Generalization and Sharpening of Zygmund-type Inequality for Polar Derivative of a Polynomial

Reingachan N , Maisnam Triveni Devi , Barchand Chanam

Abstract—**In this paper, we sharpen some of the known results of polar derivative of a polynomial by establishing the L^q-version of a known inequality on the polar derivative of a polynomial. Our result generalizes as well as improves upon some well-known polynomial inequalities in this direction.**

Index Terms—**polynomial, polar derivative, integral inequalities, maximum modulus.**

I. Introduction

Let $p(z)$ be a polynomial of degree n. Then, according to a famous well-known classical result due to Bernstein [6],

$$\max_{|z|=1}|p^{'}(z)| \leq n\max_{|z|=1}|p(z)|. \tag{1}$$

Inequality (1) is sharp and equality holds if $p(z)$ has all its zeros at the origin.
If $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then Erdös conjectured and later Lax [19] verified that

$$\max_{|z|=1}|p^{'}(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

Inequality (2) is best possible and equality holds for $p(z)=a+bz^n$, where $|a|=|b|$.

For the class of polynomials $p(z)$ of degree n not vanishing in $|z|<k$, $k\geq 1$, Malik [20] proved

$$\max_{|z|=1}|p^{'}(z)| \leq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{3}$$

Next, Bidkham and Dewan [7] generalized inequality (3) and obtained

Theorem 1. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for $1\leq R\leq k$,*

$$\max_{|z|=R}|p^{'}(z)| \leq \frac{n(R+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|p(z)|. \tag{4}$$

The result is best possible and equality in (4) holds for $p(z)=\left(\frac{z+k}{1+k}\right)^n$.

Aziz and Zargar [5] considered the class of polynomials $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, not vanishing in $|z|<k$, $k\geq 1$, and proved the following generalization of Theorem 1.

Theorem 2. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for $0<r\leq R\leq k$,*

$$\max_{|z|=R}|p^{'}(z)| \leq \frac{nR^{\mu-1}(R^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}\max_{|z|=r}|p(z)|. \tag{5}$$

The result is best possible and equality in (5) holds for $p(z)=(z^n+k^n)^{\frac{n}{\mu}}$, where n is a multiple of μ.

As an improvement and generalization of Theorem 2, Aziz and Shah [4] proved

Theorem 3. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for $0<r\leq R\leq k$,*

$$\max_{|z|=R}|p^{'}(z)| \leq \frac{nR^{\mu-1}(R^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}\times \left\{\max_{|z|=r}|p(z)|-\min_{|z|=k}|p(z)|\right\}. \tag{6}$$

The result is best possible and equality in (6) holds for $p(z)=(z^n+k^n)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Further, by involving some of the coefficients of the polynomial, Chanam and Dewan [9] obtained an improvement of Theorem 3.

Theorem 4. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k, k>0$, then for $0<r\leq R\leq k$,*

$$\max_{|z|=R}|p^{'}(z)| \leq n\times \left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)}\right\}$$
$$\times exp\left\{n\int_r^R\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\}$$
$$\times\left\{\max_{|z|=r}|p(z)|-\min_{|z|=k}|p(z)|\right\}. \tag{7}$$

For a polynomial $p(z)$ of degree n, we now define the polar derivative of $p(z)$ with respect to a real or complex number α as

$$D_\alpha p(z)=np(z)+(\alpha-z)p^{'}(z).$$

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This polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p^{'}(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p^{'}(z),$$

uniformly with respect to z for $|z|\leq R$, $R>0$.

Aziz [2] was the first to extend some of the above inequalities to polar derivative. He, in fact, extended inequality (3) to polar derivative by proving that, if $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, and for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\leq n\left(\frac{|\alpha|+k}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{8}$$

Dividing both sides of inequality (8) by $|\alpha|$ and letting $|\alpha|\to\infty$, we obtain inequality (3).

Over the last four decades, a large number of results concerning the polar derivative of polynomials was obtained by many different authors. More information on classical results and polar derivatives can be found in the books of Milovanović et al. [23], Rahman and Schmeisser [31] and Marden [21]. We can also see in the literature (for example, refer [11], [16], [18], [22], [25], [26], [27], [28], [33], [34], [35]) the latest research and development in this direction.

If we examine inequalities (2) due to Erdös-Lax [19] onwards to inequality (7) of Theorem 4, it is concluded that these inequalities give upper bound estimates of the maximum modulus of the ordinary derivative of a polynomial on a bigger circle in terms of the maximum modulus of the polynomial itself on a smaller circle, where both the circles are prescribed on the zero free open disc and its boundary. Similar further extensions for the polar derivative of a polynomial were made by Dewan and Singh [13] by extending Theorems 2 and 3 into polar derivative as follows.

Theorem 5. *If $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for every real or complex number α with $|\alpha|\geq R$ and $0<r\leq R\leq k$,*

$$\max_{|z|=R}|D_\alpha p(z)|\leq\frac{n(R^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}\times (k^\mu+|\alpha|R^{\mu-1})\max_{|z|=r}|p(z)|. \tag{9}$$

Theorem 6. *If $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for every real or complex number α with $|\alpha|\geq R$ and $0<r\leq R\leq k$,*

$$\max_{|z|=R}|D_\alpha p(z)|\leq\frac{n(R^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}\times \Bigg[(k^\mu+|\alpha|R^{\mu-1})\max_{|z|=r}|p(z)|- \left\{(k^\mu+|\alpha|R^{\mu-1})-\frac{(r^\mu+k^\mu)^{\frac{n}{\mu}}}{(R^\mu+k^\mu)^{\frac{n}{\mu}-1}}\right\}\min_{|z|=k}|p(z)|\Bigg]. \tag{10}$$

Similarly, Bidkham et al.[8] extended Theorem 4 to polar derivative and proved

Theorem 7. *If $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for every real or complex number α with $|\alpha|\geq R$ and $0<r\leq R\leq k$,*

$$\max_{|z|=R}|D_\alpha p(z)|\leq\frac{n}{1+s_0^{'}(\mu)}\Bigg[\left(\frac{|\alpha|}{R}+s_0^{'}(\mu)\right)\times exp\left\{n\int_r^R A_t dt\right\}\max_{|z|=r}|p(z)| +\left(s_0^{'}(\mu)+1-\left(\frac{|\alpha|}{R}+s_0^{'}(\mu)\right)exp\left\{n\int_r^R A_t dt\right\}\right)m\Bigg] \tag{11}$$

where $m=\min_{|z|=k}|p(z)|$,

$$A_t=\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}, \tag{12}$$

and

$$s_0^{'}(\mu)=\left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{(\frac{\mu}{n})\frac{|a_\mu|Rk^{\mu-1}}{|a_0|-m}+1}{(\frac{\mu}{n})\frac{|a_\mu|k^{\mu+1}}{(|a_0|-m)R}+1}\right\}. \tag{13}$$

It was Zygmund [36] who extended Bernstein's inequality (1) into L^q version for $q\geq 1$, whereas for $0<q<1$ was proved by Arestov [1] that

$$\left\{\int_0^{2\pi}|p^{'}(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq n\left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}.$$

Inequality (2) due to Erdös and Lax [19] was extended into L^q-setting by de-Brujin [10] for $q\geq 1$ and Rahman [30] for $0<q<1$ by establishing

$$\left\{\int_0^{2\pi}|p^{'}(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq\frac{n}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}\times \left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{14}$$

Later, similar extension into L^q analogue of inequality (3) due to Malik [20] was made by Gardner and Weems [15] and independently by Rather [32] and proved

$$\left\{\int_0^{2\pi}|p^{'}(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq\frac{n}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|k+e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}\times \left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{15}$$

It is of interest to obtain L^q inequalities for the polar derivative of a polynomial. In this direction, for the first time

Govil et al. [17] generalized inequality (14) due to de-Brujin [10] and Rahman [30] for polar derivative version by proving

Theorem 8. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for $q\geq 1$ and for every real or complex number α with $|\alpha|\geq 1$,*

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq n(|\alpha|+1)F_q\times$$

$$\left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},$$

where

$$F_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}|1+e^{i\alpha}|^q d\alpha\right\}^{\frac{-1}{q}}.$$

Next, Aziz et al. [3] extended inequality (15) for the polar derivative as

Theorem 9. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, then for $k\geq 1$, then for $q\geq 1$ and for every real or complex number α with $|\alpha|\geq 1$,*

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq n(|\alpha|+k)G_q$$

$$\left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},$$

where

$$G_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}|k+e^{i\alpha}|^q d\alpha\right\}^{\frac{-1}{q}}.$$

Recently, improved bounds of Theorem 9 were proved by Milovanović and Mir [22].

Very recently, Theorems 5 and 6 were extended to L^q analogue by Maisnam et al. [12] as follows.

Theorem 10. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for every $q>0$ and for every real or complex number α with $|\alpha|\geq R$, and for $0<r\leq R\leq k$,*

$$\left\{\int_0^{2\pi}|D_\alpha p(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq$$

$$\frac{n\left\{\left(\frac{k}{R}\right)^\mu+\frac{|\alpha|}{R}\right\}}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|\left(\frac{k}{R}\right)^\mu+e^{i\gamma}|^q d\gamma\right\}^{\frac{1}{q}}}$$

$$\times\left[\int_0^{2\pi}\left\{|p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^\mu+k^\mu}M(p,t)dt\right\}^q d\theta\right]^{\frac{1}{q}},\tag{16}$$

where

$$M(p,t)=\max_{|z|=t}|p(z)|.$$

Theorem 11. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for every $q>0$ and for every real or complex numbers α, β with $|\alpha|\geq R$, $|\beta|<1$ and for $0<r\leq R\leq k$,*

$$\left\{\int_0^{2\pi}\left|D_\alpha p(Re^{i\theta})+n\beta m\right|^q d\theta\right\}^{\frac{1}{q}}\leq$$

$$\frac{n\left\{\left(\frac{k}{R}\right)^\mu+\frac{|\alpha|}{R}\right\}}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|\left(\frac{k}{R}\right)^\mu+e^{i\gamma}|^q d\gamma\right\}^{\frac{1}{q}}}$$

$$\times\left[\int_0^{2\pi}\left\{|p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{k^\mu+t^\mu}M(p,t)dt\right.\right.$$

$$\left.\left.-\int_r^R\frac{nt^{\mu-1}}{k^\mu+t^\mu}mdt-|\beta|m\right\}^q d\theta\right]^{\frac{1}{q}},\tag{17}$$

where $M(p,t)=\max\limits_{|z|=t}|p(z)|$, $m=\min\limits_{|z|=k}|p(z)|$.

II. Lemmas

We need the following lemmas to prove our theorem.

Lemma 12. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then*

$$\max_{|z|=1}|p^{'}(z)|\leq\frac{n}{1+s_0(\mu)}\left\{\max_{|z|=1}|p(z)|-m\right\},\tag{18}$$

where $m=\min\limits_{|z|=k}|p(z)|$

and

$$s_0(\mu)=k^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}+1}\right\}.$$

The above lemma is due to Gardner et al. [14].

Lemma 13. *If $p(z)=a_0+\sum\limits_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for any complex number α with $|\alpha|\geq 1$, and for $q>0$*

$$\left\{\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq n(|\alpha|+T_0(\mu))C_\gamma(T_0(\mu))\times$$

$$\left\{\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},\tag{19}$$

where

$$T_0(\mu)=k^{\mu+1}\left\{\frac{(\frac{\mu}{n})\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{(\frac{\mu}{n})\frac{|a_\mu|}{|a_0|}k^{\mu+1}+1}\right\}$$

and

$$C_\gamma(T_0(\mu))=\left\{\frac{1}{2\pi}\int_0^{2\pi}|T_0(\mu)+e^{i\beta}|^q d\beta\right\}^{\frac{-1}{q}}.$$

This lemma is due to Mir and Ahmad [24].

Lemma 14. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zero in* $|z| < k, k > 0$, *then for* $0 < r \le R \le k$,

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + n \times \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}} \{M(p,t)-m\}\, dt, \quad (20)$$

and

$$M(p,r) + n \times \left[\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)} \{M(p,t)-m\}\, dt\right]$$

$$\le \{M(p,r)-m\} \times exp\left\{n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\} +m, \quad (21)$$

where $m = \min_{|z|=k}|p(z)|$, $M(p,t) = \max_{|z|=t}|p(z)|$ and $M(p,r) = \max_{|z|=r}|p(z)|$.

Proof: Since $p(z)$ has no zero in $|z| < k, k > 0$, then for $0 < t \le k, P(z) = p(tz)$ has no zero in $|z| < \frac{k}{t}, \frac{k}{t} \ge 1$. Thus on using Lemma 12 to $|p(z)|$, we have

$$\max_{|z|=1}|p'(z)| \le \frac{n}{1+(\frac{k}{t})^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}t^\mu(\frac{k}{t})^{\mu-1}+1}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}t^\mu(\frac{k}{t})^{\mu+1}+1}\right\}}$$

$$\times\left\{\max_{|z|=1}|p(z)| - \min_{|z|=\frac{k}{t}}|P(z)|\right\}$$

where

$$m = \min_{|z|=\frac{k}{t}}|P(z)| = \min_{|z|=\frac{k}{t}}|p(tz)| = \min_{|z|=k}|p(z)|.$$

Which gives

$$t\max_{|z|=t}|p'(z)| \le n\times \left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\frac{k^{\mu+1}}{t}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\frac{k^{\mu+1}}{t}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\frac{k^{2\mu}}{t^\mu}+\frac{k^{\mu+1}}{t^{\mu+1}}}\right\} \times\left\{\max_{|z|=1}|p(tz)|-m\right\},$$

which is equivalent to

$$\max_{|z|=t}|p'(z)| \le n\times \left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}}\right\} \times\left\{\max_{|z|=t}|p(z)|-m\right\}. \quad (22)$$

Now, for $0 < r \le R \le k$ and $0 \le \theta < 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| \le \int_r^R |p'(te^{i\theta})|dt$$

which implies

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})|dt. \quad (23)$$

Since

$$\int_r^R |p'(te^{i\theta})|dt \le \int_r^R \max_{|z|=t}|p'(z)|dt,$$

using inequality (22) in (23), we get the first inequality (20) of Lemma 14.

Further, taking maximum over θ in inequality (20), we have

$$\max_{|z|=R}|p(z)| \le \max_{|z|=r}|p(z)| + n\times \int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1}+t^\mu}{t^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu+k^{2\mu}t)+k^{\mu+1}} \times\{M(p,t)-m\}\,dt. \quad (24)$$

Now, let us denote the right hand side of inequality (24) by $\phi(R)$. Then

$$\phi'(R) = n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}\right\} \times\{M(p,R)-m\}. \quad (25)$$

Using $M(p,R) \le \phi(R)$, equality (25) can be written as

$$\phi'(R) - n\{A_R\}\times\{\phi(R)-m\} \le 0. \quad (26)$$

where $A_R = \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^{\mu-1}+R^\mu}{R^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}R^\mu+k^{2\mu}R)+k^{\mu+1}}$.
Multiplying both sides of (26) by $exp\{-n\int A_R dR\}$, we get

$$\frac{d}{dR}\left[\{\phi(R)-m\}\,exp\left\{-n\int A_R dR\right\}\right] \le 0. \quad (27)$$

It is concluded from (27) that the function

$$\{\phi(R)-m\}\,exp\left\{-n\int A_R dR\right\}$$

is a non-increasing function of R in $(0,k]$.
Hence for $0 < r \le R \le k$,

$$\{\phi(r)-m\}\,exp\left\{-n\int A_t dr\right\} \ge \{\phi(R)-m\}\,exp\left\{-n\int A_t dR\right\},$$

where A_t is as defined in (12),
which is equivalent to

$$\{\phi(r)-m\}\,exp\left\{n\int_r^R A_t dt\right\} \ge \{\phi(R)-m\}. \quad (28)$$

Since $\phi(r) = M(p,r)$ and using the value of $\phi(R)$ in (28), we get

$$M(p,r) + n\left[\int_r^R A_t\{M(p,t)-m\}\,dt\right]$$

$$\leq \{M(p,r)-m\}\, exp\left\{n\int_r^R A_t dt\right\}+m.$$

This completes the proof of inequality (21) of Lemma 14. ■

The following lemma is due to Govil and Kumar [16].

Lemma 15. *If $a \geq 1, b \geq c \geq 1$, and $q>0$, then*

$$\frac{a+b}{\left\{\int_0^{2\pi}|e^{i\theta}+b|^q d\theta\right\}^{\frac{1}{q}}} \leq \frac{a+c}{\left\{\int_0^{2\pi}|e^{i\theta}+c|^q d\theta\right\}^{\frac{1}{q}}}. \tag{29}$$

Lemma 16. *If $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z|<k, k>0$, then*

$$\frac{\mu}{n}\frac{|a_\mu|k^\mu}{|a_0|-m} \leq 1, \tag{30}$$

where $m=\min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [14].

Lemma 17. *If $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z|<k, k>0$, then $0<R\leq k$,*

$$\frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu}+k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^\mu+R^{\mu+1}} \geq 1. \tag{31}$$

Proof: Since $p(z)\neq 0$ in $|z|<k, k>0$, then for $0<R\leq k$, the polynomial $P(z)=p(Rz)\neq 0$ in $|z|<\frac{k}{R}, \frac{k}{R}\geq 1$. Applying Lemma 16 to the polynomial $P(z)$, we have

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^\mu \leq 1. \tag{32}$$

Since $R\leq k$, we have

$$0 \leq R^\mu k - Rk^\mu \leq k^{\mu+1}-R^{\mu+1}. \tag{33}$$

Multiplying (32) and (33) sidewise, we have

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^\mu(R^\mu k-Rk^\mu) \leq (k^{\mu+1}-R^{\mu+1}),$$

which is equivalent to (31) and the proof of Lemma 17 is completed. ■

Lemma 18. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k, k>0$, then*

$$|p(z)| \geq m \qquad \text{for } |z|\leq k, \tag{34}$$

where $m=\min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [14].

Lemma 19. *The function*

$$g(x)=k^{t+1}\left\{\frac{\frac{t}{n}\frac{|a_t|}{x}k^{t-1}+1}{\frac{t}{n}\frac{|a_t|}{x}k^{t+1}+1}\right\} \tag{35}$$

where $k\geq 1, t>0, n\in\mathbb{N}$, is a non-decreasing function of $x>0$.

Proof: The proof follows simply by first derivative test. ■

III. Main Result

In this paper, we generalize and strengthen some of the previously mentioned inequalities by establishing the L^q inequality of Theorem 7, where the value of k is also extended from $k\geq 1$ to $k>0$. Moreover, our result reduces to several interesting generalisations and improvements of known inequalities in this direction. More precisely, we prove

Theorem 20. *If $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n$, is a polynomial of degree n having no zero in $|z|<k$, $k>0$, then for every $q>0$ and for every real or complex numbers α, β with $|\alpha|\geq R$, $|\beta|<1$ and for $0<r\leq R\leq k$,*

$$\left\{\int_0^{2\pi}|D_\alpha p(Re^{i\theta})+n\beta m|^q d\theta\right\}^{\frac{1}{q}} \leq n\left(\frac{|\alpha|}{R}+s_0^{'}(\mu)\right)\times$$

$$C_\gamma(s_0^{'}(\mu))\left[\int_0^{2\pi}\left\{|p(re^{i\theta})|+n\int_r^R A_t \times\right.\right.$$

$$\left.\left.\{M(p,t)-m\}\,dt-|\beta|m\right\}^q d\theta\right]^{\frac{1}{q}} \tag{36}$$

where $M(p,t)=\max_{|z|=t}|p(z)|$, $m=\min_{|z|=k}|p(z)|$, A_t and $s_0^{'}(\mu)$ is as defined in (12) *and* (13) *and*

$$C_\gamma(s_0^{'}(\mu))=\left\{\frac{1}{2\pi}\int_0^{2\pi}|s_0^{'}(\mu)+e^{i\gamma}|^q d\gamma\right\}^{\frac{-1}{q}}. \tag{37}$$

Proof: Since the polynomial $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n$, has no zero in $|z|<k$, $k>0$, therefore, for every real or complex number β with $|\beta|<1$, by Rouche's Theorem, the polynomial $p(z)+\beta m$, where $m=\min_{|z|=k}|p(z)|$, has no zero in $|z|<k$, $k>0$. Let $0<r\leq R\leq k$, then the polynomial $P(z)=p(Rz)+\beta m$ has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\geq 1$, and hence applying Lemma 13 with $\delta=\frac{\alpha}{R}$ such that $\frac{|\alpha|}{R}\geq 1$,

$$\left\{\int_0^{2\pi}|D_{\frac{\alpha}{R}}\left\{p(Re^{i\theta})+\beta m\right\}|^q d\theta\right\}^{\frac{1}{q}} \leq n\left\{\frac{|\alpha|}{R}+S_R(\mu)\right\}$$

$$\times\, C_\gamma(S_R(\mu))\left\{\int_0^{2\pi}|p(Re^{i\theta})+\beta m|^q d\theta\right\}^{\frac{1}{q}}, \tag{38}$$

where

$$S_R(\mu)=\left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0-\lambda m|}\left(\frac{k}{R}\right)^{\mu-1}+1}{\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0-\lambda m|}\left(\frac{k}{R}\right)^{\mu+1}+1}\right\}$$

and

$$C_\gamma(S_R(\mu))=\left\{\frac{1}{2\pi}\int_0^{2\pi}|S_R(\mu)+e^{i\gamma}|^q d\gamma\right\}^{\frac{-1}{q}}.$$

Using Lemma 18, $|p(z)|>m$ for $|z|<k$, i.e., in particular, $|a_0|>m$. Since $|\lambda|<1$, we have $|\lambda|m<m<|a_0|$, and therefore

$$|a_0-\lambda m| \geq |a_0|-|\lambda|m > |a_0|-m.$$

Using the fact of Lemma 19, we have $S_R(\mu) \geq s_0^{'}(\mu)$, where

$$s_0^{'}(\mu) = \left(\frac{k}{R}\right)^{\mu+1} \left\{ \frac{\frac{\mu}{n}\frac{|a_\mu| R k^{\mu-1}}{|a_0|-m} + 1}{\frac{\mu}{n}\frac{|a_\mu| k^{\mu+1}}{(|a_0|-m)R} + 1} \right\}, \tag{39}$$

and by Lemma 17, $s_0^{'}(\mu) \geq 1$.
Now using the fact of Lemma 15, (38) becomes

$$\left\{\int_0^{2\pi} |D_{\frac{\alpha}{R}}\left\{p(Re^{i\theta}) + \beta m\right\}|^q d\theta\right\}^{\frac{1}{q}} \leq n\left\{\frac{|\alpha|}{R} + s_0^{'}(\mu)\right\} C_\gamma(s_0^{'}(\mu)) \times \left\{\int_0^{2\pi} |p(Re^{i\theta}) + \beta m|^q d\theta\right\}^{\frac{1}{q}}. \tag{40}$$

Since

$$\begin{aligned} D_{\frac{\alpha}{R}} p(Re^{i\theta}) &= n\left\{p(Re^{i\theta}) + \beta m\right\} \\ &+ (\tfrac{\alpha}{R} - e^{i\theta}) R p^{'}(Re^{i\theta}) \\ &= D_\alpha p(Re^{i\theta}) + n\beta m, \end{aligned}$$

(40) is equivalent to

$$\left\{\int_0^{2\pi} |D_\alpha p(Re^{i\theta}) + n\beta m|^q d\theta\right\}^{\frac{1}{q}} \leq n\left\{\frac{|\alpha|}{R} + s_0^{'}(\mu)\right\} C_\gamma(s_0^{'}(\mu)) \times \left\{\int_0^{2\pi} |p(Re^{i\theta}) + \beta m|^q d\theta\right\}^{\frac{1}{q}}. \tag{41}$$

Now, we choose the argument of β suitably such that

$$|p(Re^{i\theta}) + \beta m| = |p(Re^{i\theta})| - |\beta| m. \tag{42}$$

If we use equality (42) in the right hand side of (41), we get

$$\left\{\int_0^{2\pi} |D_\alpha p(Re^{i\theta}) + n\beta m|^q d\theta\right\}^{\frac{1}{q}} \leq n\left\{\frac{|\alpha|}{R} + s_0^{'}(\mu)\right\} C_\gamma(s_0^{'}(\mu)) \times \left\{\int_0^{2\pi} \left[|p(Re^{i\theta})| + |\beta| m\right]^q d\theta\right\}^{\frac{1}{q}}. \tag{43}$$

By using inequality (20) of Lemma 14 in (43), we have

$$\left\{\int_0^{2\pi} |D_\alpha p(Re^{i\theta}) + n\beta m|^q d\theta\right\}^{\frac{1}{q}} \leq n\left\{\frac{|\alpha|}{R} + s_0^{'}(\mu)\right\} C_\gamma(s_0^{'}(\mu)) \times \left[\int_0^{2\pi} \left\{|p(re^{i\theta})| + n\int_r^R A_t \times \left\{M(p,t) - m\right\} dt - |\beta| m\right\}^q d\theta\right]^{\frac{1}{q}}, \tag{44}$$

and this completes the proof of Theorem 20. ∎

Remark 21. *Taking limit as $q \to \infty$ in (36), we have*

$$\max_{|z|=R} |D_\alpha p(z) + n\beta m| \leq \frac{n}{1 + s_0^{'}(\mu)}\left(\frac{|\alpha|}{R} + s_0^{'}(\mu)\right) \times \left[M(p,r) + n\int_r^R A_t \left\{M(p,t) - m\right\} dt - |\beta| m\right], \tag{45}$$

where A_t is as defined in (12).
Using the simple fact

$$|D_\alpha p(z) + n\beta m| \geq |D_\alpha p(z)| - n|\beta| m,$$

and inequality (21) *of* Lemma 14 *in inequality* (45)*, we have*

$$\max_{|z|=R} |D_\alpha p(z)| \leq \frac{n}{1 + s_0^{'}(\mu)}\left(\frac{|\alpha|}{R} + s_0^{'}(\mu)\right) \times \left[\left\{M(p.r) - m\right\} exp\left\{n\int_r^R A_t dt\right\} + m - |\beta| m\right] + n|\beta| m,$$

which on taking limit as $|\beta| \to 1$ becomes inequality (11) *of* Theorem 7.

Remark 22. *Dividing both sides of inequality* (36) *of* Theorem 20 *by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we have*

Corollary 23. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for every $q > 0$ and for every real or complex number β with $|\beta| < 1$, and for $0 < r \leq R \leq k$,*

$$\left\{\int_0^{2\pi} |p^{'}(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq \frac{n}{R} C_\gamma(s_0^{'}(\mu)) \times \left[\int_0^{2\pi} \left\{|p(re^{i\theta})| + n\int_r^R A_t \times \left\{M(p,t) - m\right\} dt - |\beta| m\right\}^q d\theta\right]^{\frac{1}{q}}, \tag{46}$$

where $M(p,t) = \max_{|z|=t} |p(z)|$, A_t is as defined in (12) *and $C_\gamma(s_0^{'}(\mu))$ is as defined in* Theorem 20.

Remark 24. *Taking simultaneous limit as $q \to \infty$ and $|\beta| \to 1$ in* (46) *of* Corollary 23*, we have*

$$\max_{|z|=R} |p^{'}(z)| \leq n\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^{\mu-1} + R^\mu}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\left(k^{\mu+1} R^\mu + k^{2\mu} R\right)}\right\} \times \left[\max_{|z|=r} |p(z)| + n\int_r^R A_t \left\{M(p,t) - m\right\} dt - m\right],$$

which on applying inequality (21) *of* Lemma 14 *gives inequality* (7) *of* Theorem 4.

Remark 25. *Putting $R = r$ in* Corollary 23*, we obtain a generalized L^q extension of* Lemma 12 *proved by* Gardner et al. [14].

Corollary 26. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for every $p > 0$ and for every real or complex number β with $|\beta| < 1$, and for $r \leq k$,*

$$\left\{\int_0^{2\pi} \left|p^{'}(re^{i\theta})\right|^q d\theta\right\}^{\frac{1}{q}} \leq \frac{\frac{n}{r}}{\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|s_0^{'}(\mu) + e^{i\gamma}\right|^q d\gamma\right\}^{\frac{1}{q}}} \times$$

$$\left[\int_{0}^{2\pi}\left\{|p(re^{i\theta})|-|\beta|m\right\}^{q}d\theta\right]^{\frac{1}{q}},$$

where $s_0^{'}(\mu)$ *is as defined in* Theorem 20*,*
which for $r=1$ *and* $|\beta|\to 1$ *gives* L^q *analogue of inequality* (18) *of* Lemma 12.

Remark 27. Theorem 20 *improves as well as generalizes both* Theorems 10 *and* 11 *of* Maisnam et al. [12].

Remark 28. *Further, if* $R=r=k=\mu=1$*,* Theorem 20 *yields an improved* L^q*-version in polar derivative of the* L^q*-inequality* (14) *in ordinary derivative proved by* de-Brujin [10].

REFERENCES

[1] V. V. Arestov, “On inequalities for trigonometric polynomials and their derivative”, *IZV. Akad. Nauk. SSSR. Ser. Math.*, vol. 45, pp. 3-22, 1981.
[2] A. Aziz, “Inequalities for the polar derivative of a polynomial”, *J. Approx. Theory.*, vol. 55(2), pp. 183-193, 1988.
[3] A. Aziz, N. A. Rather and Q. Aliya, “L_q norm inequalities for the polar derivative of a polynomial”, *Math. Inequal. Appl.*, vol. 11(2), pp. 283-296, 2008.
[4] A. Aziz and W. M. Shah, “Inequalities for a polynomial and its derivative”, *Math. Inequal. Appl.*, vol. 7(3), pp. 379-391, 2004.
[5] A. Aziz and B. A. Zargar, “Inequalities for a polynomial and its derivative”, *Math. Inequal. Appl.*, vol. 1(4), pp. 543-550, 1998.
[6] S. Bernstein, “Lecons Sur Les Propriétés extrémales et la meilleure approximation desfunctions analytiques d’une fonctions reele”, Gauthier-Villars Paris, 1926.
[7] M. Bidkham and K. K. Dewan, “Inequalities for polynomial and its derivative”, *J. Math. Anal. Appl.*, vol. 166(2), pp. 319-324, 1992.
[8] M. Bidkham, M. Shakeri and M. Eshaghi Gordji, “Inequalities for the polar derivative of a polynomial”, *J. Inequal. Appl.*, vol. 1, pp. 1-9, 2009.
[9] B. Chanam and K. K. Dewan, “Inequalities for a polynomial and its derivative”, *J. Math. Anal. Appl.*, vol. 336, pp. 171-179, 2007.
[10] N. G. De-Brujin, “Inequalities concerning polynomials in the complex domain”, *Nederl. Akad. Wetench. Proc. Ser. A*, vol. 50, pp. 1265-1272, 1947.
[11] K. B. Devi, K. Krishnadas and B. Chanam “Some inequalities on polar derivative of a polynomial”, *Nonlinear Funct. Anal. Appl.*, vol. 27(1), pp. 141-148, 2022.
[12] M. T. Devi, K. Krishnadas, N. Reingachan and B. Chanam, “Integral inequalities for polar derivative of a polynomial”, *J. Anal.*, 2021. https://doi.org/10.1007/s41478-021-00332-7.
[13] K. K. Dewan and B. Singh, “Inequalities for the polar derivative of a polynomial”, *J. Comb. Inf. and Syst. Sci.*, vol. 31, pp. 83-90, 2006.
[14] R. B. Gardner, N. K. Govil and A. Weems, “Some results concerning rate of growth of polynomials”, *East Journal on Approximations*, vol. 10, pp. 301-312, 2004.
[15] R. B. Gardner and A. Weems, “A Bernstein-type of L^q inequality for a certain class of polynomials”, *J. Math. Anal. Appl.*, vol. 219, pp. 472-478, 1998.
[16] N. K. Govil and P. Kumar, “On L^q inequalities involving polar derivative of a polynomial”, *Acta Math. Hung.*, vol. 152(1), pp. 130-139, 2017.
[17] N. K. Govil, G. Nyuydinkong and B. Tameru, “Some L^q inequalities for the polar derivative of a polynomial”, *J. Math. Anal. Appl.*, vol. 254(2), pp. 618- 626, 2001.
[18] P. Kumar, “On Zygmund-type inequalities involving polar derivative of a lacunary-type polynomial”, *Bull. Math. Soc. Sci. Math. Roum.*, vol. 61(110), pp. 163-172, 2019.
[19] P. D. Lax, “Proof of a conjecture of P. Erdös on the derivative of a polynomial”, *Bull. Amer. Math. Soc.*, vol. 50, pp. 509-513, 1944.
[20] M. A. Malik, “On the derivative of a polynomial”, *J. London Math. Soc.*, vol. 2(1), pp. 57-60, 1969.
[21] M. Marden, “Geometry of Polynomials”, Math. Surveys, vol. 3, Amer. Math. Soc., Providence 3, 1966.
[22] G. V. Milovanović and A. Mir, “Generalizations of Zygmund-type integral inequalities for the polar derivatives of a complex polynomials”, *J. Inequal. Appl.*, vol. 136, 2020.
[23] G. V. Milovanović, D. S. Mitrinović and Th. Rassias, “Topics in polynomials: Extremal problems, Inequalities, Zeros”, World Scientific, Singapore, 1994.
[24] A. Mir and A. Ahmad, “Bernstein-type integral inequalities for a certain class of polynomials”, *Mediterr. J. Math.*, vol. 17, Art. 177, 2020.
[25] A. Mir and I. Hussian, “On the Erdöx-Lax inequality concerning polynomials”, *C. R. Acad. Sci. Paris, Ser.I* vol. 355, pp. 1055-1062, 2017.
[26] A. Mir and A. Wani, “Polynomials with polar derivatives”, *Funct. Approx. Comment. Math.*, vol. 55(2), pp. 139-144, 2016.
[27] A. Mir, “Bernstein-type integral inequalities for a certain class of polynomials”, *Mediterr. J. Math.*, vol. 16, Art. 143, 2019.
[28] A. Mir, “Generalizations of Bernstein and Turán-type inequalities for the polar derivative of a complex polynomial”, *Mediterr. J. Math.*, vol. 17, Art. 14, 2020.
[29] M. A. Qazi, “On the maximum modulus of polynomials”, *Proc. Amer. Math. Soc.*, vol. 115(2), pp. 337-343, 1992.
[30] Q. I. Rahman, “Functions of exponential type”, *Trans. Amer. Math. Soc.*, vol. 135, pp. 295-309, 1969.
[31] Q. I. Rahman and G. Schmeisser, “Analytic Theory of Polynomials”, Oxford University Press, London, 2002.
[32] N. A. Rather, “Extremal properties and location of zeros of polynomials”, Ph.D. Thesis, University of Kashmir, 1998.
[33] N. Reingachan and B. Chanam “Bernstein and Turán type inequalities for the polar derivative of a polynomial”, *Nonlinear Funct. Anal. Appl.*, vol. 28(1), pp. 287-294, 2023.
[34] N. Reingachan, R. Soraisam and B. Chanam “Some inequalities on polar derivative of a polynomial”, *Nonlinear Funct. Anal. Appl.*, vol. 27(4), pp. 797-805, 2022.
[35] A. Zireh, “On the polar derivative of a polynomial”, *Bull. Iran. Math. Soc.*, vol. 40(4), pp. 967-976, 2014.
[36] A. Zygmund, “A remark on conjugate series”, *Proc. London. Math. Soc.*, vol. 34, pp. 392-400, 1932.