Existence and Iterative Algorithms of Solutions for Lotka-Volterra Competition Model

Li-Li Shi, Yan-Qiu Chen

Abstract—**The Lotka-Volterra competition model consisting of two equations is studied. The existence and uniqueness of solutions on an infinite interval are proved by using the Schauder fixed point theorem, Gronwall's inequality and some special analytical techniques. Some conditions of existence for positive solutions are obtained. Iterative algorithms and error estimations for solving this model are established. The results of this paper can be generalized to the cases consisting of more than two equations.**

Index Terms—**Lotka-Volterra model; Gronwall's inequality; existence and uniqueness; iterative algorithms; infinite interval**

I. Introduction

THE Lotka-Volterra competition model was initially proposed independently by American mathematician Alfred J. Lotka and Italian mathematician Vito Volterra around 1925. It is usually used to describe interactions between species, including predator-prey, competition, and symbiosis relationships. In the past few decades, many researchers have improved and expanded this model, attempting to better describing and analyzing various interactions in ecosystems or other similar systems through different methods and perspectives. This model can be applied not only to biological systems, but also to various other fields such as market competition, financial derivatives, energy consumption, and environmental pollution. We review a few recent results for this model (especially, the practical applications). The existence results of the fractional-order model and positive periodic solutions were proved ([1], [2]); the traveling wave solutions and almost periodic solution were presented ([3], [4]); the study of fractal dimension analysis and control of Julia set were considered [5]; the dynamical study was investigated ([6], [7]). Lotka-Volterra method has been applied to many fields, such as China's manufacturing industry [8], product competition between smart TVs and flat panel TVs [9], Taiwan's retail industry [10], the stock index futures market [11], a scenario analysis for inter-port interactions [12], energy consumption forecasting [13], and pollution shares [14].

In practical applications, the most common Lotka-Volterra competition model may be written as

$$\begin{cases} \dfrac{du}{dt} = u(t)\left[\sigma_1(t) - b_1(t)u(t) - c_1(t)v(t)\right] \\ \quad := f(t, u(t), v(t)), t > 0, \\ \dfrac{dv}{dt} = v(t)\left[\sigma_2(t) - b_2(t)v(t) - c_2(t)u(t)\right] \\ \quad := g(t, u(t), v(t)), t > 0, \\ u(0) = u_0 > 0, v(0) = v_0 > 0. \end{cases} \tag{1}$$

See, for instance, [8]–[10], where $u(t), v(t)$ represent the numbers of two species at time t, $\sigma_1(t)$, $\sigma_2(t)$ are the parameters related to the number of species $u(t)$ and $v(t)$; $b_1(t)$, $b_2(t)$ represent the own limiting coefficients of species $u(t)$ and $v(t)$; $c_1(t)$, $c_2(t)$ are the interaction coefficients between species $u(t)$ and $v(t)$, respectively. Generally, $\sigma_i(t), b_i(t), c_i(t)$ are assumed to satisfy the following basic assumption:

$(\mathbf{P})$ $\sigma_i(t), b_i(t), c_i(t)$ are all continuous, and $\sigma_i(t), b_i(t), c_i(t) : [0, \infty) \to (0, \infty)(i = 1, 2)$.

Remark 1. If $\sigma_i(t), b_i(t), c_i(t)$ are positive constants, then $(\mathbf{P})$ is satisfied naturally.

For (1), the following three problems must be addressed:

(a) If (u, v) is a solution of (1), it should be nonnegative or positive, that is, $u(t) \geq 0$, $v(t) \geq 0$ or $u(t) > 0$, $v(t) > 0$ for time t within an existence interval.

(b) If (1) has a solution, what is the maximum existence interval of that? Is this solution unique?

(c) If (1) has a solution, can it be approximated by iterative sequences?

However, a solution of (1) obtained by existing methods may not necessarily satisfy (a). Meanwhile, we remark that the usual methods of positive solutions for self mapping defined on a cone [15], [16] cannot be used to treat (1) due to the nonlinear terms with sign-changing in (1). Regarding (b), seeking the maximum existence interval of solutions for (1) is very interesting and important under the basic condition $(\mathbf{P})$. To the best of our knowledge, there is little study on them (especially (c)).

In this paper, under the basic assumption $(\mathbf{P})$, by using the Schauder fixed-point theorem, the Gronwall's inequality and some special analytical techniques, we prove the existence and uniqueness of solutions for (1) and obtain that the maximum existence interval of solutions for (1) is the infinite interval $[0, \infty)$. Also, the iterative algorithms and error estimations of solutions are established. Finally, we point out that the results obtained in this paper can be extended to the Lotka-Volterra models consisting of more than two equations.

This paper is organized as follows: In Section 2, we make some preliminaries. In Section 3, we prove the main results of the existence and uniqueness of solutions and

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positive solutions. In Section 4, we establish some iterative algorithms and error estimations of solutions. In Section 5, we summarize the conclusions of this paper and give some comments.

II. Some preliminaries

Let $T \in (0,\infty)$ be a constant. We denote by $C[0,T]$ the Banach space consisting of all continuous functions on $[0,T]$ with the norm $\|u\| = \max\{|u(t)| : t \in [0,T]\}$. $C_+[0,T] = \{u : u(t) \in C[0,T], u(t) \geq 0, t \in [0,T]\}$ and $C^1[0,T] = \{u : u \in C[0,T], u(t)$ has the first order continuous derivative on $[0,T]\}$.

Let $u, v \in (C_+[0,T] \setminus \{0\}) \cap C^1[0,T]$ and (u,v) satisfies (1). Then (u,v) is called to be a solution of (1); If $u(t) > 0$, $v(t) > 0$ for $t \in [0,T]$, (u,v) is called to be a positive solution.

Integrating (1) from 0 to t, we obtain the following integral system:

$$\begin{cases} u(t) = \int_0^t f(s, u(s), v(s))ds + u_0, t \geq 0, \\ v(t) = \int_0^t g(s, u(s), v(s))ds + v_0, t \geq 0. \end{cases} \tag{2}$$

It is easy to know that (u,v) is a solution of (1) if and only if $u, v \in C_+[0,T] \setminus \{0\}$ and (u,v) satisfy (2). Hence, we only need to study the existence and uniqueness of nonnegative solutions in $C[0,T]$ for the integral system (2).

The following inequality is the famous Gronwall's one in differential equations [17].

Gronwall's Inequality. Let $x \in C^1[0,T]$ and $a(t) \in C[0,T]$. If $\frac{dx(t)}{dt} \leq a(t)x(t), t > 0, x(0) = x_0$, then $x(t) \leq x_0 e^{\int_0^t a(s)ds}, 0 \leq t \leq T$.

From the Gronwall's inequality, we can easily know that if (1) has a solution, then $u_0 > 0, v_0 > 0$ by $\frac{du}{dt} \leq \sigma_1(t)u(t), \frac{dv}{dt} \leq \sigma_2(t)v(t), t \geq 0$.

Let X be a Banach space, $D \subset X$ be a nonempty set. The mapping $S : D \to X$ is called to be a compact mapping if

(i) S is continuous;
(ii) $S(\Omega)$ is relatively compact for any bounded set $\Omega \subseteq D$.

The following theorem is the known Schauder fixed point theorem [15], [16].

Schauder Fixed Point Theorem. Let D be a bounded non-empty closed convex set and S map D into D. If S is a compact mapping, then S has a fixed point in D, that is, there exists $x \in D$ such that $Sx = x$.

The following function is used to construct the special mapping, which plays a crucial role in this study.

Let $k \in (0,\infty)$ be a constant. We define function

$$r_k(z) := \begin{cases} k, & \text{if } z > k, \\ z, & \text{if } 0 \leq z \leq k, \\ 0, & \text{if } z < 0. \end{cases} \tag{3}$$

It is obvious that $r_k(z)$ is continuous in $(-\infty,\infty)$, $0 \leq r_k(z) \leq k$ for $-\infty < z < \infty$ and it is easy to verify

$$|r_k(z_2) - r_k(z_1)| \leq |z_2 - z_1| \text{ for } z_2, z_1 \in (-\infty,\infty). \tag{4}$$

It is easy to verify that the following proposition is true and the proof is omitted.

Proposition 1. *Let*

$$h_1(t,u,v) = u[\sigma_1(t) - b_1(t)u - c_1(t)v],$$
$$h_2(t,u,v) = v[\sigma_2(t) - b_2(t)v - c_2(t)u]$$

for $t, u, v \in [0,\infty)$. *Then*

$$\begin{aligned} &h_1(t,u,v) - h_1(t,z,w) \\ &= (\sigma_1(t) - b_1(t)(u+z) \\ &\quad - c_1(t)v)(u-z) - c_1(t)z(v-w), \\ &h_2(t,u,v) - h_2(t,z,w) \\ &= (\sigma_2(t) - b_2(t)(v+w) \\ &\quad - c_2(t)u)(v-w) - c_2(t)w(u-z). \end{aligned}$$

Notation

$$k = \max\left\{u_0 e^{\int_0^T \sigma_1(s)ds}, v_0 e^{\int_0^T \sigma_2(s)ds}\right\}. \tag{5}$$

Let $E = C[0,T] \times C[0,T]$ with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$. We define a mapping $S : E \to E$ by $S(u,v)(t) = (A(u,v)(t), B(u,v)(t))$, where

$$\begin{cases} A(u,v)(t) = \int_0^t f(s, r_k[u(s)], r_k[v(s)])\, ds + u_0, \\ B(u,v)(t) = \int_0^t g(s, r_k[u(s)], r_k[v(s)])\, ds + v_0, \end{cases}$$

$0 \leq t \leq T$.

Theorem 2. *For any fixed* $T \in (0,\infty)$, S *has a fixed point in* E, *that is, there exists* $(u_*, v_*) \in E$ *such that*

$$\begin{cases} u_*(t) = \int_0^t f(s, r_k[u_*(s)], r_k[v_*(s)])\, ds + u_0, \\ v_*(t) = \int_0^t g(s, r_k[u_*(s)], r_k[v_*(s)])\, ds + v_0, \end{cases} \tag{6}$$

$0 \leq t \leq T$.

Proof: Since $0 \leq r_k(z) \leq k$, we see

$$\begin{aligned} &|f(t, r_k[u(t)], r_k[v(t)])| \\ &\leq r_k[u(t)] (\sigma_1(t) + b_1(t) r_k[u(t)] + c_1(t) r_k[v(t)]) \\ &\leq k(\sigma_1(t) + k b_1(t) + k c_1(t)), \\ &|g(t, r_k[u(t)], r_k[v(t)])| \\ &\leq r_k[v(t)] (\sigma_2(t) + b_2(t) r_k[v(t)] + c_2(t) r_k[u(t)]) \\ &\leq k(\sigma_2(t) + k b_2(t) + k c_2(t)). \end{aligned}$$

The continuity of $\sigma_i(t), b_i(t), c_i(t)$ on $[0,T]$ implies for any $(u,v) \in E$,

$$f(t, r_k[u(t)], r_k[v(t)]), g(t, r_k[u(t)], r_k[v(t)])$$

are bounded on $[0,T]$, that is, there exists a constant $M > 0$ such that

$$\begin{aligned} &|f(t, r_k[u(t)], r_k[v(t)])| \leq M, \\ &|g(t, r_k[u(t)], r_k[v(t)])| \leq M, t \in [0,T]. \end{aligned}$$

Let $R = MT + u_0 + v_0, D = \{(u,v) \in X : \|u\| \leq R, \|v\| \leq R\}$. Then

$$\begin{aligned} |A(u,v)(t)| &\leq \int_0^T |f(s, r_k[u(s)], r_k[v(s)])|\, ds + u_0 \\ &\leq MT + u_0 \leq R, \\ |B(u,v)(t)| &\leq \int_0^T |g(s, r_k[u(s)], r_k[v(s)])|\, ds + v_0 \\ &\leq MT + v_0 \leq R, \end{aligned}$$

we know that S maps a bounded closed convex set D into D. From the continuity of f and g, a standard argument shows that S is compact, and there exists $(u_*, v_*) \in E$ such that (6) holds by the Schauder fixed point theorem. ∎

III. Existence and uniqueness of solutions on $[0, \infty)$

We are in a position to prove the existence and uniqueness of solutions for (1) and conclude that the maximum existence interval of solutions for (1) is the infinite interval $[0, \infty)$ under the basic assumption $(\mathbf{P})$. Some conditions of existence for positive solutions are obtained.

Theorem 3. *The fixed point (u_*, v_*) in Theorem 2 is a nonnegative solution of* (2).

Proof: **Step 1.** $u_*(t) \geq 0, v_*(t) \geq 0, 0 \leq t \leq T$.

If there exists $t_0 \in (0, T)$ such that $u_*(t_0) < 0$. Since $u_*(0) = u_0 > 0$ and $r_k(z) = 0, z \leq 0$, there must exists $[a, b] \subseteq (0, t_0)\,(a < b)$ satisfying

$$u_*(a) = 0, u_*(t) < 0, t \in (a, b].$$

Therefore $r_k[u_*(t)] = 0 (t \in [a, b])$, and we can get

$$\begin{aligned}\frac{du_*(t)}{dt} &= f(t, r_k[u_*(t)], r_k[v_*(t)]) \\ &= f(t, 0, r_k[v_*(t)]) = 0, t \in [a, b].\end{aligned}$$

This implies $u_*(t) = u_*(a) = 0, t \in (a, b]$, it is a contradiction. Therefore $u_*(t) \geq 0, t \in [0, T]$.

If there exists $t_0 \in (0, T)$ such that $v_*(t_0) < 0$. Similarly, we can obtain $v_*(t) \geq 0, t \in [0, T]$ and the details are omitted.

Step 2. $u_*(t) \leq k, v_*(t) \leq k (t \in [0, T])$.

By (6), we know

$$\begin{aligned}\frac{du_*(t)}{dt} &\leq \sigma_1(t) r_k[u_*(t)] \leq \sigma_1(t) u_*(t), t > 0, \\ \frac{dv_*(t)}{dt} &\leq \sigma_2(t) r_k[v_*(t)] \leq \sigma_2(t) v_*(t), t > 0, \\ u_*(0) &= u_0 > 0, v_*(0) = v_0 > 0.\end{aligned}$$

Utilizing the Gronwall's differential inequality, we have

$$u_*(t) \leq u_0 e^{\int_0^t \sigma_1(s) ds}, v_*(t) \leq v_0 e^{\int_0^t \sigma_2(s) ds}, t > 0. \tag{7}$$

This, together with step 1, implies $0 \leq u_*(t) \leq k, 0 \leq v_*(t) \leq k (t \in [0, T]), r_k[u_*(t)] = u_*(t), r_k[v_*(t)] = v_*(t)$.

Step 3. $u_*(t)$ and $v_*(t)$ belong to $C_+[0, T] \setminus \{0\}$.

If $u_*(t) \equiv 0$ or $v_*(t) \equiv 0$, by (6), we get $u_0 = 0$ or $v_0 = 0$, it is a contradiction.

The proof is completed. ∎

Theorem 4. *The nonnegative solution of* (2) *in $C[0, T]$ is unique.*

Proof: Let (u^*, v^*) be another nonnegative solution of (2) in $C[0, T]$. We prove that $(u^*, v^*) = (u_*, v_*)$.

Let

$$k_1(t) = \sigma_1(t) - c_1(t) v_*(t) - b_1(t)(u_*(t) + u^*(t)),$$

$$k_2(t) = \sigma_2(t) - c_2(t) u_*(t) - b_2(t)(v_*(t) + v^*(t)).$$

Since $r_i(t), \sigma_i(t), c_i(t)$ are continuous functions on $[0, T]$, we may assume that

$$|k_1(t)| \leq M_1, c_1(t)\,|u^*(t)| \leq M_2,$$

$$|k_2(t)| \leq M_3, c_2(t)\,|v^*(t)| \leq M_4,$$

for $t \in [0, T]$, where M_1, M_2, M_3, M_4 are constants.

Dividing $[0, T]$ into N equal parts such that

$$\rho_1 = (M_1 + M_2) h < 1, \rho_2 = (M_3 + M_4) h < 1,$$

where $h = \frac{T}{N}$.

Let

$$\begin{aligned}\|u_* - u^*\|_h &= \max\{|u_*(t) - u^*(t)|, 0 \leq t \leq h\}, \\ \|v_* - v^*\|_h &= \max\{|v_*(t) - v^*(t)|, 0 \leq t \leq h\}.\end{aligned}$$

Therefore, we have by Proposition 1 (setting $(u, v) = (u_*, v_*), (z, w) = (u^*, v^*)$) when $0 \leq t \leq h$

$$\begin{aligned}&|u_*(t) - u^*(t)| \\ &= \left|\int_0^t [f(s, u_*(s), v_*(s)) - f(s, u^*(s), v^*(s))] ds\right| \\ &= \left|\int_0^t [k_1(s)(u_*(s) - u^*(s)) \right. \\ &\quad \left. - c_1(s) u^*(s)(v_*(s) - v^*(s))] ds\right| \\ &\leq \int_0^t |k_1(s)(u_*(s) - u^*(s)) \\ &\quad - c_1(s) u^*(s)(v_*(s) - v^*(s))| ds \\ &\leq (M_1 \|u_* - u^*\|_h + M_2 \|v_* - v^*\|_h) h \\ &\leq (M_1 + M_2) h \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\} \\ &= \rho_1 \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\},\end{aligned}$$

$$\begin{aligned}&|v_*(t) - v^*(t)| \\ &= \left|\int_0^t [g(s, u_*(s), v_*(s)) - g(s, u^*(s), v^*(s))] ds\right| \\ &= \left|\int_0^t [k_2(s)(v_*(s) - v^*(s)) \right. \\ &\quad \left. - c_2(s) v^*(s)(u_*(s) - u^*(s))] ds\right| \\ &\leq \int_0^t |k_2(s)(v_*(s) - v^*(s)) \\ &\quad - c_2(s) v^*(s)(u_*(s) - u^*(s))| ds \\ &\leq (M_3 \|v_* - v^*\|_h + M_4 \|u_* - u^*\|_h) h \\ &\leq (M_3 + M_4) h \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\} \\ &= \rho_2 \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\}.\end{aligned}$$

Hence

$$\begin{aligned}\|u_* - u^*\|_h &\leq \rho_1 \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\} \\ \|v_* - v^*\|_h &\leq \rho_2 \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\}\end{aligned}$$

and

$$\begin{aligned}&\max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\} \\ &\leq \rho \max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\},\end{aligned}$$

where $\rho = \max\{\rho_1, \rho_2\} < 1$. This implies $\max\{\|u_* - u^*\|_h, \|v_* - v^*\|_h\} = 0$ and $u_*(t) =$

$u^*(t), v_*(t) = v^*(t), 0 \le t \le h$. Repeating this process on $[(i-1)h, ih](i = 2, 3, ..., N)$, we can conclude $u_*(t) = u^*(t), v_*(t) = v^*(t)$ on $[0, T]$. Therefore, (u_*, v_*) is a unique nonnegative solution of (2) on $[0, T]$. ■

The following theorem shows that the existence interval of solutions for (1) is the infinite interval $[0, \infty)$.

Theorem 5. (2) *has a unique nonnegative solution on* $C[0, \infty)$.

Proof: For any natural number $N \ge 1$, we denote by $u_*^{(N)}$ and $v_*^{(N)}$ the unique nonnegative solution on $[0, N]$ obtained in Theorem 4. The uniqueness of the solution shows that $u_*^{(k)}(t) = u_*^{(N+1)}(t), v_*^{(k)}(t) = v_*^{(N+1)}(t), 0 \le t \le k, k = 1, 2, \ldots, N$. Therefore

$$\tilde{u}(t) = \begin{cases} u_*^{(1)}(t), 0 \le t \le 1, \\ u_*^{(2)}(t), 1 < t \le 2, \\ \qquad \cdots\cdots \\ u_*^{(N)}(t), N-1 < t \le N, \\ \qquad \cdots\cdots \end{cases}$$

$$\tilde{v}(t) = \begin{cases} v_*^{(1)}(t), 0 \le t \le 1, \\ v_*^{(2)}(t), 1 < t \le 2, \\ \qquad \cdots\cdots \\ v_*^{(N)}(t), N-1 < t \le N, \\ \qquad \cdots\cdots \end{cases}$$

are well defined on $[0, \infty)$. Clearly, $\tilde{u}(t), \tilde{v}(t) \in C[0, \infty)$. By Theorem 4, we obtain immediately that $(\tilde{u}, \tilde{v})$ is a unique nonnegative solution of (2) on $C[0, \infty)$.

Theorem is proved. ■

Corollary 1. *Let* $\sigma_i^*, d_i : [0, \infty) \to (0, \infty)$ *and be continuous* $(i = 1, 2)$. *If* $\sigma_i^*(t) > d_i(t)(t \in [0, \infty)$, *then Lotka-Volterra competition system of the following form*

$$\begin{cases} \frac{du}{dt} = u(t)\left[\sigma_1^*(t) - b_1 u(t) - c_1(t)v(t)\right] - d_1(t)u(t), t > 0 \\ \frac{dv}{dt} = v(t)\left[\sigma_2^*(t) - b_2 v(t) - c_2(t)u(t)\right] - d_2(t)v(t), t > 0 \\ \qquad u(0) = u_0 > 0, v(0) = v_0 > 0 \end{cases}$$

has a unique solution on $C[0, \infty)$.

Let $\sigma_i(t) = \sigma_i^*(t) - d_i(t)$. We can rewrite it as (1) and obtain the desired result.

In the end of this section, we present some conditions of existence for positive solutions.

Theorem 6. (2) *has a unique positive solution on* $C[0, \tilde{T})$ *if*

$$\sigma_1(t) - b_1(t)u_0 e^{\int_0^t \sigma_1(s)ds} - c_1(t)v_0 e^{\int_0^t \sigma_2(s)ds} \ge 0,$$

$$\sigma_2(t) - b_2(t)v_0 e^{\int_0^t \sigma_2(s)ds} - c_2(t)u_0 e^{\int_0^t \sigma_1(s)ds} \ge 0,$$

for $t \in [0, \tilde{T})$, *where* $\tilde{T} > 0$ *may take* ∞.

Proof: By (7), we

$$u_*(t) \le u_0 e^{\int_0^t \sigma_1(s)ds}, v_*(t) \le v_0 e^{\int_0^t \sigma_2(s)ds}, t > 0.$$

This implies

$$f(t, u_*(t), v_*(t)) = \sigma_1(t) - b_1(t)u_*(t) - c_1(t)v_*(t) \ge 0,$$

$$g(t, u_*(t), v_*(t)) = \sigma_2(t) - b_2(t)v_*(t) - c_2(t)u_*(t) \ge 0,$$

for $t \in [0, \tilde{T})$ and $u_*(t) > 0$, $v_*(t) > 0$ in $[0, \tilde{T})$ by (2). The proof is completed. ■

Remark 2. There are many functions to satisfy the conditions of Theorem 6, for example, $\sigma_i(t) = e^{-t}$, $b_i(t) = e^{-it}$, $c_i(t) = e^{-3t}$, $u_0 < e^{-1}/2$, $v_0 < e^{-1}/2$, $i = 1, 2$, $\tilde{T} = \infty$.

IV. Iterative algorithms and error estimations

In this section, we establish the iterative algorithms and error estimations of solutions for (2).

Let $T \in (0, \infty)$ and

$$m_1 = \max\{\sigma_1(s) + 2b_1(s)k + c_1(s)k : t \in [0, T]\},$$

$$m_2 = \max\{\sigma_2(s) + 2b_2(s)k + c_2(s)k : t \in [0, T]\},$$

$$m = \max\{m_1, m_2\},$$

$$e_n = \sum_{j=0}^{n} \frac{(mT)^j}{j!},$$

where k is in (5). We define the iterative sequences as follows

$$u_n(t) = A\left(r_k[u_{n-1}], r_k[v_{n-1}]\right)(t), n = 1, 2, ... \quad (8)$$

$$v_n(t) = B\left(r_k[u_{n-1}], r_k[v_{n-1}]\right)(t), n = 1, 2, ... \quad (9)$$

$$u_0(t) = u_0, v_0(t) = v_0, t \in [0, T].$$

Theorem 7. $u_*(t) = \lim_{n\to\infty} u_n(t)$, $v_*(t) = \lim_{n\to\infty} v_n(t)$ *on* $t \in [0, T]$ *and*

$$|u_*(t) - u_n(t)| \le 2k\left(e^{mT} - e_{n-1}\right),$$

$$|v_*(t) - v_n(t)| \le 2k\left(e^{mT} - e_{n-1}\right),$$

for $t \in [0, T]$.

Proof:

Let

$$\alpha_n(t) = \sigma_1(t) - c_1(t)r_k[v_n(t)] - b_1(t)(r_k[u_n(t)] + r_k[u_{n-1}(t)]),$$

$$\beta_n(t) = \sigma_2(t) - c_2(t)r_k[u_n(t)] - b_2(t)(r_k[v_n(t)] + r_k[v_{n-1}(t)]).$$

Since $0 \le r_k(z) \le k$ for any $z \in (-\infty, \infty)$, we know $|\alpha_n(t)| \le m_1$, $|\beta_n(t)| \le m_2$ on $[0, T]$ and $0 \le c_1(t)r_k[u_{n-1}(t)] \le m_1$, $0 \le c_2(t)r_k[v_{n-1}(t)] \le m_2$ on $[0, T]$.

We prove

$$|u_n(t) - u_{n-1}(t)| \le 2k\frac{(mT)^{n-1}}{(n-1)!}, \quad 0 \le t \le T, n \ge 1, \quad (10)$$

$$|v_n(t) - v_{n-1}(t)| \le 2k\frac{(mT)^{n-1}}{(n-1)!}, \quad 0 \le t \le T, n \ge 1. \quad (11)$$

From $(n \geq 2)$, we have by Proposition 1(setting $(u,v) = (u_n, v_n), (z,w) = (u_{n-1}, v_{n-1})$)

$$
\begin{aligned}
&|u_{n+1}(t) - u_n(t)| \\
&\leq \int_0^t |f(s, r_k[u_n(s)], r_k[v_n(s)]) \\
&\quad - f(s, r_k[u_{n-1}(s)], r_k[v_{n-1}(s)])| \, ds \\
&= \int_0^t |\alpha_n(s)(r_k[u_n(s)] - r_k[u_{n-1}(s)]) \\
&\quad - c_1(s) r_k[u_{n-1}(s)](r_k[v_n(s)] - r_k[v_{n-1}(s)])| \, ds \\
&\leq \int_0^t (|\alpha_n(s)||r_k[u_n(s)] - r_k[u_{n-1}(s)]| \\
&\quad + c_1(s) r_k[u_{n-1}(s)]|r_k[v_n(s)] - r_k[v_{n-1}(s)]|) ds \\
&\leq m_1 \int_0^t (|r_k[u_n(s)] - r_k[u_{n-1}(s)]| \\
&\quad + |r_k[v_n(s)] - r_k[v_{n-1}(s)]|) ds,
\end{aligned}
$$

$$
\begin{aligned}
&|v_{n+1}(t) - v_n(t)| \\
&\leq \int_0^t |g(s, r_k[u_n(s)], r_k[v_n(s)]) \\
&\quad - g(s, r_k[u_{n-1}(s)], r_k[v_{n-1}(s)])| \, ds \\
&= \int_0^t |\beta_n(s)(r_k[v_n(s)] - r_k[v_{n-1}(s)]) \\
&\quad - c_2(s) r_k[v_{n-1}(s)](r_k[u_n(s)] - r_k[u_{n-1}(s)])| ds \\
&\leq \int_0^t (|\beta_n(s)||r_k[v_n(s)] - r_k[v_{n-1}(s)]| \\
&\quad + c_2(s) r_k[v_{n-1}(s)]|r_k[u_n(s)] - r_k[u_{n-1}(s)]|) ds \\
&\leq m_2 \int_0^t (|r_k[u_n(s)] - r_k[u_{n-1}(s)]| \\
&\quad + |r_k[v_n(s)] - r_k[v_{n-1}(s)]|) ds,
\end{aligned}
$$

we have

$$
\begin{aligned}
&|u_{n+1}(t) - u_n(t)| + |v_{n+1}(t) - v_n(t)| \\
&\leq (m_1 + m_2) \int_0^t (|r_k[u_n(s)] - r_k[u_{n-1}(s)]| \\
&\quad + |r_k[v_n(s)] - r_k[v_{n-1}(s)]|) ds \\
&= m \int_0^t (|r_k[u_n(s)] - r_k[u_{n-1}(s)]| \\
&\quad + |r_k[v_n(s)] - r_k[v_{n-1}(s)]|) ds.
\end{aligned}
\tag{12}
$$

Noting that

$$
\begin{aligned}
&|r_k[u_n(t)] - r_k[u_{n-1}(t)]| \leq k, \\
&|r_k[v_n(t)] - r_k[v_{n-1}(t)]| \leq k,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&|u_{n+1}(t) - u_n(t)| + |v_{n+1}(t) - v_n(t)| \\
&\leq m \int_0^t 2k ds = 2kmt.
\end{aligned}
\tag{13}
$$

By $|r_k[u(t)] - r_k[v(t)]| \leq |u(t) - v(t)|$ for $u, v \in C[0,T]$ (see (4)) and (12), we see

$$
\begin{aligned}
&|u_{n+1}(t) - u_n(t)| + |v_{n+1}(t) - v_n(t)| \\
&\leq m \int_0^t (|r_k[u_n(s)] - r_k[u_{n-1}(s)]| \\
&\quad + |r_k[v_n(s)] - r_k[v_{n-1}(s)]|) ds \\
&\leq m \int_0^t (|u_n(s) - u_{n-1}(s)| + |v_n(s) - v_{n-1}(s)|) ds.
\end{aligned}
\tag{14}
$$

Let $\delta_n(t) = |u_n(t) - u_{n-1}(t)| + |v_n(t) - v_{n-1}(t)|$. By combining (13) and (14), we get $\delta_{n+1}(t) \leq 2kmt, \delta_{n+1}(t) \leq m \int_0^t e_n(s) ds, n \geq 2$. By induction, we have

$$
\delta_n(t) \leq 2k \frac{(mt)^{n-1}}{(n-1)!} \leq 2k \frac{(mT)^{n-1}}{(n-1)!}, 0 \leq t \leq T, n \geq 2.
\tag{15}
$$

(15) implies that (10) and (11) hold.

By (10) and (11), we see for $p \geq 1$ and $n \geq 2$

$$
\begin{aligned}
|u_{n+p}(t) - u_n(t)| &\leq \sum_{j=n+1}^{n+p} |u_j(t) - u_{j-1}(t)| \\
&\leq \sum_{j=n}^{n+p} \delta_j(t) \leq 2k \sum_{j=n+1}^{n+p} \frac{(mT)^{j-1}}{(j-1)!} \\
&= 2k \sum_{j=n}^{n+p-1} \frac{(mT)^j}{j!},
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
|v_{n+p}(t) - v_n(t)| &\leq \sum_{j=n+1}^{n+p} |v_j(t) - v_{j-1}(t)| \\
&\leq \sum_{j=n}^{n+p} \delta_j(t) \leq 2k \sum_{j=n+1}^{n+p} \frac{(mT)^{j-1}}{(j-1)!} \\
&= 2k \sum_{j=n}^{n+p-1} \frac{(mT)^j}{j!}.
\end{aligned}
\tag{17}
$$

The inequalities (16)-(17) imply that $\{u_n(t)\}$, $\{v_n(t)\}$ converge uniformly on $[0,T]$ the limits are denoted by

$$
\tilde{u}(t) = \lim_{n \to \infty} u_n(t), \tilde{v}(t) = \lim_{n \to \infty} v_n(t).
$$

Then $\tilde{u}$, $\tilde{v} \in C[0,T]$. Letting $n \to \infty$ in (16) and (17), we have

$$
\tilde{u} = A(r_k[\tilde{u}], r_k[\tilde{v}]), \tilde{v} = B(r_k[\tilde{u}], r_k[\tilde{v}]).
$$

By Theorem 3 and Theorem 4, we have $(\tilde{u}, \tilde{v}) = (u_*, v_*)$. Letting $p \to \infty$ in ((16),(17)), we obtain

$$
\begin{aligned}
|u_*(t) - u_n(t)| \leq 2k \textstyle\sum_{j=n}^{\infty} \frac{(mT)^j}{j!} = 2k\left(e^{mT} - e_{n-1}\right), \\
|v_*(t) - v_n(t)| \leq 2k \textstyle\sum_{j=n}^{\infty} \frac{(mT)^j}{j!} = 2k\left(e^{mT} - e_{n-1}\right).
\end{aligned}
$$

■

V. Conclusions and generalization

Under the basic assumption $(\mathbf{P})$, we prove the existence and uniqueness of solutions and positive solutions of (1) and the existence interval of the solutions is $[0,\infty)$, the existence interval is the best answer we expect, see Theorem 5 and Theorem 6. The iterative algorithms and error estimations are established (Theorem 7). It is well known that studying the existence of solutions on unbounded domains is difficult. The feature of this study is that the existence interval of solutions is the infinite interval and solutions can be approximated by iterative sequences. Hence, the results of this paper will be a solid theoretical support for future applications.

The results of this paper can be generalized to systems consisting of more than two equations $(L-V)_n$:

$$\begin{cases} \frac{du_1}{dt} = u_1(t)\left[\sigma_1(t) - a_{11}(t)u_1(t) - a_{12}(t)u_2(t) - \right. \\ \qquad \left. \ldots - a_{1n}(t)u_n(t)\right] \\ \qquad := f_1\left(t, u_1(t), u_2(t), \ldots, u_n(t)\right), \\ \frac{du_2}{dt} = u_1(t)\left[\sigma_2(t) - a_{21}(t)u_1(t) - a_{22}(t)u_2(t) - \right. \\ \qquad \left. \ldots - a_{2n}(t)u_n(t)\right] \\ \qquad := f_2\left(t, u_1(t), u_2(t), \ldots, u_n(t)\right), \\ \qquad \ldots\ldots \\ \frac{du_n}{dt} = u_n(t)\left[\sigma_n(t) - a_{n1}(t)u_1(t) - a_{n2}(t)u_2(t) - \right. \\ \qquad \left. \ldots - a_{nn}(t)u_n(t)\right] \\ \qquad := f_n\left(t, u_1(t), u_2(t), \ldots, u_n(t)\right), \\ \qquad u_i(0) = u_i > 0, i = 1, 2, \ldots, n, \end{cases}$$

for $t \geq 0$, where σ_i, a_{ij} satisfy the basic assumption $(\mathbf{P})$, that is, σ_i, a_{ij}: $[0, \infty) \to (0, \infty)$ are continuous, $i, j = 1, 2, \ldots, n$, we may prove that the same results as Theorem 5, Theorem 6 and Theorem 7 hold and provide only a framework of main results.

Let $T \in (0, \infty)$ and

$$k = \max\left\{u_i^{(0)} e^{\int_0^T \sigma_i(s)ds} : i = 1, 2, \ldots, n\right\}.$$

We define

$$\begin{aligned} & S\left(u_1, u_2, \ldots, u_n\right) \\ & = \left(A_1\left(u_1, u_2, \ldots, u_n\right), A_2\left(u_1, u_2, \ldots, u_n\right),\right. \\ & \quad \left. \ldots, A_n\left(u_1, u_2, \ldots, u_n\right)\right), \end{aligned}$$

where

$$\begin{aligned} & A_i\left(u_1, u_2, \ldots, u_n\right)(t) \\ = & \int_0^t f_i\left(s, r_k[u_1(s)], r_k[u_2(s)], \ldots, r_k[u_n(s)]\right) ds + u_i^{(0)}, \end{aligned}$$

for $(u_1, u_2, \ldots, u_n) \in C[0, T], i = 1, 2, \ldots, n,$

$$\begin{aligned} u_i^{(k)}(t) = A_i\left(u_1^{(k-1)}, u_2^{(k-1)}, \ldots, u_n^{(k-1)}\right)(t), \\ k = 1, 2, \ldots, n. \end{aligned} \tag{18}$$

We can prove

Theorem 8. *S has a fixed point $(u_1^*, u_2^*, \ldots, u_n^*)$ $(u_i^* \in C[0, T], i = 1, 2, \ldots, n)$ and $(u_1^*, u_2^*, \ldots, u_n^*)$ is a unique solution of $(L - V)_n$.*

Theorem 9. (1) *The system $(L - V)_n$ has a unique solution on $C[0, \infty)$.* (2) *The system $(L - V)_n$ has a unique positive solution on $C[0, \tilde{T})$ if*

$$\sigma_i(t) - \Sigma_{j=1}^n a_{ij} u_j^{(0)} e^{\int_0^t \sigma_j(s)ds} \geq 0$$

for $t \in [0, \tilde{T})$, where $\tilde{T} > 0$ may take ∞.

Theorem 10. *$\{(u_1^{(k)}, u_2^{(k)}, \ldots, u_n^{(k)})\}$ converges uniformly to $(u_1^*, u_2^*, \ldots, u_n^*)$ on $[0, T]$, where $u_*^{(i)}(t) = \lim_{k \to \infty} u_i^{(k)}(t), i = 1, 2, \ldots, n$.*

We may establish error estimations that are similar to Theorem 7, all details (including the proof of Theorem 8 to Theorem 10) are omitted due to the duplication of most work.

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