell}(\mathbf{Z}, \mathbf{u}) \times \mathbf{Z}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t = 0$$
$$-\rho\int_{\Theta}\mathbf{u}_t \cdot \boldsymbol{\psi}_t\ \mathrm{d}x\mathrm{d}t + \int_{\Theta}\mathbf{h} \cdot \boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t$$
$$+\int_{\Theta}\Big(\mathcal{S}(\mathbf{u}) + \frac{1}{2}\mathcal{L}(\mathbf{Z})\Big) \cdot \epsilon(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t = 0;$$

- *$\mathbf{Z}(0, x) = \mathbf{Z}_0(x)$ and $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ in the trace sense;*

- *for all $T > 0$, we have:*

$$\begin{aligned} &\frac{\gamma}{2}\int_{\boldsymbol{D}}|\Lambda^\alpha \mathbf{Z}(T)|^2 \, \mathrm{d}x + \int_\Theta |\mathbf{Z}_t|^2 \, \mathrm{d}x\mathrm{d}t \\ &+\tfrac{\gamma\rho}{2}\int_{\boldsymbol{D}} |\mathbf{u}_t(T)|^2 \, \mathrm{d}x + \tfrac{\gamma\beta}{4}\int_{\boldsymbol{D}} |\nabla \mathbf{u}(T)|^2 \, \mathrm{d}x \\ &\leq \frac{\gamma}{2}\int_{\boldsymbol{D}} |\Lambda^\alpha \mathbf{Z}_0|^2 \, \mathrm{d}x + \frac{\gamma\rho}{2}\int_{\boldsymbol{D}} |\mathbf{u}_1|^2 \, \mathrm{d}x \\ &+\tfrac{3\gamma\tau}{4}\int_{\boldsymbol{D}} |\nabla \mathbf{u}_0|^2 \, \mathrm{d}x + C(\boldsymbol{D},\beta,\zeta,\gamma,\mathbf{h}), \end{aligned} \tag{11}$$

where $C(\boldsymbol{D},\beta,\zeta,\gamma,\mathbf{h}) \geq 0$.

The paper's main result is as follows.

Theorem III.2. *Consider $\alpha \in (1, \frac{3}{2})$, $\mathbf{Z}_0 \in \mathbf{H}^\alpha(\boldsymbol{D})$ such that $|\mathbf{Z}_0| = 1$ a.e., $\mathbf{u}_0 \in \mathbf{H}^1_0(\boldsymbol{D})$, $\mathbf{u}_1 \in \mathbf{L}^2(\boldsymbol{D})$ and $\mathbf{h} \in \mathbf{L}^2(\Theta)$. Hence there exists a weak solution of the problem* (6)-(7)-(8) *in the sense the Definition $III.1$.*

Proof: See section 5.

IV. A few technical lemmas

This section is dedicated to introduce a few lemmas which are going to be used in the rest of this paper.

Lemma IV.1. *Assume these three spaces E, F et G are Banach and satisfy $E \subset F \subset G$ where the injections are continuous with compact embedding $E \hookrightarrow F$ and E,G are reflexive. Denote*

$$H := \{u | u \in L^{k_0}(0,T;E), u_t = \frac{du}{dt} \in L^{k_1}(0,T;G)\}$$

where T is finite and $1 < k_i < \infty$, $i = 0, 1$. Then H, equipped with the norm

$$||u||_{L^{k_0}(0,T;E)} + ||u_t||_{L^{k_1}(0,T;G)},$$

is a Banach space and the embedding $H \hookrightarrow L^{k_0}(0,T;F)$ is compact.

Proof: (see[15], (page 57)).

Lemma IV.2. *For a bounded open set of $\mathbb{R}^d_x \times \mathbb{R}_t, Q$. y_n and y in $L^k(Q), 1 < k < \infty$ so that y_n satisfies $||y_n||_{L^k(Q)} \leq C, y_n \to y$ a.e. in Q, then $y_n \rightharpoonup y$ weakly in $L^k(Q)$.*

Proof: (see [15], page 12)

Lemma IV.3. *(Commutator estimates). Assume that $l > 0$ and $k \in (1, +\infty)$. If $u, v \in \mathcal{S}$ (the Schwartz class) therefore*

$$\begin{aligned} &\|\Lambda^l(uv) - u\Lambda^l v\|_{L^k} \\ &\leq C\Big(\|\nabla u\|_{L^{k_1}}\|k\|_{\dot{W}^{l-1,k_2}} + \|u\|_{\dot{W}^{l,k_3}}\|v\|_{L^{k_4}}\Big) \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\|\Lambda^l(uv)\|_{L^k} \\ &\leq C\Big(\|u\|_{L^{k_1}}\|v\|_{\dot{W}^{l,k_2}} + \|u\|_{\dot{W}^{l,k_3}}\|v\|_{L^{k_4}}\Big) \end{aligned} \tag{13}$$

with $k_2, k_3 \in (1, +\infty)$ such that $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{k_3} + \frac{1}{k_4}$.

Proof: (see [5], [12], [13])

Lemma IV.4. *Suppose that $k > j > 1$ and $\frac{1}{k} + \frac{l}{d} = \frac{1}{j}$. Assume that $\Lambda^l h \in L^j$, then $h \in L^k$ and there exist a constant $C > 0$ such that*

$$\|h\|_{L^k} \leq C\|\Lambda^l h\|_{L^j}.$$

Proof: (see [20],[25])

we conclude this section by this lemma.

Lemma IV.5. *If u and v belong to $H^{2\alpha}_{per}(D) := \{u \in L^2(D)/\Lambda^{2\alpha}u \in L^2(D)\}$, then*

$$\int_D \Lambda^{2\alpha}u \cdot v \, dx = \int_D \Lambda^\alpha u \cdot \Lambda^\alpha v \, dx.$$

Proof: (see[11])

V. Global existence of weak solutions

In this section, we are going to use the Faedo-Galerkin/Penalty method to demonstrate the previous theorem III.2 .

A. Penalty problem

For a fixed parameter, $\varepsilon > 0$. We consider the following penalty problem

$$\begin{cases} \mathbf{Z}^\varepsilon_t - \eta\mathbf{Z}^\varepsilon \times \mathbf{Z}^\varepsilon_t + \gamma\Lambda^{2\alpha}\mathbf{Z}^\varepsilon + \gamma\boldsymbol{\ell}(\mathbf{Z}^\varepsilon,\mathbf{u}^\varepsilon) \\ +\dfrac{|\mathbf{Z}^\varepsilon|^2 - 1}{\varepsilon}\mathbf{Z}^\varepsilon = 0 \\ \rho\, \mathbf{u}^\varepsilon_{tt} - \mathrm{div}\Big(\mathcal{S}(\mathbf{u}^\varepsilon) + \dfrac{1}{2}\mathcal{L}(\mathbf{Z}^\varepsilon)\Big) + \mathbf{h} = 0, \end{cases} \tag{14}$$

in $\Theta = \mathbf{D} \times (0,T)$, with initial and boundary conditions are as follows.

$$\mathbf{Z}^\varepsilon(\cdot, 0) = \mathbf{Z}_0, \;\; |\mathbf{Z}_0| = 1 \quad a.e,$$

$$\mathbf{u}^\varepsilon(\cdot, 0) = \mathbf{u}_0, \;\; \mathbf{u}^\varepsilon_t(\cdot, 0) = \mathbf{u}_1, \;\; \text{in} \;\; \mathbf{D},$$

$$\mathbf{u}^\varepsilon = 0 \;\; \text{on} \;\; \Sigma.$$

Applying Faedo-Galerkin method: consider an orthonormal basis $\{g_i\}_{i\in\mathbb{N}}$ of $L^2(\mathbf{D})$ consisting of all the eigenfunctions for the operator $\Lambda^{2\alpha}$ ([21], Ch.II proves the existence of such orthonormal basis)

$$\Lambda^{2\alpha}g_i = \alpha_i g_i, i = 1, 2, ...$$

under periodic boundary conditions, and an orthonormal basis $\{f_i\}_{i\in\mathbb{N}}$ of $L^2(\mathbf{D})$ consisting of all the eigenfunctions for the operator $-\Delta$

$$\begin{cases} -\Delta f_i = \beta_i f_i, i = 1, 2, ... \\ f_i = 0 \;\; \text{on} \;\; \partial\mathbf{D}. \end{cases}$$

We obtain the next approximate problem in $\Theta = \mathbf{D} \times (0,T)$

$$\begin{cases} \mathbf{Z}^{\varepsilon,N}_t - \eta\mathbf{Z}^{\varepsilon,N} \times \mathbf{Z}^{\varepsilon,N}_t + \gamma\Lambda^{2\alpha}\mathbf{Z}^{\varepsilon,N} + \gamma\boldsymbol{\ell}(\mathbf{Z}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N}) \\ +\dfrac{|\mathbf{Z}^{\varepsilon,N}|^2 - 1}{\varepsilon}\mathbf{Z}^{\varepsilon,N} = 0 \\ \rho\, \mathbf{u}^{\varepsilon,N}_{tt} - \mathrm{div}\Big(\mathcal{S}(\mathbf{u}^{\varepsilon,N}) + \dfrac{1}{2}\mathcal{L}(\mathbf{Z}^{\varepsilon,N})\Big) + \mathbf{h}^N = 0, \end{cases} \tag{15}$$

where the initial and boundary conditions are as follow:

$$\mathbf{u}^{\varepsilon,N}(\cdot, 0) = \mathbf{u}^N(\cdot, 0), \;\; \mathbf{u}^{\varepsilon,N}_t(\cdot, 0) = \mathbf{u}^N_t(\cdot, 0),$$

$$\mathbf{Z}^{\varepsilon,N}(\cdot,0)=\mathbf{Z}^{N}(\cdot,0),\ \text{in } \mathbf{D},$$
$$\mathbf{u}^{\varepsilon,N}=0 \quad \text{on} \quad \Sigma=\partial\mathbf{D}\times(0,T).$$

and

$$\int_{\mathbf{D}}\mathbf{u}^{N}(x,0)f_i(x)\ \mathrm{d}x=\int_{\mathbf{D}}\mathbf{u}_0(x)f_i(x)\ \mathrm{d}x\ ,$$
$$\int_{\mathbf{D}}\mathbf{u}_t^{N}(x,0)f_i(x)\ \mathrm{d}x=\int_{\mathbf{D}}\mathbf{u}_1(x)f_i(x)\ \mathrm{d}x,$$
$$\int_{\mathbf{D}}\mathbf{Z}^{N}(x,0)g_i(x)\ \mathrm{d}x=\int_{\mathbf{D}}\mathbf{Z}_0(x)g_i(x)\ \mathrm{d}x.$$

and the vector $\mathbf{h}^N$ satisfies

$$\int_{\mathbf{D}}\mathbf{h}^{N}(x,t)f_i(x)\ \mathrm{d}x=\int_{\mathbf{D}}\mathbf{h}(x,t)f_i(x)\ \mathrm{d}x,$$

We are interested to find the approximate solutions $(\mathbf{Z}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})$ of (15) under the following form

$$\mathbf{Z}^{\varepsilon,N}=\sum_{i=1}^{N}\mathbf{a}_i(t)g_i(x) \quad , \quad \mathbf{u}^{\varepsilon,N}=\sum_{i=1}^{N}\mathbf{b}_i(t)f_i(x),$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are $\mathbb{R}^3$-valued vectors.

Multiplying each scalar of the first and second equations of (15) by g_i and f_i respectively, and then integrating in $\mathbf{D}$ we obtain a system of ordinary differential equations where the unknowns are $(\mathbf{a}_i(t),\mathbf{b}_i(t)), i=1,2,...,N$. Based on standard ordinary differential equations theory, we can easily prove the existence of local solutions of the problem. This latter can be extended on $[0,T]$ by using a priori estimates. Therefore, multiplying the first and the second equations of (15) by $\mathbf{Z}_t^{\varepsilon,N}$ and ε, N_t respectively, and integrating in $\mathbf{D}$, we obtain

$$\begin{cases}\int_{\mathbf{D}}|\mathbf{Z}_t^{\varepsilon,N}|^2\mathrm{d}x+\gamma\int_{\mathbf{D}}\Lambda^{2\alpha}\mathbf{Z}^{\varepsilon,N}\cdot\mathbf{Z}_t^{\varepsilon,N}\mathrm{d}x\\ +\gamma\int_{\mathbf{D}}\boldsymbol{\ell}(\mathbf{Z}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})\cdot\mathbf{Z}_t^{\varepsilon,N}\mathrm{d}x\\ +\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}|^2-1)^2\mathrm{d}x=0\\ \frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}|\mathbf{u}_t^{\varepsilon,N}|^2\mathrm{d}x-\int_{\partial\mathbf{D}}\Big(\mathcal{S}(\mathbf{u}^{\varepsilon,N})+\frac{1}{2}\mathcal{L}(\mathbf{Z}^{\varepsilon,N})\Big)\mathbf{n}\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x\\ +\int_{\mathbf{D}}\Big(\mathcal{S}(\mathbf{u}^{\varepsilon,N})+\frac{1}{2}\mathcal{L}(\mathbf{Z}^{\varepsilon,N})\Big)\cdot\nabla\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x\\ +\int_{\mathbf{D}}\mathbf{h}^N\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x=0\end{cases}$$

where $\mathbf{n}$ is the outer unit normal at the boundary $\partial\mathbf{D}$. On the other hand (note that $\zeta_{ijkl}=\zeta_{jikl}$)

$$\gamma\int_{\mathbf{D}}\boldsymbol{\ell}(\mathbf{Z}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})\cdot\mathbf{Z}_t^{\varepsilon,N}\ \mathrm{d}x$$
$$=\gamma\int_{\mathbf{D}}\zeta_{ijkl}Z_j^{\varepsilon,N}\dot{Z}_i^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x$$
$$=\frac{\gamma}{2}\int_{\mathbf{D}}\zeta_{ijkl}(Z_j^{\varepsilon,N}\dot{Z}_i^{\varepsilon,N}+Z_i^{\varepsilon,N}\dot{Z}_j^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x$$
$$=\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\mathrm{d}x$$
$$-\frac{\gamma}{2}\int_{\mathbf{D}}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}_t^{\varepsilon,N})\mathrm{d}x,$$

and since both tensor $\mathcal{S}$ and $\mathcal{L}$ are symmetric, we have

$$\int_{\mathbf{D}}\Big(\mathcal{S}(\mathbf{u}^{\varepsilon,N})+\frac{1}{2}\mathcal{L}(\mathbf{Z}^{\varepsilon,N})\Big)\cdot\epsilon(\mathbf{u}_t^{\varepsilon,N})\ \mathrm{d}x$$
$$=\int_{\mathbf{D}}\Big(\mathcal{S}(\mathbf{u}^{\varepsilon,N})+\frac{1}{2}\mathcal{L}(\mathbf{Z}^{\varepsilon,N})\Big)\cdot\nabla\mathbf{u}_t^{\varepsilon,N}\ \mathrm{d}x,$$

and

$$\int_{\mathbf{D}}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}_t^{\varepsilon,N})\ \mathrm{d}x$$
$$=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x.$$

By using the Lemma $IV.5$, we can write

$$\begin{cases}\int_{\mathbf{D}}|\mathbf{Z}_t^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}|\Lambda^{\alpha}\mathbf{Z}^{\varepsilon,N}|^2\ \mathrm{d}x\\ +\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x\\ -\frac{\gamma}{2}\int_{\mathbf{D}}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}_t^{\varepsilon,N})\ \mathrm{d}x\\ +\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x=0\\ \frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}|\mathbf{u}_t^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x\\ +\frac{1}{2}\int_{\mathbf{D}}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}_t^{\varepsilon,N})\ \mathrm{d}x+\int_{\mathbf{D}}\mathbf{h}^N\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x=0\end{cases}$$

Mutiplying second equation by γ and make sum with the first equation, we obtain

$$\int_{\mathbf{D}}|\mathbf{Z}_t^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}|\Lambda^{\alpha}\mathbf{Z}^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$+\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x+\frac{\gamma\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}|\mathbf{u}_t^{\varepsilon,N}|^2\ \mathrm{d}x$$
$$+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x$$
$$+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{D}}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\ \mathrm{d}x$$
$$+\gamma\int_{\mathbf{D}}\mathbf{h}^N\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x=0.$$

Then, integrating in time,

$$\begin{aligned}
&\int_\Theta |\mathbf{Z}_t^{\varepsilon,N}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\gamma}{2}\int_\mathbf{D} |\Lambda^\alpha \mathbf{Z}^{\varepsilon,N}(T)|^2 \, \mathrm{d}x \\
&+\tfrac{1}{4\varepsilon}\int_\mathbf{D}(|\mathbf{Z}^{\varepsilon,N}(T)|^2-1)^2 \, \mathrm{d}x \\
&+\frac{\gamma\rho}{2}\int_\mathbf{D} |\mathbf{u}_t^{\varepsilon,N}(T)|^2 \, \mathrm{d}x \\
&+\tfrac{\gamma}{2}\int_\mathbf{D} \sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})(T) \, \mathrm{d}x \\
&+\frac{\gamma}{2}\int_\mathbf{D} \zeta_{ijkl} Z_i^{\varepsilon,N} Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})(T) \, \mathrm{d}x \qquad (16)\\
&= -\gamma\int_\Theta \mathbf{h}^N \cdot \mathbf{u}_t^{\varepsilon,N}\mathrm{d}x + \tfrac{\gamma}{2}\int_\mathbf{D} |\Lambda^\alpha \mathbf{Z}^N(0)|^2 \, \mathrm{d}x \\
&+\frac{1}{4\varepsilon}\int_\mathbf{D}(|\mathbf{Z}^N(0)|^2-1)^2 \, \mathrm{d}x + \frac{\gamma\rho}{2}\int_\mathbf{D} |\mathbf{u}_t^N(0)|^2 \, \mathrm{d}x \\
&+\frac{\gamma}{2}\int_\mathbf{D} \sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^N)\epsilon_{kl}(\mathbf{u}^N)(0) \, \mathrm{d}x \\
&+\frac{\gamma}{2}\int_\mathbf{D} \zeta_{ijkl} Z_i^N Z_j^N \epsilon_{kl}(\mathbf{u}^N)(0) \, \mathrm{d}x.
\end{aligned}$$

Hence, we denote the left-hand side and the right-hand side of (16) by $\mathcal{A}^{\varepsilon,N}(T)$ and $\mathcal{A}^N(0)$ respectively.

Assuming there exist a positive parameter, ζ, satisfies $\frac{2\zeta}{9} > \sup_{ijkl}|\zeta_{ijkl}|$ by using Young's inequality, omitting superscripts, we have

$$\begin{aligned}
|\zeta_{ijkl}Z_iZ_j\epsilon_{kl}(\mathbf{u})| &\le \frac{2\zeta}{9}|Z_i||Z_j||\epsilon_{kl}(\mathbf{u})| \\
&\le \frac{2\zeta}{9}\Big(\frac{\zeta}{\beta}|Z_i|^2|Z_j|^2 + \frac{\beta}{4\zeta}|\epsilon_{kl}(\mathbf{u})|^2\Big).
\end{aligned}$$

From where

$$\begin{aligned}
&\sum_{ijkl}|\zeta_{ijkl}Z_iZ_j\epsilon_{kl}(\mathbf{u})| \\
&\le \frac{2\zeta}{9}\Big(\frac{9\zeta}{\beta}\sum_i|Z_i|^2\sum_j|Z_j|^2 + \frac{9\beta}{4\zeta}\sum_{kl}|\epsilon_{kl}(\mathbf{u})|^2\Big) \\
&= 2\zeta\Big(\frac{\zeta}{\beta}\big(\sum_i|Z_i|^2\big)^2 + \frac{\beta}{4\zeta}\sum_{kl}|\epsilon_{kl}(\mathbf{u})|^2\Big) \\
&= \frac{2\zeta^2}{\beta}|\mathbf{Z}|^4 + \frac{\beta}{2}\sum_{kl}|\epsilon_{kl}(\mathbf{u})|^2.
\end{aligned}$$

Therefore, according to the idea introduced in [22] we have

$$\begin{aligned}
&\frac{\gamma}{2}\Big|\int_\mathbf{D}\zeta_{ijkl}Z_iZ_j\epsilon_{kl}(\mathbf{u})\mathrm{d}x\Big| \\
&= \frac{\gamma}{2}\Big|\int_\mathbf{D}\sum_{ijkl}\zeta_{ijkl}Z_iZ_j\epsilon_{kl}(\mathbf{u})\mathrm{d}x\Big| \\
&\le \frac{\gamma}{2}\int_\mathbf{D}\sum_{ijkl}|\zeta_{ijkl}Z_iZ_j\epsilon_{kl}(\mathbf{u})|\mathrm{d}x \\
&\le \frac{\zeta^2\gamma}{\beta}\int_\mathbf{D}|\mathbf{Z}|^4\mathrm{d}x + \frac{\beta\gamma}{4}\int_\mathbf{D}\sum_{kl}|\epsilon_{kl}(\mathbf{u})|^2\mathrm{d}x \\
&= \frac{\zeta^2\gamma}{\beta}\int_\mathbf{D}\Big(|\mathbf{Z}|^2-1+1\Big)^2\mathrm{d}x + \frac{\beta\gamma}{4}\int_\mathbf{D}\sum_{kl}|\epsilon_{kl}(\mathbf{u})|^2\mathrm{d}x \\
&\le \frac{2\zeta^2\gamma}{\beta}\int_\mathbf{D}\Big(|\mathbf{Z}|^2-1\Big)^2\mathrm{d}x + \frac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}) \\
&\quad+\frac{\beta\gamma}{4}\int_\mathbf{D}\sum_{kl}|\epsilon_{kl}(\mathbf{u})|^2\mathrm{d}x \\
&\le \frac{2\zeta^2\gamma}{\beta}\int_\mathbf{D}\Big(|\mathbf{Z}|^2-1\Big)^2\mathrm{d}x + \frac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}) \\
&\quad+\frac{\gamma}{4}\int_\mathbf{D}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{u})\mathrm{d}x.
\end{aligned}$$

by using (4). Now, for $\varepsilon < \frac{\beta}{16\zeta^2\gamma}$ we have

$$\begin{aligned}
&\frac{\gamma}{2}\Big|\int_\mathbf{D}\zeta_{ijkl}mZiZ_j\epsilon_{kl}(\mathbf{u})\mathrm{d}x\Big| \\
&\le \frac{1}{8\varepsilon}\int_\mathbf{D}\Big(|\mathbf{Z}|^2-1\Big)^2\mathrm{d}x + \frac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}) \\
&\quad+\frac{\gamma}{4}\int_\mathbf{D}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{u})\mathrm{d}x.
\end{aligned}$$

Which implies

$$\begin{aligned}
&\frac{\gamma}{2}\int_\mathbf{D}\zeta_{ijkl}Z_i^NZ_j^N\epsilon_{kl}(\mathbf{u}^N)(0)\mathrm{d}x \\
&\le \frac{1}{8\varepsilon}\int_\mathbf{D}\Big(|\mathbf{Z}^N(0)|^2-1\Big)^2\mathrm{d}x \\
&+\frac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}) + \frac{\gamma}{4}\int_\mathbf{D}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^N)\epsilon_{kl}(\mathbf{u}^N)(0)\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{8\varepsilon}\int_\mathbf{D}\Big(|\mathbf{Z}^{\varepsilon,N}(T)|^2-1\Big)^2\mathrm{d}x - \frac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}) \\
&-\frac{\gamma}{4}\int_\mathbf{D}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})(T)\mathrm{d}x \\
&\le \frac{\gamma}{2}\int_\mathbf{D}\zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})(T)\mathrm{d}x.
\end{aligned}$$

Based on the definition of $\mathcal{A}^{\varepsilon,N}(T)$ and $\mathcal{A}^N(0)$ we have

$$\begin{aligned}
&\int_\Theta |\mathbf{Z}_t^{\varepsilon,N}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\gamma}{2}\int_\mathbf{D} |\Lambda^\alpha \mathbf{Z}^{\varepsilon,N}(T)|^2 \, \mathrm{d}x \\
&+\tfrac{1}{8\varepsilon}\int_\mathbf{D}(|\mathbf{Z}^{\varepsilon,N}(T)|^2-1)^2 \, \mathrm{d}x + \frac{\gamma\rho}{2}\int_\mathbf{D}|\mathbf{u}_t^{\varepsilon,N}(T)|^2 \, \mathrm{d}x \\
&+\tfrac{\gamma}{4}\int_\mathbf{D}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})(T) \, \mathrm{d}x \\
&-\tfrac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}) \le \mathcal{A}^{\varepsilon,N}(T),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{A}^N(0) \le -\gamma\int_\Theta \mathbf{h}^N\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x + \frac{\gamma}{2}\int_\mathbf{D}|\Lambda^\alpha\mathbf{Z}^N(0)|^2 \, \mathrm{d}x \\
&+\frac{3}{8\varepsilon}\int_\mathbf{D}(|\mathbf{Z}^N(0)|^2-1)^2 \, \mathrm{d}x + \frac{\gamma\rho}{2}\int_\mathbf{D}|\mathbf{u}_t^N(0)|^2 \, \mathrm{d}x \\
&+\frac{3\gamma}{4}\int_\mathbf{D}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^N)\epsilon_{kl}(\mathbf{u}^N)(0) \, \mathrm{d}x + \frac{2\zeta^2\gamma}{\beta}vol(\mathbf{D}).
\end{aligned}$$

Since $\mathcal{A}^{\varepsilon,N}(T) = \mathcal{A}^N(0)$, we have

$$\begin{aligned}
&\int_\Theta |\mathbf{Z}_t^{\varepsilon,N}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\gamma}{2}\int_\mathbf{D}|\Lambda^\alpha\mathbf{Z}^{\varepsilon,N}(T)|^2 \, \mathrm{d}x \\
&+\frac{1}{8\varepsilon}\int_\mathbf{D}(|\mathbf{Z}^{\varepsilon,N}(T)|^2-1)^2 \, \mathrm{d}x + \frac{\gamma\rho}{2}\int_\mathbf{D}|\mathbf{u}_t^{\varepsilon,N}(T)|^2 \, \mathrm{d}x
\end{aligned}$$

$$+\frac{\gamma}{4}\int_{\mathbf{D}}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})(T)\ \mathrm{d}x$$
$$\leq -\gamma\int_{\Theta}\mathbf{h}^N\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x+\frac{\gamma}{2}\int_{\mathbf{D}}|\Lambda^\alpha\mathbf{Z}^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3}{8\varepsilon}\int_{\mathbf{D}}(|\mathbf{Z}^N(0)|^2-1)^2\ \mathrm{d}x+\frac{\gamma\rho}{2}\int_{\mathbf{D}}|\mathbf{u}_t^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3\gamma}{4}\int_{\mathbf{D}}\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^N)\epsilon_{kl}(\mathbf{u}^N)(0)\ \mathrm{d}x+\frac{4\zeta^2\gamma}{\beta}vol(\mathbf{D}).$$

Now, we define the functional

$$\mathcal{B}^{\varepsilon,N}(T)$$
$$=\int_{\Theta}|\mathbf{Z}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{\gamma}{2}\int_{\mathbf{D}}|\Lambda^\alpha\mathbf{Z}^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\tfrac{1}{8\varepsilon}\textstyle\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x+\frac{\gamma\rho}{2}\int_{\mathbf{D}}|\mathbf{u}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\tfrac{\gamma\beta}{4}\textstyle\int_{\mathbf{D}}|\nabla\mathbf{u}^{\varepsilon,N}|^2(T)\ \mathrm{d}x,$$

then

$$\mathcal{B}^{\varepsilon,N}(0)$$
$$=\frac{\gamma}{2}\int_{\mathbf{D}}|\Lambda^\alpha\mathbf{Z}^{\varepsilon,N}(0)|^2\ \mathrm{d}x+\frac{1}{8\varepsilon}\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}(0)|^2-1)^2\ \mathrm{d}x$$
$$+\frac{\gamma\rho}{2}\int_{\mathbf{D}}|\mathbf{u}_t^{\varepsilon,N}(0)|^2\ \mathrm{d}x+\frac{\gamma\beta}{4}\int_{\mathbf{D}}|\nabla\mathbf{u}^{\varepsilon,N}|^2(0)\ \mathrm{d}x.$$

Moreover

$$-\gamma\int_{\Theta}\mathbf{h}^N\cdot\mathbf{u}_t^{\varepsilon,N}\mathrm{d}x\leq\frac{\gamma\rho}{2}||u_t^{\varepsilon,N}||^2_{\mathbf{L}^2(\Theta)}+\frac{\gamma}{2\rho}||\mathbf{h}^N||^2_{\mathbf{L}^2(\Theta)},$$
$$\int_{\mathbf{D}}|\nabla\mathbf{u}^{\varepsilon,N}(T)|^2\ \mathrm{d}x\leq\int_{\mathbf{D}}\sum_{kl}|\epsilon_{kl}(\mathbf{u}^{\varepsilon,N}(T))|^2\mathrm{d}x,$$

and under the assumption $\sigma_{ijkl}\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{u})\leq\tau|\nabla\mathbf{u}|^2$ (for a positive constant τ), we have

$$\int_{\Theta}|\mathbf{Z}_t^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{\gamma}{2}\int_{\mathbf{D}}|\Lambda^\alpha\mathbf{Z}^{\varepsilon,N}(T)|^2\ \mathrm{d}x$$
$$+\frac{1}{8\varepsilon}\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}(T)|^2-1)^2\ \mathrm{d}x+\frac{\gamma\rho}{2}\int_{\mathbf{D}}|\mathbf{u}_t^{\varepsilon,N}(T)|^2\ \mathrm{d}x \quad (17)$$
$$+\frac{\gamma\beta}{4}\int_{\mathbf{D}}|\nabla\mathbf{u}^{\varepsilon,N}|^2(T)\ \mathrm{d}x\leq\frac{\gamma}{2}\int_{\mathbf{D}}|\Lambda^\alpha\mathbf{Z}^N(0)|^2\ \mathrm{d}x$$
$$+\frac{3}{8\varepsilon}\int_{\mathbf{D}}(|\mathbf{Z}^N(0)|^2-1)^2\ \mathrm{d}x+\frac{\gamma\rho}{2}\int_{\mathbf{D}}|\mathbf{u}_t^N(0)|^2\ \mathrm{d}x$$
$$+\frac{\gamma}{2\rho}||\mathbf{h}^N||^2_{\mathbf{L}^2(\Theta)}+\frac{3\gamma\tau}{4}\int_{\mathbf{D}}|\nabla\mathbf{u}^N(0)|^2\ \mathrm{d}x+\frac{4\zeta^2\gamma}{\beta}vol(\mathbf{D}),$$

which implies

$$\mathcal{B}^{\varepsilon,N}(T)$$
$$\leq\int_0^T\mathcal{B}^{\varepsilon,N}(t)\ \mathrm{d}t+3\mathcal{B}^{\varepsilon,N}(0)+\frac{\gamma}{2\rho}||\mathbf{h}^N||^2_{\mathbf{L}^2(\Theta)}+\frac{4\zeta^2\gamma}{\beta}vol(\mathbf{D}),$$

Based on Gronwall lemma, we obtain

$$\mathcal{B}^{\varepsilon,N}(T)\leq e^T\big(3\mathcal{B}^{\varepsilon,N}(0)+\frac{\gamma}{2\rho}||\mathbf{h}^N||^2_{\mathbf{L}^2(\Theta)}+\frac{4\zeta^2\gamma}{\beta}vol(\mathbf{D})\big).$$

Since $\mathbf{u}_0\in\mathbf{H}_0^1(\mathbf{D})$, $\mathbf{u}_1\in\mathbf{L}^2(\mathbf{D})$ and $\mathbf{Z}_0\in\mathbf{H}^\alpha(\mathbf{D})$ which is embedded into $\mathbf{L}^4(\mathbf{D})$ for $1<\alpha<\frac{3}{2}$ the right hand side is uniformly bounded. Therefore, for constants C_1, C_2, C_3, C_4 and $C(\mathbf{h})$ independent of N

$$\int_{\mathbf{D}}(|\mathbf{Z}^N(0)|^2-1)^2\ \mathrm{d}x$$
$$=\int_{\mathbf{D}}|\mathbf{Z}^N(0)|^4\ \mathrm{d}x-2\int_{\mathbf{D}}|\mathbf{Z}^N(0)|^2\ \mathrm{d}x+vol(\mathbf{D})$$
$$\leq||\mathbf{Z}^N(0)||^4_{\mathbf{L}^4(\mathbf{D})}+vol(\mathbf{D})$$
$$\leq C_1||\mathbf{Z}^N(0)||^4_{\mathbf{H}^\alpha(\mathbf{D})}+C_2$$
$$\leq C_3,$$
$$\int_{\mathbf{D}}|\nabla\mathbf{u}^N(0)|^2\ \mathrm{d}x=\int_{\mathbf{D}}|\nabla\mathbf{u}^N(0)-\nabla\mathbf{u}_0+\nabla\mathbf{u}_0|^2\ \mathrm{d}x$$
$$\leq 2\int_{\mathbf{D}}|\nabla\mathbf{u}^N(0)-\nabla\mathbf{u}_0|^2\ \mathrm{d}x+2\int_{\mathbf{D}}|\nabla\mathbf{u}_0|^2\ \mathrm{d}x$$
$$\leq 2||\mathbf{u}^N(0)-\mathbf{u}_0||^2_{\mathbf{H}_0^1(\mathbf{D})}+2||\mathbf{u}_0||^2_{\mathbf{H}_0^1(\mathbf{D})}$$
$$\leq C_4,$$

and

$$||\mathbf{h}^N||^2_{\mathbf{L}^2(\Theta)}=||\mathbf{h}^N-\mathbf{h}+\mathbf{h}||^2_{\mathbf{L}^2(\Theta)}$$
$$\leq 2||\mathbf{h}^N-\mathbf{h}||^2_{\mathbf{L}^2(\Theta)}+2||\mathbf{h}||^2_{\mathbf{L}^2(\Theta)}\leq C(\mathbf{h}),$$

thanks to the strong convergences $\mathbf{Z}^N(.,0)\to\mathbf{Z}_0$ in $\mathbf{H}^\alpha(\mathbf{D})$, $\mathbf{u}^N(.,0)\to\mathbf{u}_0$ in $\mathbf{H}_0^1(\mathbf{D})$ and $\mathbf{h}^N\to\mathbf{h}$ in $\mathbf{L}^2(Q)$. For the other term ($\mathbf{u}_t^N(0)$), the estimate can be achieved in a similarly way using course the strong convergence $\mathbf{u}_t^N(.,0)\to\mathbf{u}_1$ in $\mathbf{L}^2(\mathbf{D})$. Additionally, (for C being a constant which does not depend on ε and N)

$$\int_{\mathbf{D}}|\mathbf{Z}^{\varepsilon,N}|^2\ \mathrm{d}x=\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}|^2-1+1)\ \mathrm{d}x$$
$$\leq\frac{1}{2}\int_{\mathbf{D}}(|\mathbf{Z}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x+C.$$

Hence, for the fixed $\varepsilon>0$ we have

$$\begin{aligned}
&(\mathbf{Z}^{\varepsilon,N})_N\ \text{ is bounded in }\ L^\infty(0,T;\mathbf{H}^\alpha(\mathbf{D})),\\
&(\mathbf{Z}_t^{\varepsilon,N})_N\ \text{ is bounded in }\ L^2(0,T;\mathbf{L}^2(\mathbf{D})),\\
&(|\mathbf{Z}^{\varepsilon,N}|^2-1)_N\ \text{ is bounded in }\ L^\infty(0,T;L^2(\mathbf{D})),\\
&(\mathbf{u}^{\varepsilon,N})_N\ \text{ is bounded in }\ L^2(0,T;\mathbf{H}_0^1(\mathbf{D})),\\
&(\mathbf{u}_t^{\varepsilon,N})_N\ \text{ is bounded in }\ L^2(0,T;\mathbf{L}^2(\mathbf{D})).
\end{aligned}\quad (18)$$

Notice that, (18) is owing to the Poincaré lemma. Additionally, Based on the classical compactness results, we obtain the next convergences to a two subsequences further notes $(\mathbf{Z}^{\varepsilon,N})$ and $(\mathbf{u}^{\varepsilon,N})$

$$\begin{aligned}
&\mathbf{Z}^{\varepsilon,N}\rightharpoonup\mathbf{Z}^\varepsilon\quad\text{weakly in}\quad L^2(0,T;\mathbf{H}^\alpha(\mathbf{D})),\\
&\mathbf{Z}_t^{\varepsilon,N}\rightharpoonup\mathbf{Z}_t^\varepsilon\quad\text{weakly in}\quad\mathbf{L}^2(\Theta),\\
&\mathbf{Z}^{\varepsilon,N}\to\mathbf{Z}^\varepsilon\quad\text{strongly in}\quad L^2(0,T,\mathbf{H}^\beta(\mathbf{D}))\\
&\text{and a.e. for }\ 0\leq\beta<\alpha\\
&|\mathbf{Z}^{\varepsilon,N}|^2-1\rightharpoonup\xi\quad\text{weakly in}\quad L^2(\Theta),\\
&\mathbf{u}^{\varepsilon,N}\rightharpoonup\mathbf{u}^\varepsilon\quad\text{weakly in}\quad L^2(0,T;\mathbf{H}_0^1(\mathbf{D})),\\
&\mathbf{u}_t^{\varepsilon,N}\rightharpoonup\mathbf{u}_t^\varepsilon\quad\text{weakly in}\quad\mathbf{L}^2(\Theta),\\
&\mathbf{u}^{\varepsilon,N}\to\mathbf{u}^\varepsilon\quad\text{strongly in}\quad\mathbf{L}^2(\Theta).
\end{aligned}\quad (19)$$

The convergence (19) is concluded owing to Lemma $IV.1$. Due to Lemma $IV.2$, we can reach that $\xi = |\mathbf{Z}^\varepsilon|^2-1$. Based on the Sobolev embedding $H^\alpha(\Theta) \hookrightarrow L^4(\Theta)$, since $1<\alpha<\frac{3}{2}$, the following compactness results are obtained.

$$Z_i^{\varepsilon,N} Z_j^{\varepsilon,N} \to Z_i^\varepsilon Z_j^\varepsilon \text{ strongly in } L^2(\Theta), \tag{20}$$

and

$$Z_i^{\varepsilon,N}\phi_j \to Z_i^\varepsilon \phi_j \text{ strongly in } L^2(\Theta).$$

According to the previous estimation, we can pass to the limit as $N\to\infty$ and to reach the sought-after outcomes. therefore, consider the variational formulation of (15)

$$\begin{cases} \int_\Theta \mathbf{Z}_t^{\varepsilon,N}\cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t - \eta\int_\Theta (\mathbf{Z}^{\varepsilon,N}\times \mathbf{Z}_t^{\varepsilon,N})\cdot\phi\ \mathrm{d}x\mathrm{d}t \\ +\gamma\int_\Theta \Lambda^\alpha \mathbf{Z}^{\varepsilon,N}\cdot\Lambda^\alpha\boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ +\gamma\int_\Theta \zeta_{ijkl}Z_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\boldsymbol{\phi}_i\ \mathrm{d}x\mathrm{d}t \\ +\int_\Theta \frac{|\mathbf{Z}^{\varepsilon,N}|^2-1}{\varepsilon}\mathbf{Z}^{\varepsilon,N}\cdot\boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t = 0 \\ -\rho\int_\Theta \mathbf{u}_t^{\varepsilon,N}\cdot\boldsymbol{\psi}_t\ \mathrm{d}x\mathrm{d}t + \int_\Theta \sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon,N})\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t \\ +\frac{1}{2}\int_\Theta \zeta_{ijkl}Z_i^{\varepsilon,N}Z_j^{\varepsilon,N}\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t + \int_\Theta \mathbf{h}^N\cdot\boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t = 0, \end{cases} \tag{21}$$

for all $\boldsymbol{\phi}\in L^2(0,T;\mathbf{H}^\alpha(\mathbf{D}))$ and $\boldsymbol{\psi}\in\mathbf{H}_0^1(\Theta)$. Taking $N\to\infty$ in (21), we get

$$\begin{cases} \int_\Theta \mathbf{Z}_t^{\varepsilon}\cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t - \eta\int_\Theta (\mathbf{Z}^{\varepsilon}\times \mathbf{Z}_t^{\varepsilon})\cdot\phi\ \mathrm{d}x\mathrm{d}t \\ +\gamma\int_\Theta \Lambda^\alpha \mathbf{Z}^{\varepsilon}\cdot\Lambda^\alpha\boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ +\gamma\int_\Theta \zeta_{ijkl}Z_j^{\varepsilon}\epsilon_{kl}(\mathbf{u}^{\varepsilon})\boldsymbol{\phi}_i\ \mathrm{d}x\mathrm{d}t \\ +\int_\Theta \frac{|\mathbf{Z}^{\varepsilon}|^2-1}{\varepsilon}\mathbf{Z}^{\varepsilon}\cdot\boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t = 0 \\ -\rho\int_\Theta \mathbf{u}_t^{\varepsilon}\cdot\boldsymbol{\psi}_t\ \mathrm{d}x\mathrm{d}t + \int_\Theta \sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^{\varepsilon})\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t \\ +\frac{1}{2}\int_\Theta \zeta_{ijkl}Z_i^{\varepsilon}Z_j^{\varepsilon}\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t + \int_\Theta \mathbf{h}\cdot\boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t = 0, \end{cases} \tag{22}$$

for all $\boldsymbol{\phi}\in L^2(0,T;\mathbf{H}^\alpha(\mathbf{D}))$ and $\boldsymbol{\psi}\in\mathbf{H}_0^1(\Theta)$. We proved the next proposition.

Proposition V.1. *Given $\mathbf{Z}_0\in\mathbf{H}^\alpha(\boldsymbol{D})$ such that $|\mathbf{Z}_0|=1$ a.e., $\mathbf{u}_0\in\mathbf{H}_0^1(\boldsymbol{D})$ and $\mathbf{u}_1\in\mathbf{L}^2(\boldsymbol{D})$. Then there exists a solution $\mathbf{Z}^\varepsilon$, for any positive ε small enough and any fixed time T, to the problem (14) in the sense of distributions. Additionally, we obtain the energy estimate below.*

$$\begin{aligned} &\int_\Theta |\mathbf{Z}_t^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t + \frac{\gamma}{2}\int_{\boldsymbol{D}}|\Lambda^\alpha\mathbf{Z}^\varepsilon(T)|^2\ \mathrm{d}x \\ &+\frac{1}{8\varepsilon}\int_{\boldsymbol{D}}(|\mathbf{Z}^\varepsilon(T)|^2-1)^2\ \mathrm{d}x + \frac{\gamma\rho}{2}\int_{\boldsymbol{D}}|\mathbf{u}_t^\varepsilon(T)|^2\ \mathrm{d}x \\ &+\frac{\gamma\beta}{4}\int_{\boldsymbol{D}}|\nabla\mathbf{u}^\varepsilon|^2(T)\ \mathrm{d}x \le \frac{\gamma}{2}\int_{\boldsymbol{D}}|\Lambda^\alpha\mathbf{Z}_0|^2\ \mathrm{d}x \\ &+\frac{\gamma\rho}{2}\int_{\boldsymbol{D}}|\mathbf{u}_1|^2\ \mathrm{d}x + \frac{3\gamma\tau}{4}\int_{\boldsymbol{D}}|\nabla\mathbf{u}_0|^2\ \mathrm{d}x + \frac{4\zeta^2\gamma}{\beta}vol(\boldsymbol{D}) + C(\mathbf{h}). \end{aligned} \tag{23}$$

Remark V.2. *We can easly get (23) by taking the lower semicontinuous limit in (17).*

B. Convergence of approximate solutions

In this subsection, our goal is to pass to the limit in ε ($\varepsilon\to 0$). According to the estimate (23) we get

$$\begin{gathered} (\mathbf{Z}^\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(0,T;\mathbf{H}^\alpha(\mathbf{D})), \\ (\mathbf{Z}_t^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\mathbf{D})), \\ (|\mathbf{Z}^\varepsilon|^2-1)_\varepsilon \text{ is bounded in } L^\infty(0,T;L^2(\mathbf{D})), \\ (\mathbf{u}^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0,T;\mathbf{H}_0^1(\mathbf{D})), \\ (\mathbf{u}_t^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\mathbf{D})). \end{gathered}$$

There exist two subsequences further noted $(\mathbf{Z}^\varepsilon)$ and $(\mathbf{u}^\varepsilon)$ such that the next convergences hold

$$\begin{aligned} &\mathbf{Z}^\varepsilon \rightharpoonup \mathbf{Z} \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}^\alpha(\mathbf{D})), \\ &\mathbf{Z}_t^\varepsilon \rightharpoonup \mathbf{Z}_t \quad \text{weakly in} \quad L^2(0,T;\mathbf{L}^2(\mathbf{D})), \\ &\mathbf{Z}^{\varepsilon,N} \to \mathbf{Z}^\varepsilon \quad \text{strongly in} \quad L^2(0,T,\mathbf{H}^\beta(\mathbf{D})) \\ &\qquad \text{and a.e. for } 0\le\beta<\alpha \\ &|\mathbf{Z}^\varepsilon|^2-1\to 0 \quad \text{strongly in } L^2(\Theta) \text{ and a.e.} \\ &\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}_0^1(\mathbf{D})), \\ &\mathbf{u}_t^\varepsilon \rightharpoonup \mathbf{u}_t \quad \text{weakly in} \quad \mathbf{L}^2(\Theta), \\ &\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{strongly in} \quad \mathbf{L}^2(\Theta). \end{aligned} \tag{24}$$

$|\mathbf{Z}|=1$ a.e. is a consequence of the convergence (24). To pass to the limit $\varepsilon\to 0$ in (22) , consider $\boldsymbol{\phi}=\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi}$ where $\boldsymbol{\varphi}\in\mathbf{C}^\infty(\overline{\Theta})$. As $\boldsymbol{\phi}$ is in $L^2(0,T;\mathbf{H}^\alpha(\mathbf{D}))$, there holds

$$\begin{cases} \int_\Theta \mathbf{Z}_t^\varepsilon\cdot(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t - \eta\int_\Theta(\mathbf{Z}^\varepsilon\times\mathbf{Z}_t^\varepsilon)\cdot(\mathbf{Z}^\varepsilon\times\varphi)\ \mathrm{d}x\mathrm{d}t \\ +\gamma\int_\Theta \Lambda^\alpha\mathbf{Z}^\varepsilon\cdot\Lambda^\alpha(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t \\ +\gamma\int_\Theta \zeta_{ijkl}Z_j^\varepsilon\epsilon_{kl}(\mathbf{u}^\varepsilon)(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})_i\ \mathrm{d}x\mathrm{d}t = 0 \\ -\rho\int_\Theta \mathbf{u}_t^\varepsilon\cdot\boldsymbol{\psi}_t\ \mathrm{d}x\mathrm{d}t + \int_\Theta\sigma_{ijkl}\epsilon_{ij}(\mathbf{u}^\varepsilon)\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t \\ +\frac{1}{2}\int_\Theta \zeta_{ijkl}Z_i^\varepsilon Z_j^\varepsilon\epsilon_{kl}(\boldsymbol{\psi})\ \mathrm{d}x\mathrm{d}t + \int_\Theta \mathbf{h}\cdot\boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t = 0. \end{cases} \tag{25}$$

Recent convergences that we have established and a result like the one in (20),Allows us to pass to the limit in (25) as $\varepsilon\to 0$ (except for the second and the third term of the first equation).
For the first equation's second term,
we set $\Psi_\varepsilon = \int_\Theta(\mathbf{Z}^\varepsilon\times\mathbf{Z}_t^\varepsilon)\cdot(\mathbf{Z}^\varepsilon\times\varphi)\ \mathrm{d}x\mathrm{d}t$.
we have

$$\Psi_\varepsilon = \int_\Theta |\mathbf{Z}_t^\varepsilon|^2\mathbf{Z}_t^\varepsilon\cdot\varphi\ \mathrm{d}x\mathrm{d}t - \int_\Theta(\mathbf{Z}^\varepsilon\cdot\varphi)(\mathbf{Z}^\varepsilon\cdot\mathbf{Z}_t^\varepsilon)\ \mathrm{d}x\mathrm{d}t.$$

On the one hand

$$\int_\Theta |\mathbf{Z}^\varepsilon|^2\mathbf{Z}_t^\varepsilon\cdot\varphi\ \mathrm{d}x\mathrm{d}t$$

$$= \int_\Theta (|\mathbf{Z}^\varepsilon|^2-1)\mathbf{Z}_t^\varepsilon\cdot\varphi\ \mathrm{d}x\mathrm{d}t + \int_\Theta \mathbf{Z}_t^\varepsilon\cdot\varphi\ \mathrm{d}x\mathrm{d}t.$$

$$\to \int_\Theta \mathbf{Z}_t\cdot\varphi\ \mathrm{d}x\mathrm{d}t.$$

On the other hand

$$\int_\Theta (\mathbf{Z}^\varepsilon\cdot\varphi)(\mathbf{Z}^\varepsilon\cdot\mathbf{Z}_t^\varepsilon)\ \mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Theta \partial_t(|\mathbf{Z}^\varepsilon|^2-1)\mathbf{Z}^\varepsilon\cdot\varphi\ \mathrm{d}x\mathrm{d}t.$$

$$= \frac{1}{2}\left[\int_{\mathbf{D}}(|\mathbf{Z}^\varepsilon|^2-1)\mathbf{Z}^\varepsilon\cdot\varphi\ \mathrm{d}x\right]_0^T$$

$$-\frac{1}{2}\int_\Theta (|\mathbf{Z}^\varepsilon|^2-1)\partial_t(\mathbf{Z}^\varepsilon\cdot\varphi)\ \mathrm{d}x\mathrm{d}t.$$

Now choose φ so that $\varphi=0$ in $t=0$ and $t=T$. thereafter

$$\left[\int_{\mathbf{D}}(|\mathbf{Z}^\varepsilon|^2-1)\mathbf{Z}^\varepsilon\cdot\varphi\ \mathrm{d}x\right]_0^T = 0$$

Thus,

$$\int_\Theta (\mathbf{Z}^\varepsilon\cdot\varphi)(\mathbf{Z}^\varepsilon\cdot\mathbf{Z}_t^\varepsilon)\ \mathrm{d}x\mathrm{d}t = -\frac{1}{2}\int_\Theta (|\mathbf{Z}^\varepsilon|^2-1)\partial_t(\mathbf{Z}^\varepsilon\cdot\varphi)\ \mathrm{d}x\mathrm{d}t.$$

$$= -\frac{1}{2}\int_\Theta (|\mathbf{Z}^\varepsilon|^2-1)\partial_t\mathbf{Z}^\varepsilon\cdot\varphi\ \mathrm{d}x\mathrm{d}t$$

$$-\frac{1}{2}\int_\Theta (|\mathbf{Z}^\varepsilon|^2-1)\mathbf{Z}^\varepsilon\cdot\partial_t\varphi\ \mathrm{d}x\mathrm{d}t.$$

$$\to 0$$

Hence

$$\Psi_\varepsilon \to \int_\Theta \mathbf{Z}_t\cdot\varphi\ \mathrm{d}x\mathrm{d}t.$$

Now for the third term of the first equation, the convergence is not obvious since we encounter nonlocal operators Λ^α and in this case the classical methods are not applied anymore. However, commutator estimates (Lemma $IV.3$) provide us proper tools, to which the success in the following owes a lot. We start firstly by showing that $\Lambda^\alpha(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})\in\mathbf{L}^2(\Theta)$, therefore using the multiplicative estimates (13) in Lemma $IV.3$ to $\mathbf{Z}^\varepsilon$ and φ (for $l=\alpha, k=2, k_1=\frac{2d}{d-2\alpha}, k_2=\frac{d}{\alpha}, k_3=2$ and $k_4=\infty$) for C being a constant which does not depend on ε, we find

$$\|\Lambda^\alpha(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})\|_{\mathbf{L}^2(\mathbf{D})}$$

$$\leq C\Big(\|\mathbf{Z}^\varepsilon\|_{\mathbf{L}^{k_1}(\mathbf{D})}\|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,k_2}(\mathbf{D})} + \|\mathbf{Z}^\varepsilon\|_{\dot{\mathbf{H}}^{\alpha}(\mathbf{D})}\|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})}\Big)$$

$$= C\Big(\|\mathbf{Z}^\varepsilon\|_{\mathbf{L}^{k_1}(\mathbf{D})}\|\Lambda^\alpha\boldsymbol{\varphi}\|_{\mathbf{L}^{k_2}(\mathbf{D})} + \|\Lambda^\alpha\mathbf{Z}^\varepsilon\|_{\mathbf{L}^2(\mathbf{D})}\|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})}\Big),$$

In Lemma $IV.4$, we take $h=\mathbf{Z}^\varepsilon, j=2, l=\alpha, k=k_1$ then $\|\mathbf{Z}^\varepsilon\|_{L^{k_1}(\mathbf{D})}\leq C_1\|\Lambda^\alpha\mathbf{Z}^\varepsilon\|_{L^2(\mathbf{D})}$. Therefore,

$$\|\Lambda^\alpha(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})\|_{\mathbf{L}^2(\mathbf{D})}$$

$$\leq C\Big(C_1\|\Lambda^\alpha\mathbf{Z}^\varepsilon\|_{\mathbf{L}^2(\mathbf{D})}\|\Lambda^\alpha\boldsymbol{\varphi}\|_{\mathbf{L}^{k_2}(\mathbf{D})} + \|\Lambda^\alpha\mathbf{Z}^\varepsilon\|_{\mathbf{L}^2(\mathbf{D})}\|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})}\Big)$$

$$\leq C\|\Lambda^\alpha\mathbf{Z}^\varepsilon\|_{\mathbf{L}^2(\mathbf{D})}\Big(C_1\|\Lambda^\alpha\boldsymbol{\varphi}\|_{\mathbf{L}^{k_2}(\mathbf{D})} + \|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})}\Big)$$

$$\leq C_2$$

where the constants C_1, C_2 and C are independent of ε.

In the following, we are interested in studying the convergence of the term

$$\mathfrak{I}_\varepsilon := \int_\Theta \Lambda^\alpha\mathbf{Z}^\varepsilon\cdot\Lambda^\alpha(\mathbf{Z}^\varepsilon\times\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t.$$

Let $\mathfrak{I} := \int_\Theta \Lambda^\alpha\mathbf{Z}\cdot\Lambda^\alpha(\mathbf{Z}\times\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t$. We will show that $\mathfrak{I}_\varepsilon\to\mathfrak{I}$ as $\varepsilon\to 0$. For this, we introduce the commutator (see [17])

$$\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}) := \Lambda^\alpha(\mathbf{Z}\times\boldsymbol{\varphi}) - \boldsymbol{\varphi}\times\Lambda^\alpha\mathbf{Z}$$

in the first place, notice that $\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\in\mathbf{L}^2(Q)$. therefore, using (12) with $k_1=\infty, k_2=2, k_3=\frac{d}{\beta}$ and $k_4=\frac{2d}{d-2\beta}$ with $\beta=\alpha-1$ (note that we have $\dot{\mathbf{H}}^\beta(\mathbf{D})\hookrightarrow\mathbf{L}^{k_4}(\Omega)$ for the choice of k_4), we find

$$\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\|_{\mathbf{L}^2(\mathbf{D})}$$

$$\leq C_1\Big(\|\nabla\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})}\|\mathbf{Z}\|_{\dot{\mathbf{H}}^\beta(\mathbf{D})} + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,k_3}(\mathbf{D})}\|\mathbf{Z}\|_{\mathbf{L}^{k_4}(\mathbf{D})}\Big)$$

$$\leq C_1\Big(\|\nabla\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})}\|\mathbf{Z}\|_{\dot{\mathbf{H}}^\beta(\mathbf{D})} + C_2\|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,k_3}(\mathbf{D})}\|\mathbf{Z}\|_{\dot{\mathbf{H}}^\beta(\mathbf{D})}\Big)$$

$$\leq C\|\mathbf{Z}\|_{\mathbf{H}^\beta(\mathbf{D})}\Big(\|\nabla\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\mathbf{D})} + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,k_3}(\mathbf{D})}\Big)$$

$$\leq C'\|\mathbf{Z}\|_{\mathbf{H}^\beta(\mathbf{D})}.$$

Once again

$$\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon-\mathbf{Z})\|_{\mathbf{L}^2(\mathbf{D})}\leq C\|\mathbf{Z}^\varepsilon-\mathbf{Z}\|_{\mathbf{H}^\beta(\mathbf{D})}.$$

Therefore

$$\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon-\mathbf{Z})\|_{\mathbf{L}^2(\Theta)}\leq C\|\mathbf{Z}^\varepsilon-\mathbf{Z}\|_{L^2(0,T;\mathbf{H}^\beta(\mathbf{D}))}.$$

Finally, since $\Lambda^\alpha\mathbf{Z}\cdot(\Lambda^\alpha\mathbf{Z}\times\boldsymbol{\varphi})=0$ we have

$$\mathfrak{I}_\varepsilon = \textstyle\int_\Theta \Lambda^\alpha\mathbf{Z}^\varepsilon\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon)\ \mathrm{d}x\mathrm{d}t \quad \text{and}$$
$$\mathfrak{I} = \int_\Theta \Lambda^\alpha\mathbf{Z}\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\ \mathrm{d}x\mathrm{d}t.$$

Then

$$|\mathfrak{I}_\varepsilon-\mathfrak{I}|$$

$$= |\int_\Theta \Lambda^\alpha\mathbf{Z}^\varepsilon\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon)\ \mathrm{d}x\mathrm{d}t - \int_\Theta \Lambda^\alpha\mathbf{Z}\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\ \mathrm{d}x\mathrm{d}t|$$

$$= |\int_\Theta \Lambda^\alpha\mathbf{Z}^\varepsilon\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon-\mathbf{Z})\ \mathrm{d}x\mathrm{d}t + \int_\Theta \Lambda^\alpha(\mathbf{Z}^\varepsilon-\mathbf{Z})\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\ \mathrm{d}x\mathrm{d}t|$$

$$\leq \int_\Theta |\Lambda^\alpha\mathbf{Z}^\varepsilon\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon-\mathbf{Z})|\ \mathrm{d}x\mathrm{d}t + |\int_\Theta \Lambda^\alpha(\mathbf{Z}^\varepsilon-\mathbf{Z})\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\ \mathrm{d}x\mathrm{d}t|$$

$$\leq C\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z}^\varepsilon-\mathbf{Z})\|_{\mathbf{L}^2(\Theta)} + |\int_\Theta \Lambda^\alpha(\mathbf{Z}^\varepsilon-\mathbf{Z})\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\ \mathrm{d}x\mathrm{d}t|$$

$$\leq C'\|\mathbf{Z}^\varepsilon-\mathbf{Z}\|_{L^2(0,T;\mathbf{H}^\beta(\mathbf{D}))} + |\int_\Theta \Lambda^\alpha(\mathbf{Z}^\varepsilon-\mathbf{Z})\cdot\Gamma_{\boldsymbol{\varphi}}(\mathbf{Z})\ \mathrm{d}x\mathrm{d}t|$$

$$\to 0$$

Let $\varepsilon \to 0$ in (25) , we obtain

$$\begin{cases} \int_{\Theta} \mathbf{Z}_t \cdot (\mathbf{Z} \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t - \eta \int_{\Theta} \mathbf{Z}_t \cdot \varphi \, \mathrm{d}x\mathrm{d}t. \\ +\gamma \int_{\Theta} \Lambda^{\alpha} \mathbf{Z} \cdot \Lambda^{\alpha} (\mathbf{Z} \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t \\ +\gamma \int_{\Theta} \zeta_{ijkl} Z_j \epsilon_{kl}(\mathbf{u}) (\mathbf{Z} \times \boldsymbol{\varphi})_i \, \mathrm{d}x\mathrm{d}t = 0 \\ -\rho \int_{\Theta} \mathbf{u}_t \cdot \boldsymbol{\psi}_t \, \mathrm{d}x\mathrm{d}t + \int_{\Theta} \sigma_{ijkl} \epsilon_{ij}(\mathbf{u}) \epsilon_{kl}(\boldsymbol{\psi}) \, \mathrm{d}x\mathrm{d}t \\ +\frac{1}{2} \int_{\Theta} \zeta_{ijkl} Z_i Z_j \epsilon_{kl}(\boldsymbol{\psi}) \, \mathrm{d}x\mathrm{d}t + \int_{\Theta} \mathbf{h} \cdot \boldsymbol{\psi} \, \mathrm{d}x\mathrm{d}t = 0, \end{cases}$$

for all $\boldsymbol{\varphi} \in \mathbf{C}^{\infty}(\overline{\Theta})$ and $\boldsymbol{\psi} \in \mathbf{H}_0^1(\Theta)$. Note that , (11) can be easily get from (23). Then the Theorem III.2 is proved.

VI. Concluding remarks

In this study, a three-dimensional mathematical model is proposed to describe the magneto elastic interactions in order to obtain the global existence of weak solutions. A fractional generalization of the harmonic map heat flow and an evolution equation for the displacement are used to describe the model. Due to the non-local collinearities in the model and the special structure of the magnetization equation, we employed the commutator estimate and some calculus inequalities of fractional order to prove the convergence of the approximate solutions. We intend to pursue our investigation by studying this problem with a fractional in time derivative and to establish an existing result.

References

[1] F. Alouges and A. Soyeur. *On global weak solutions for Landau-Lifshitz equations: Existence and non uniqueness. Nonlinear Anal.*, 18:1071–1084, (1992)

[2] Y. Chen. *The weak solutions to the evolution problems of harmonic maps. Math. Z.*, 201(1):69–74, (1989)

[3] M. Chipot, I. Shafrir, V. Valente, and G. Vergara Caffarelli. *A nonlocal problem arising in the study of magneto-elastic interactions. Boll. Unione Mat. Ital. (9) 1,* no. 1, 197–221, (2008)

[4] M. Chipot, I. Shafrir, V. Valente, and G. Vergara Caffarelli. *On a hyperbolic-parabolic system arising in magnetoelasticity. J. Math. Anal. Appl. 352,* no. 1, 120–131, (2009)

[5] R. R. Coifman and Y. Meyer. *Nonlinear harmonic analysis, operator theory and P.D.E. Beijing Lectures in Harmonic Analysis, Princeton University Press, Ann. of Math. Stud.* pp.3-45, (1986)

[6] F. Demengel and G. Demengel. *Espaces Fonctionnels, utilisation dans la résolution des équations aux dérivées partielles. EDP Sciences, Les Ulis; CNRS Éditions,* Paris, (2007)

[7] I. Ellahiani, E.H. Essoufi and M. Tilioua. *Global existence of weak solutions to a fractional model in magnetoelastic interactions. Abstract and Applied Analysis,* Article ID 9238948 (2016)

[8] I. Ellahiani, E.H. Essoufi and M. Tilioua. *Global existence of weak solutions to a three-dimensional fractional model in magneto-viscoelastic interactions. Bound. Value Probl.,* Paper No. 122, 20 pp. (2017)

[9] L. C. Evans. *Partial differential equations, Grad. Stud. Math. vol 19, Amer. Soc. Providence,* RI (1998).

[10] B. Guo and M. Hong. *The weak solutions to the evolution problems of harmonic maps. Calc. Var. Partial Differential Equations* 1(3):311–334, (1993)

[11] B. Guo and M. Zeng. *Solutions for the fractional Landau-Lifshitz equation. J. Math. Anal. Appl,* 361(1): 131–138, (2010)

[12] T. Kato. *Liapunov Functions and Monotonicity in the Navier-Stokes Equations. Lecture Notes in Mathematics, 1450, Berlin: Springer-Verlag,* (1990)

[13] T. Kato and G. Ponce. *Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure Appl. Math.,* 41:891–907, (1988)

[14] J. L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. volume 1 Dunod,* Paris (1968)

[15] J. L. Lions. *Quelques Méthodes de Résolution des Problèmes aux limites Non linéaires. Dunod, Paris* (1969)

[16] F. Lin and C. Wang. *The analysis of harmonic maps and their heat flows. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ,* (2008)

[17] X. Pu and B. Guo. *The fractional Landau-Lifshitz-Gilbert equation and the heat flow of harmonic maps. Calc. Var.*, 42(1):1-19, (2011)

[18] X. Pu and B. Guo. *Well-posedness for the fractional Landau-Lifshitz equation without Gilbert damping. Calc. Var. Partial Differential Equations,* 46(3-4): 441–460, (2013)

[19] J. Simon. *Compact sets in the space $L^p(0, T; B)$. Ann. Math. Pura. Appl.,* 146:65–96, (1987)

[20] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, NJ: Princeton University Press,* (1970)

[21] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Applied Mathematical Sciences, 68. Springer-Verlag, New York,* (1997)

[22] V. Valente and G. Vergara Caffarelli. *On the dynamics of magneto-elastic interactions: existence of solutions and limit behaviors. Asymptotic Analysis* 51:319-333, (2007)

[23] A. Visintin. *On the Landau-Lifshitz equation for ferromagnetism. Japan J. Appl. Math.,* 2:69–84, (1985)

[24] Linna Li, Yuze Li and Jianke Zhang . *Fractional-order SS1IR Model for Predicting Public Opinion Dissemination in Social Networks. Engineering Letters.,* vol. 31, no.1, pp239-245, 2023

[25] Meiling Li, and Yunqing Wang . *Existence and Iteration of Monotone Positive Solutions for Fractional Boundary Value Problems with RieszCaputo Derivative. Engineering Letters.,* vol. 29, no.2, pp327-331,2021