On Multiparameter Spectral Theory of 3-Parameter Aeroelastic Flutter Problems

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***Abstract*—In this paper, we consider the aeroelastic flutter problem (AEFP) in terms of matrix equations. We provide a general framework on the spectral theory of three parameter AEFP in tensor product space under the auspices of the techniques of multiparameter eigenvalue problems (MEP) in matrix form. It is to noted that, the AEFP of an undamped system can be converted to a linear singular two parameter eigenvalue problem (2PEP) in double dimension and by quasi-linearization technique it can be converted to a linear three parameter eigenvalue problem (3PEP) of the same dimension. The de-facto way to find the numeric of MEP is by solving associated joint generalised eigenvalue problem (GEP) using conventional numerical method, provided the problem being nonsingular. Since the transformed version of AEFP in matrix form is singular, so the usual solution techniques for MEP can not be applied to address AEFP and it necessitates to adopt alternate numerical techniques for the same. In the current paper, it is intended to address singular AEFP under certain assumptions. The paper also serves as a report on the Kronecker product method, so called Delta method developed by Atkinson for finding finite eigenvalues of singular three parameter AEFP.**

***Index Terms*—Aeroelasticity, Aeroelastic flutter problem, three parameter eigenvalue problems, Kronecker Product, generalized eigenvalue problems.**

I. Introduction

AEROELASTIC flutter is a phenomenon that can occur in aircraft and other structures when the interaction between aerodynamic forces and structural dynamics causes self-sustained oscillations [5], [6], [30]. These oscillations can potentially lead to structural failure if not properly managed. The flutter problem arises when elastic deformation of structure couples with the aerodynamic forces acting on it. As the airflow passes over the structure, it induces vibrations that can become self-amplifying and unstable. This can cause the structure to vibrate violently or even break up. In the general study of fluid-structure interaction, Aeroelasticity is being seen as a subfield that is primarily concerned with the working fluid in air. Moreover, aeroelasticity is a field that combines the fields of aerodynamics with structural dynamics [12]. It is worth mentioning that, aeroelastic flutter problem is a critical design consideration in the aerospace sector. Understanding and managing this problem is extremely important to ensure the safety and performance of aircraft and other structures exposed to aerodynamic forces. The capacity to forecast and manage aeroelastic instability is one of the fundamental goals of modern aeroelasticity. In this case, instability is the event in which the structure in question becomes self-exiting, flutter in dynamic way (an-oscillatory) and divergence in static way (a nonoscillatory). Divergence is the subsidiary form of flutter, and we will use flutter to refer to instability. The term dynamic flutter will be used to describe flutter in oscillatory instability.

In linear aeroelastic system, the term flutter or divergence can be formulated for the stability criterion as

$$Im(\chi) > 0, \quad for \quad stability \tag{1}$$

where, χ are the time-eigenvalues of the system, transformed according to the fourier transform $h(t) = \bar{h}e^{i\chi t}$, for the system coordinate h. The eigenvalues in the top half plane of the argand plane are stable, whereas those in the lower half plane are unstable. When the system characteristics (airspeed, air density, and so on) are on the point of transitioning from stability to instability or vice versa, then flutter occurs i.e, when

$$Im(\chi) = 0 \tag{2}$$

In a given system, there are multiple flutter points, each of which is characterized by a modal frequency and air value, with the air density and other parameters being fixed. Flutter points are always ordered as air speed increases, with the first flutter point occurring at the lowest (positive) air speed value. As a result, dynamic flutter and divergence points are ordered individually, which either occur at negative air speed or irrelevant to frequency. Only the first flutter point and the first divergence point are commonly used in the industrial field. The generalised laplase transform method [4] can be used to find flutter points and for stability analysis of AEFP pseudospectral continuation approach is found in [8]. The solution method for nonlinear MEP which yields during the analysis of aeroelastic flutter are available in [7].

The current paper combines the theory of numerical linear algebra and aeroelastic stability analysis, and it offers a feasible ways to calculate useful aeroelastic stability parameters. It considers a multiparameter spectral theory-based approach for locating and assessing stability boundaries in parametric systems. The paper is organized as follows: Section II contains some basic definitions, which will be used in the section to follow. Section III contains the general framework of AEFP in terms of matrix equations. Similarly, in section IV linearization of AEFP are considered. In section V the structure of Singular 3-parameter AEFP is discussed and its numerical illustration is presented in section VI. Finally, in section VII a conclusion is drawn on the whole works.

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II. PRELIMINARIES

The notations and basic definitions presented in this section will be used throughout the paper. The set of real numbers and set of complex numbers will be denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Th symbol $\otimes$ stands for usual Kronecker product.

Definition 1: [15] The Kronecker Product ($\otimes$) for two matrices A and B is defined by $A \otimes B = a_{ij}B$, where a_{ij} are the elements of A in i^{th} row and j^{th} column.

Definition 2: [28] For any scalar $\lambda \in \mathbb{C}$ and nonzero vector x, the GEP is to find the pair (λ, x) that satisfy the matrix equation $Ax = \lambda Bx$, where A and B are the square matrices of same size over $\mathbb{C}$.

Definition 3: [19] The linear multiparameter eigenvalue problem is to find the scalars $\lambda_i \in \mathbb{C}$ and the corresponding non zero vectors $w_i \in \mathbb{C}^{n_i}$ form set of k coupled equations such that

$$(A_i - \sum_{j=1}^{k} \lambda_j A_{ij}) w_i = 0 \tag{3}$$

where $A_i, A_{ij} \in \mathbb{C}^{n_i \times n_i}$; $i, j = 1, \ldots, k$. The pair $(\lambda_1, \ldots, \lambda_k)$ is called eigenvalue, if for some λ_i the system (3) has a solution for $0 \neq w_i; i = 1, \ldots, k$. Then, the corresponding tensor product $w = w_1 \otimes w_2 \otimes \ldots, \otimes w_k$ is called the eigenvector (right). Linear 2PEP and 3PEP are particular case of the system (3) when $k = 1$ and $k = 2$ respectively.

III. AEFP IN MATRIX FORM

Pons et al., [5] transformed AEFP in the frequency domain into a 2PEP of the form given by

$$A(\chi, p)x = 0 \tag{4}$$

$$\bar{A}(\chi, p)\bar{x} = 0 \tag{5}$$

where, $\chi \in \mathbb{C}$ is a structural eigenvalue parameter, $p \in \mathbb{R}$ is aerodynamic parameter and $x \in \mathbb{C}$ is eigenvector and P is a matrix of order $n \times n$ matrix over the complex field $\mathbb{C}$. The problem is to find the p such that the imaginary part of eigenvalues χ being zero. The equivalent condition of the equation (2) is to exist solution under $\chi \in \mathbb{R}$ on the stability of the boundary. As the parameters χ, p are unaffected by the conjugation, this operation enforces these conditions. This procedure has been utilized in the analysis of delay differential equations [17], Hopf bifurcation prediction [18] and in aeroelastic or other structural stability problems [6]. Here section model form [5], [6] with Theodorsen aerodynamics is considered as an initial trial system for numerical experiments. This model has two degrees of freedom (2DOF) plunge h and twist θ. In time domain, its governing equations are

$$m\ddot{h} + d_h\dot{h} + k_h h - mx_\theta\ddot{\theta} = -L(t) \tag{6}$$

$$I_P\ddot{\theta} + d_\theta\dot{\theta} + k_\theta\theta - mx_\theta\ddot{h} = M(t) \tag{7}$$

where d_h and d_θ denotes section plunge and twist damping coefficients respectively. Similarly k_h and k_θ represents section plunge and twist stiffnesses respectively. m and I_P denotes section mass and polar moment of inertia and x_θ is the section's static imbalance. Taking Fourier transform, $[h(t), \theta(t)] = [\hat{h}, \hat{\theta}]e^{i\chi t}$ of this model yields the following equations

TABLE I: Value of dimensionless parameters

Sl.No.	Parameters	Value
1	Mass ratio - μ	$+20$
2	Radius of gyration - r	$+0.4899$
3	Bending damping - ς_h	$+1.4105$
4	Torsional damping - ς_θ	$+2.3508$
5	Bending nat. frequency - w_h	$+0.5642$ rad/s
6	Torsional nat. frequency - w_θ	$+1.4105$ rad/s
7	Static imbalance - r_θ	-0.1
8	Pivot point location - a	-0.2

$$(-m\chi^2 + id_h\chi + k_h)\hat{h} + mx_\theta\chi^2\hat{\theta} = L(\chi, \hat{h}, \hat{\theta}) \tag{8}$$

$$mx_\theta\chi^2\hat{h} + (-I_p\chi^2 + ld_\theta\chi + k_\theta)\hat{\theta} = M(\chi, \hat{h}, \hat{\theta}) \tag{9}$$

In the environment of frequency domain, Theodorsen's unsteady aerodynamic theory [12] is used to model the aerodynamic loads

$$L = -\chi^2(L_h\hat{h} + L_\theta\hat{\theta}) \tag{10}$$

$$M = \chi^2(M_h\hat{h} + M_\theta\hat{\theta}) \tag{11}$$

where $L_h, M_h, L_\theta, M_\theta$ are aerodynamic coefficients, which are complex functions of the reduced frequency k. Here k is an aerodynamic parameter related to the airspeed U and b is airfoil semichord given by

$$k := \frac{b\chi}{U}$$

Nondimensionalising (8) and (9), the flutter problem takes the following form [5].

$$\left(\left(M_0 + G_0 + G_1\frac{1}{k} + G_2\frac{1}{k^2}\right)\chi^2 - D_0\chi - K_0\right)x = 0 \tag{12}$$

where,

$G_0 = \frac{1}{\mu}\begin{pmatrix} 1 & a \\ a & (\frac{1}{8} + a^2) \end{pmatrix}$

$G_1 = \frac{1}{\mu}\begin{pmatrix} -2i & 2i(1-a) \\ -i(1+2a) & ia(1-2a) \end{pmatrix}$

$G_2 = \frac{1}{\mu}\begin{pmatrix} 0 & 2 \\ 0 & 1+2a \end{pmatrix}$; $M_0 = \begin{pmatrix} 1 & -r_\theta \\ -r_\theta & r^2 \end{pmatrix}$

$D_0 = \begin{pmatrix} 2i\varsigma_h\omega_h & 0 \\ 0 & 2ir^2\varsigma_\theta\omega_\theta \end{pmatrix}$; $K_0 = \begin{pmatrix} \omega_h^2 & 0 \\ 0 & r^2\omega_\theta^2 \end{pmatrix}$

For the section model, the value of dimensionless parameters μ, r, ς_h, ς_θ, w_h w_θ, r_θ and a are given in the Table III.

Equation (12) can be rearranged in terms of polynomial forms by introducing new eigenvalue parameters such as $\gamma := \frac{U}{b}$, $\tau := \frac{1}{k}$ and $\lambda := \frac{1}{\chi}$. In $\gamma - \chi$, this form becomes

$$\left((M_0 + G_0)\chi^2 + G_1\gamma\chi + G_2\gamma^2 - D_0\chi - K_0\right)x = 0 \tag{13}$$

In $\tau - \lambda$ form it becomes

$$\left((M_0 + G_0) + G_1\tau + G_2\tau^2 - D_0\lambda - K_0\lambda^2\right)x = 0 \tag{14}$$

For undamped system $D_0 = 0$

$$\left((M_0 + G_0) + G_1\tau + G_2\tau^2 - K_0\Lambda\right)x = 0 \tag{15}$$

where, $\Lambda = \lambda^2$.

IV. Linearization

Together with conjugate of the initial equation (15), the following system is considered.

$$\{(M_0+G_0)+\tau G_1+\tau^2 G_2-\Lambda K_0\}x=0 \quad (16)$$

$$\{(\bar{M_0}+\bar{G_0})+\tau\bar{G_1}+\tau^2\bar{G_2}-\Lambda\bar{K_0}\}\bar{x}=0 \quad (17)$$

which are all two parameter nonlinear eigenvalue problems. Such a problem consisting of several parameters can be converted to a linear one by various linearization process developed in [2]. After linearization, the system represented by the equations (16)-(17) results the following system

$$[A_1+\tau A_{11}+\Lambda A_{12}]\begin{pmatrix} x \\ \tau x \end{pmatrix}=0 \quad (18)$$

$$[A_2+\tau A_{21}+\Lambda A_{22}]\begin{pmatrix} \bar{x} \\ \bar{\tau}\bar{x} \end{pmatrix}=0 \quad (19)$$

where, $A_1=\begin{pmatrix} M_0+G_0 & 0 \\ 0 & -I_n \end{pmatrix}$,
$A_{11}=\begin{pmatrix} G_1 & G_2 \\ I_n & 0 \end{pmatrix}$, $A_{12}=\begin{pmatrix} -K_0 & 0 \\ 0 & 0 \end{pmatrix}$,
$A_2=\begin{pmatrix} \bar{M_0}+\bar{G_0} & 0 \\ 0 & -I_n \end{pmatrix}$, $A_{21}=\begin{pmatrix} \bar{G_1} & \bar{G_2} \\ I_n & 0 \end{pmatrix}$,
$A_{22}=\begin{pmatrix} -\bar{K_0} & 0 \\ 0 & 0 \end{pmatrix}$

The system (16)-(17) represents a linear 2-PEP of double dimension. This system can also be linearized to the same dimension using quasi-linearization approaches developed by Muhic el. al., [2], but this results in a 3-PEP. For that, we need to define a new eigenvalue parameter β satisfying the equation $\beta=\tau^2$, then the system (16)-(17) reduces to

$$\{(M_0+G_0)+\tau G_1+\beta G_2-\Lambda K_0\}x=0 \quad (20)$$

$$\{(\bar{M_0}+\bar{G_0})+\tau\bar{G_1}+\beta\bar{G_2}-\Lambda\bar{K_0}\}\bar{x}=0 \quad (21)$$

It is a linear 3-PEP having two linear equations only. A third equation, which can be derived from the nonlinear relation $\beta-\tau^2=0$, is necessary to convert it into a typical 3-PEP. In matrix form this relation reduces to

$$det\begin{pmatrix} \beta & \tau \\ \tau & 1 \end{pmatrix}=0 \quad (22)$$

Combining it with the system (16), AEFP of undamped system can be recast as

$$E_1(\lambda)x=0 \quad (23)$$

$$E_2(\lambda)\bar{x}=0 \quad (24)$$

$$E_3(\lambda)x_1=0 \quad (25)$$

where,
$E_1(\lambda)=(M_0+G_0)+\tau G_1+\beta G_2-\Lambda K_0$
$E_2(\lambda)=(M_0+G_0)+\tau G_1+\beta G_2-\Lambda K_0$
$E_3(\lambda)=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}+\tau\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+\beta\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Here, the problem is to find the 3-tuple $\lambda=(\tau,\beta,\Lambda)$ and is called eigenvalue and the corresponding tensor product $z=x\otimes\bar{x}\otimes x_1$ is called right eigenvector. Similarly, $y\otimes\bar{y}\otimes y_1$ is the left eigenvector of the system (23)-(25) if $0\neq y,\bar{y},y_1$, $y^*E_1(\lambda)=0$, $\bar{y}^*E_2(\lambda)=0$ and $y_1^*E_3(\lambda)=0$. The standard results of the problem of such kind are reported in the works of [1], [10], [11], [13], [19], [24], and the references therein. Numerical solutions of the problem are found in the works of [9], [16]. Converting the problem into joint GEPs in tensor product space of the form is the de facto way, known as Delta method [14] for spectral analysis of the problem (23)-(25) and is given by

$$\Delta_1 z=\tau\Delta_0 z \quad (26)$$

$$\Delta_2 z=\beta\Delta_0 z \quad (27)$$

$$\Delta_3 z=\Lambda\Delta_0 z \quad (28)$$

where $z=x\otimes\bar{x}\otimes x_1$ is decomposable tensor and each operator matrices Δ_i, i=0,1,2,3 is defined as follows

$$\Delta_0=\begin{vmatrix} G_1 & G_2 & -K_0 \\ \bar{G_1} & \bar{G_2} & -\bar{K_0} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{vmatrix}_\otimes \quad (29)$$

$$\Delta_1=\begin{vmatrix} M_0+G_0 & G_2 & -K_0 \\ \bar{M_0}+\bar{G_0} & \bar{G_2} & -\bar{K_0} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{vmatrix}_\otimes \quad (30)$$

$$\Delta_2=\begin{vmatrix} G_1 & M_0+G_0 & -K_0 \\ \bar{G_1} & \bar{M_0}+\bar{G_0} & -\bar{K_0} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{vmatrix}_\otimes \quad (31)$$

$$\Delta_3=\begin{vmatrix} G_1 & G_2 & M_0+G_0 \\ \bar{G_1} & \bar{G_2} & \bar{M_0}+\bar{G_0} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{vmatrix}_\otimes \quad (32)$$

The system is referred as singular or nonsingular according as the operator matrix Δ_0 specified in equation (29) is singular or nonsingular. The operator matrices $\Delta_0^{-1}\Delta_i$ for $i=1,2,3$ commute for nonsingular problem and the eigenvalues of the system (23)-(25) agree with the eigenvalues of joint GEPs of the types (26)-(28). Using the conventional numerical method available for GEPs [20], we can find the numerical solution for nonsingular problems using this relation. However, solving the problem with low-order matrices is more convenient. The major computational drawbacks are the cost of computing the operator matrices Δ_i, $i=0,1,2,3$ of size 8×8. Thus, it is necessary to adopt numerical algorithm to find the solution of the problem. The works [22], [23], [25], [26] contains more information on the numerical solutions of 3PEP.

While addressing aircraft aeroelasticity, the corresponding linear flutter problems are singular. Such a problem cannot be solved by transforming it into the corresponding joint GEPs. For the singular case, there are infinitely many eigenvalues that satisfy the equivalent systems of joint GEPs of the type (26)-(28), which makes computing appropriate eigenvalues of the problems challenging. The relationship between equations (23)-(25) and the joint GEP specified in equations (26)-(28) is less investigated for singular problems. In the extant literature, there are numerical techniques for computing some of the eigenvalues for singular problems, although they are mostly for the two-parameter case. Muhic and Plestenjak, [3] proposed a method for solving singular 2PEP by computing common regular eigenvalues of the appropriate system of two singular GEPs. In this case, if the eigenvalues of the system (23)-(25) are simple, then they agree with the finite regular eigenvalues of the system (26)-(28). Kosir and Plestenjak, [27] extended this link to general singular 2-PEPs with potentially many eigenvalues, allowing characteristic polynomials to have a nontrivial common factor. Muhic and Plestenjak, [2] also reported a numerical technique for singular 2PEPs, which included the linearization of the quadratic 2PEP and was based on Dooren's staircase algorithm [29] for computing the common regular component of 2PEPs and extracting the finite regular eigenvalues. However, for large-order matrices, this strategy is computationally inefficient. Hochstenbachet et. al., [21] provide another algorithm, which extends the Jacobi-Davidson type methods presented in their previous work [20] to the regular singular problem.

V. Singular 3-parameter AEFP

The system (23) defines a 3-parameter AEFP that is singular. As a result of the lack of a spectral theory for singular problems, extracting all eigenvalues of the problem becomes challenging. However, under certain assumptions, a linear substitution of parameters τ, β, Λ can change a singular problem into a nonsingular one [2]. Let $\rho_i,\ i = 0, 1, 2, 3$ be any scalars. Consider the homogeneous formulation of the problem

$$\{\eta_0 \left(M_0 + G_0\right) + \eta_1 G_1 + \eta_2 G_2 - \eta_3 K_0\}x = 0 \tag{33}$$

$$\{\eta_0 \left(M_0 + G_0\right) + \eta_1 G_1 + \eta_2 G_2 - \eta_3 K_0\}\,\bar{x} = 0 \tag{34}$$

$$\left\{\eta_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \eta_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \eta_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\} x_1 = 0 \tag{35}$$

such that $(\eta_0, \eta_1, \eta_2, \eta_3) \neq (0, 0, 0, 0)$. Then the system (33) is nonsingular if there exists linear combination $\Delta = \rho_0\Delta_0 + \rho_1\Delta_1 + \rho_2\Delta_2 + \rho_3\Delta_3$ such that $det(\Delta) \neq 0$ for all η_i, i=0,1,2,3 are the eigenvalues of the joint $\mathbb{GEP}$s

$$\Delta_0 z = \eta_0 \Delta z, \Delta_1 z = \eta_1 \Delta z, \Delta_2 z = \eta_2 \Delta z, \Delta_3 z = \eta_3 \Delta z \tag{36}$$

and the eigenvalues $(\eta_0, \eta_1, \eta_2, \eta_3)$ gives finite eigenvalues $(\tau, \beta, \Lambda) = (\frac{\eta_1}{\eta_0}, \frac{\eta_2}{\eta_0}, \frac{\eta_3}{\eta_0})$ of 3-parameter AEFP provided $\eta_0 \neq 0$.

VI. Numerical Illustrations

Consider the undamped section model presented in [6] of the system (16)-(17). Substituting the value of dimensionless parameters as per Table 1, the system (23)-(25) reduces it to the system (37)-(39).

$$E_1(\lambda) = \begin{pmatrix} 1.05 & 0.09 \\ 0.09 & 0.2483 \end{pmatrix} + \tau \begin{pmatrix} -0.1i & 0.12i \\ -0.03i & -0.014i \end{pmatrix} + \beta \begin{pmatrix} 0 & 0.1 \\ 0 & 0.03 \end{pmatrix} + \Lambda \begin{pmatrix} -0.3183 & 0 \\ 0 & -0.1354 \end{pmatrix} \tag{37}$$

$$E_2(\lambda) = \begin{pmatrix} 1.05 & 0.09 \\ 0.09 & 0.2483 \end{pmatrix} + \tau \begin{pmatrix} 0.1i & -0.12i \\ 0.03i & 0.014i \end{pmatrix} + \beta \begin{pmatrix} 0 & 0.1 \\ 0 & 0.03 \end{pmatrix} + \Lambda \begin{pmatrix} -0.3183 & 0 \\ 0 & -0.1354 \end{pmatrix} \tag{38}$$

TABLE II: Eigenvalues and their corresponding eigenvectors

$(\eta_0, \eta_1, \eta_2, \eta_3)$	(τ, β, Λ)
$(0.0597 + 0.0273i, -0.0522 - 0.2646i, -0.7573 + 0.8094i, -0.0131 + 0.0059i)$	$(-2.3994 - 3.3349i, -5.3637 + 16.0105i, -0.1441 + 0.1647i)$
$(0.0597 - 0.0273i, -0.0522 + 0.2646i, -0.7573 - 0.8094i, -0.0131 - 0.0059i)$	$(-2.3994 + 3.3349i, -5.3637 - 16.0105i, -0.1441 - 0.1647i)$
$(-0.0559, -0.1709, -0.5218, -0.0108)$	$(3.0572, 9.3345, 0.1932)$
$(-0.1914, -0.2383, -0.2968, -0.0898)$	$(1.2450, 1.5507, 0.4692)$
$(-0.1809, 0.0076, -0.0003, -0.6017)$	$(-0.0420, 0.0017, 3.326)$
$(59.1967, 0, 0, 29.0984)$	$(0, 0, 0.4916)$
$(0, 0, -1, 0)$	$infinite$
$(-0.1769, 0, 0, -0.5884)$	$(0, 0, 3.3262)$

TABLE III: Values of parameters λ and τ

Sl.No.	λ	τ
1	$-0.1933 - 0.4260i$	$-2.3994 - 3.3349i$
2	$-0.1933 + 0.4260i$	$-0.7573 - 0.8094i$
3	$0.4395 + 0.0000i$	$+3.05725 + 0.0000i$
4	$0.6850 + 0.0000i$	$+1.2450 + 0.0000i$
5	$1.8237 + 0.0000i$	$-0.0420 + 0.0000i$
6	$0.7011 + 0.0000i$	0
7	$1.8238 + 0.0000i$	0

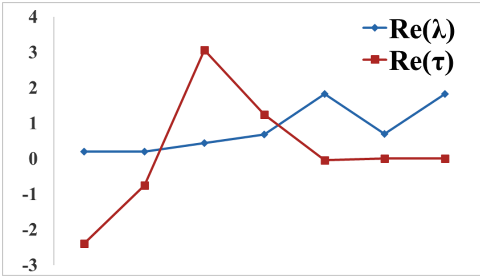

Fig. 1: λ-τ curve of undamped section model

$$E_3(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{39}$$

All the computations are performed in the environment of MATLAB $R2013a$ with Windows 8.1 operation system, Intel(R). Here, the calculated Δ_0 defined in (40) is a complex matrix of size 8×8 and is singular. Thus, the 3-parameter AEFP has 8 eigenvalues. Consider $\rho_0 = 1, \rho_1 = -3, \rho_2 = 1, \rho_3 = -2$. Then the corresponding Δ matrix involved in (36) is nonsingular.

$$\Delta_0 = \begin{pmatrix} -0.0637i & 0 & 0.0382i & 0.0318 & 0.0382i & -0.0318 & 0 & 0 \\ 0 & 0 & 0.0318 & 0 & -0.0318 & 0 & 0 & 0 \\ -0.0096i & 0 & -0.0180i & 0.0095 & 0 & 0 & 0.0162i & -0.0135 \\ 0 & 0 & 0.0096 & 0 & 0 & 0 & -0.0135 & 0 \\ -0.0096i & 0 & 0 & -0.0180i & -0.0096 & 0.0162i & 0.0135 & \\ 0 & 0 & 0 & 0 & -0.0096 & 0 & 0.0135 & 0 \\ 0 & 0 & -0.0041 & 0 & -0.0041 & 0 & -0.0038i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

VII. Conclusion

In this paper, a general framework for 3-parameter AEFP has been presented and analysed the stability boundaries in parametric system using the spectral theory of MEP. Because the problem is singular, the general Delta approach adopted by Atkinson [14] cannot be used to solve it. Only finite eigenvalues have been calculated by transforming the problem into a nonsingular one and then a nonsingular linear combination is used to find the solution of the problem. Furthermore, finding general solutions for the singular problem opens up new possibilities for developing new direct ways to address the singular k-parameter problem. This could be viewed as a promising future direction of research in the study of general k-parameter eigenvalue problem.

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