Sliding Mode Control Design and Stability Analysis of a Class of Financial Fractional-order Chaotic Mathematical Model

Jianjun Wu, Lu Xia

***Abstract*—The financial mathematical model, comprised of multiple elements, represents a complex non-linear system. Within this non-linear system, deterministic instability during operation gives rise to financial chaos phenomena such as turbulent fluctuations in the financial market and financial crises. Economic growth and social stability have been profoundly negatively affected by these events. To suppress or eliminate the disorderly condition in the fractional-order nonlinear system, and effectively stabilize and control the chaotic cyclic or erratic recurring state in the chaotic attractor, a novel sliding mode controller has been proposed in this research article. This controller aims to modify the behavior of the original economic system and achieve a new spatiotemporal order structure. By utilizing this controller, it is possible to facilitate the evolution process of the financial system from a chaotic to a regular state. Firstly, the dynamic characteristics of the fractional financial system are examined, followed by the determination of the theoretical order and coefficient range of the system. Next, a novel sliding mode control method is introduced, whose stability is examined. Finally, a numerical simulation is carried out to evaluate the effectiveness of the proposed controller. The consequences of the experiment demonstrate that the controller holds practical significance in the macro-control of financial crises.**

***Index Terms*—Fractional order, Financial mathematical model, Chaos, 0-1 test, Sliding mode control.**

I. Introduction

CHAOS is a highly complex and random-like behavior exhibited by a system, particularly a nonlinear system, that cannot determine the future state of the system based on given initial conditions. However, chaos does not imply disorder and actually contains elements of order. Chaos can also manifest within an orderly process, resulting in a seemingly random and erratic motion generated by a deterministic nonlinear dynamical system. In recent years, chaos theory has garnered significant attention and research, yielding fruitful results in various applications, including mathematics, control systems, secure communication, and economics [1-4].

Although the theory of fractional calculus has a long history, similar to that of integer calculus, it has received less attention and research because of the lack of application background. It was related theory developed more slowly than integer calculus, and was first introduced by Mandelbort in 1983. It is noted that fractal dimensions exist in nature and across numerous scientific and technological domains [5], which has contributed to the advancement of fractional calculus. In recent years, researchers have uncovered the presence of fractional-order dynamical behavior in numerous physical systems, including areas such as oscillation, turbulence, and control. As a result, fractional-order calculus, along with its associated models, has gained broader application across various scientific and engineering domains. It has become a prominent research focus in nonlinear disciplines, offering significant advancements over traditional integer-order calculus in theory and practical applications. In [6], Hegazi A S carried out a study that focused on the management and synchronization of chaotic fractional Liu systems. In [7], Gao F, et al. introduced a novel method using computational intelligence methods such as genetic programming to automatically drive the self-evolution of optimal superfractional chaos. In [8], Khennaoui A A et al. introduced three systems: Fractional Lozi Graph, Fractional Lorenz Graph and Fractional Flow Graph. They also proposed a control law with the objective of stabilizing and synchronizing these three mapping combinations. Fractional hyperchaotic economic systems were introduced by Yousefpour A et al. in [9], they suggest using a combination of adaptive terminal sliding mode control and a neural network estimator to effectively stabilize and synchronize fractional-order systems within a finite time. In [10], Chen L and Hao Y introduced a novel three-dimensional fractional-order discrete Hopfield neural network, where they utilized the left Caputo discrete delta method as a way to measure the system's responsiveness. In [11], the authors developed an efficient and reliable optimization algorithm. By utilizing this algorithm, researchers were able to successfully identify the parameters linked to chaotic dynamic behavior in various systems, including fractional-order chaos, noisy chaos, and hyperchaotic financial systems. In [12], a system of fractional differential equations is presented. This system involves new generalized Caputo fractional derivatives.

The application of fractional differential theory to system control has garnered mounting focus in the last few years. One of the key areas of focus is the stability analysis of fractional

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differential systems. This topic has captured the interest of scholars in the field, and significant research efforts have been dedicated to understanding and ensuring the stability of such systems. Drawing upon the principles of variable structure systems, sliding mode control (SMC) is a control strategy that is utilized. SMC has gained wide popularity across various fields due to its robustness to nonlinear systems, quick response, excellent temporary performance, and insensitivity to parameter changes and outside interferences. These features make it an ideal choice for applications that require stability and robustness. By effectively handling uncertainties, sliding mode control has found extensive use in diverse areas such as robotics, power systems, aerospace, and more. In [13], Yan J.J. proposed a novel approach to establish stability in continuous unified chaotic systems using discrete sliding mode control. With this new approach, only one controller is required for chaos suppression, simplifying the system design. In [14], a novel arrival rule is proposed for SMC, and the SMC signal is generated using Lyapunov stability theory for chaos control and synchronization. In [15], a global and a terminal method of SMC are presented. These approaches are specifically crafted to enable real-time tracking of variable-order fractional-order systems as well as constant-order fractional-order systems, despite the presence of uncertainties and external disturbances.

Traditionally, unstable fluctuations have been considered unfavorable in economics. Chaos has been associated with unpredictable events, which can pose challenges for decision-makers in the area of economics. Due to the inherent sensitivity and uncertainty of chaotic behavior's long-term evolution, controlling chaos has become a crucial aspect in employing chaos theory to the discipline of economics. The rise of nonlinear economics, particularly the exploration of chaotic economics, has brought about significant revolutions in economic research. It has sparked a paradigm shift in how economists analyze and interpret economic phenomena. By embracing the principles of nonlinearity and chaos, economists have gained a deeper understanding of the intricate dynamics and interconnectedness within economic systems. This has paved the way for groundbreaking insights into the emergence of complex patterns, fluctuations, and even unexpected behaviors observed in real-world economies. Chaos theory is a valuable analytical tool in economics. By employing techniques such as dynamic imbalance analysis, it becomes feasible to explore crucial dynamic features of complex economic systems, including attraction, bifurcation, mutation, and chaos. This analysis can help in controlling chaotic phenomena within the economic area or uncovering hidden laws underlying intricate economic phenomena. Chaos theory has the potential to provide valuable insights and a deeper understanding of the dynamics of economic systems. Since the 20th century, global financial crises have occurred repeatedly. In an effort to understand the underlying dynamics of financial markets, mathematical models have been devised to examine the internal structure of the financial system. These models have revealed the availability of chaos in the financial system, prompting researchers to explore various methods for controlling and restoring normalcy in financial markets. Chaos control involves intentionally influencing a chaotic system to achieve a desired state, thereby attempting to bring about the necessary order in the financial market [16-19].

In summary, research on chaos theory holds significant scientific importance and vast application prospects. Chaos theory spans across numerous disciplines in the natural and social sciences, providing an effective tool for addressing nonlinear complex problems. It is also challenging and reshaping traditional perspectives on the real world. The wide-ranging applicability of chaos theory highlights its potential to revolutionize our understanding and approach to complex systems in various domains. For the steady state analysis of the system, it is advisable to use a sliding mode control approach based on the dynamic characteristics of fractional chaos in financial markets. The subsequent sections of this paper are structured as follows: Section 2 provides a mathematical description of the fractional financial system. In Section 3, the 0-1 test is employed for dynamic analysis of the system. In Section 4, a novel sliding mode reaching law is presented with the aim of controlling the system.

II. Mathematical Description of Fractional Financial System

Based on an analysis of the laws governing macroeconomic operations [20], a chaotic financial system can be constructed. This system consists of securities sub-blocks, production sub-blocks, currency and labor sub-blocks. A chaotic financial system with only three variables can be obtained by utilizing appropriate coordinate transformations and dimension reduction techniques. Chen [21] introduces the fractional chaotic financial mathematical model:

$$\begin{cases} \dfrac{d^{\alpha}x}{dt^{\alpha}} = z+(y-a)x \\ \dfrac{d^{\beta}y}{dt^{\beta}} = 1-by-x^2 \\ \dfrac{d^{\gamma}z}{dt^{\gamma}} = -x-cz \end{cases} \tag{1}$$

In equation (1), x represents interest rate, y represents investment demand, z represents price index, $a \ge 0$ is the amount of savings, $b \ge 0$ is the investment cost, $c \ge 0$ is the flexibility of commodity demand, and $0 < \alpha, \beta, \gamma < 1$.

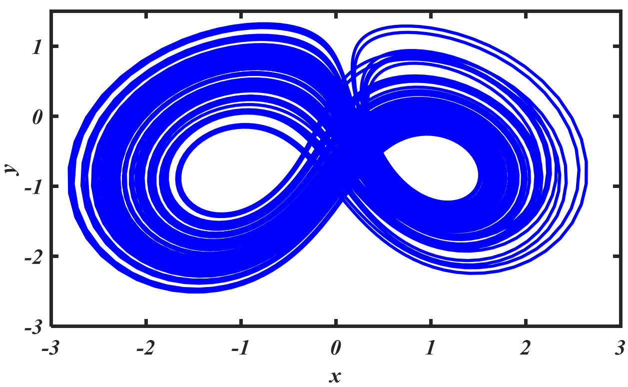

Fig. 1. Chaotic system phase diagram :x-y

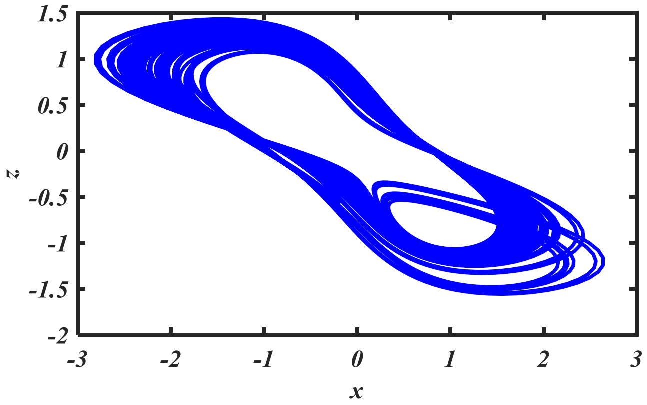

Fig. 2. Chaotic system phase diagram: x-z

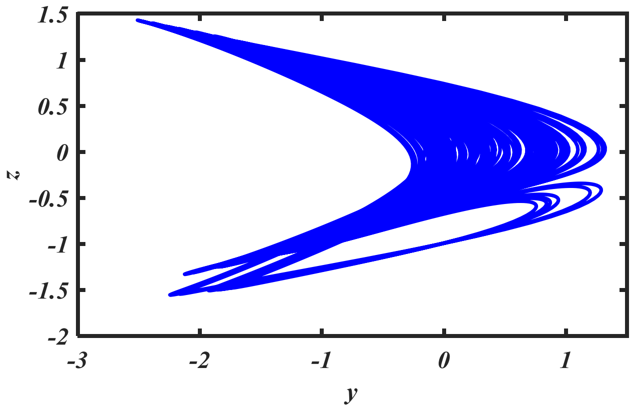

Fig. 3. Chaotic system phase diagram: z-y

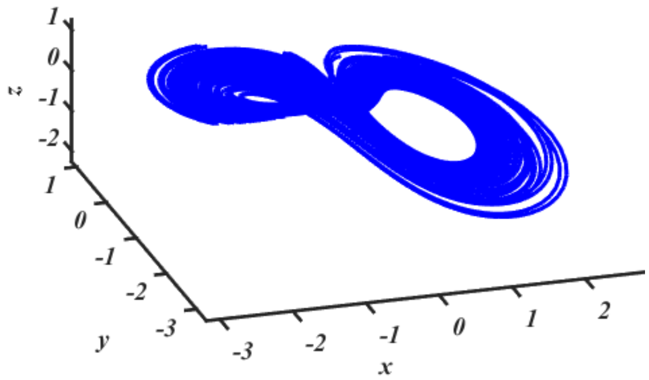

Fig. 4. Chaotic system phase diagram: x-y-z

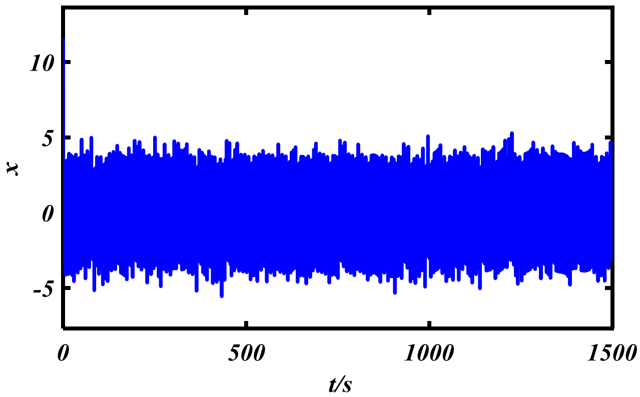

Fig. 5. Timing diagram of chaotic system: x-t

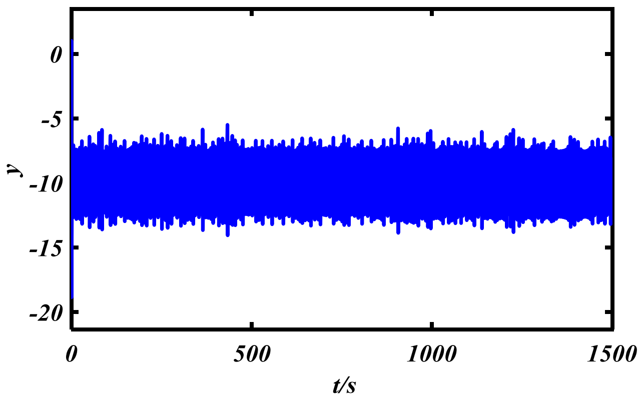

Fig. 6. Timing diagram of chaotic system: y-t

When $a=-11,\ b=0.2,\ c=0.9,\ \alpha=0.92,\ \beta=0.92,\ \gamma=0.95$, the system is in a chaotic state. The phase graph of its chaotic system is displayed in Figs. 1-4. Figs. 5-7 demonstrate the time series graph depicting the system's state variables.These diagrams show that the fractional order system is in a state of chaotic dynamics with no apparent order or pattern, which can pose a considerable risk to the financial system. This chaotic behavior can potentially have a destructive impact. It is crucial to implement macro regulation and control measures to stabilize the system's long-term violent fluctuations. Therefore, effective control methods should be devised to suppress this chaotic behavior.

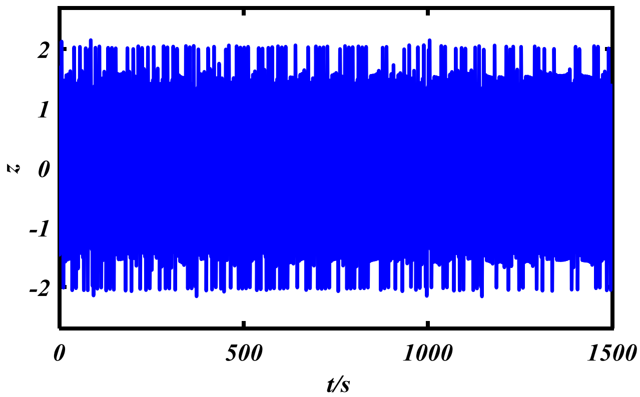

Fig. 7. Timing diagram of chaotic system: z-t

III. Financial System Dynamics Analysis

After developing the fractional financial system, it is important to analyze and describe its dynamic characteristics through mathematical analysis and numerical simulation techniques. It is also important to prove that the map used in the system exhibits chaotic behavior. This proof would indicate the presence of both periodic and chaotic motion within the financial system. The financial system exhibits a combination of regular, predictable patterns as well as unpredictable, unstable behavior. Gottwald and Melbourne [22-23] put forward a robust and productive binary approach for assessing system chaos, known as the "0-1 test".

In this approach, a positive number $c\in\left[\pi/5,4\pi/5\right]$ is selected, and numerical simulation data is used to construct a discrete set $\left\{\Phi\left(j\right)\right\}\left(j=1,2,\cdots,N\right)$. In general, n is not more than 0.1 times the span of the discrete set N. The conversion variable is defined as described below:

$$p(n)=\sum_{i=1}^{n}\phi(i)\cos ic\ ,q(n)=\sum_{i=1}^{n}\phi(i)\sin ic$$

To assess the growth characteristics of the functions $p(n)$ and $q(n)$ (e.g., their diffusion behavior), the average distance squared from $p(n)$ and $q(n)$ [MSD, M(n)] is characterized as follows:

$$M(n)=\lim_{N\to\infty}\left\{\sum_{i=1}^{N}\left[p(i+n)-p(i)\right]^2+\left[q(i+n)+q(i)\right]^2\right\}-$$
$$\left[\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\Phi(i)\right]^2\frac{1-\cos nc}{1-\cos c}$$

The convergence and divergence of functions $p(n)$ and $q(n)$ can be evaluated using $M(n)$. The asymptotic growth rate of $M(n)$, which is the defining characteristic of the dynamic system, can be obtained by fitting linear regression to the functions $\log M(n)$ and $\log n$, or alternatively, by calculating the correlation coefficient between them.

The algorithm steps are as presented:

Step1: Take the first N data points from various fractional chaotic systems and treat them as a discrete set $\phi(N)$;

Step2: Replace the discrete set $\phi(N)$ by selecting one data point every 8 points;

Step3: Incorporate $\phi(N)$ into the transformation to obtain variables $p(n)$ and $q(n)$. Represent them as the trajectory plot of variable $p(n)$ - $q(n)$;

Step4: Obtain the graph of the mean square displacement $M(n)$ as it varies with n, and the progressive growth rate K of variable $M(n)$ from variables $p(n)$ and $q(n)$.

Step5: Compute the median of all K as the median value of K. When K converges to 1, the discrete set $\phi(N)$ displays chaotic characteristics. K converges to 0, and the discrete set $\phi(N)$ displays non-chaotic characteristics.

Judgment rules:

If the $p(n)$-$q(n)$ diagram displays a random Brownian motion pattern, $M(n)$ evolves linearly with time, and K is close to 1, it will be classified as a chaotic time series. If the $p(n)$-$q(n)$ diagram displays a bounded periodic ring, $M(n)$ is confined, and K is close to 0, it will be classified as something else. It is a non-chaotic time series, either periodic or exhibiting period-doubling. To avoid potential resonance between c and the Fourier decomposition of the time series during the calculation process, only 100 random numbers between $c\in[\pi/5,4\pi/5]$ are selected for analysis.The end result value is the median of K.

A. Determine the range of differential order α,β,γ

When $a=-4,b=0.2,c=0.9$, apply the 0-1 test to generate 40 corresponding iterative sequences for α,β,γ in the interval [0.8,1]. At the same time, taking the order α,β,γ as the abscissa, the interval is 0.005.For each α,β,γ generated discrete sequence $\{x(i)\}$ with a data span of 3000, bring in the noise-free 0-1 test to calculate the corresponding K value as the ordinate, complete the K diagram of the model iteration sequence, as shown in the Figs. 8-9.

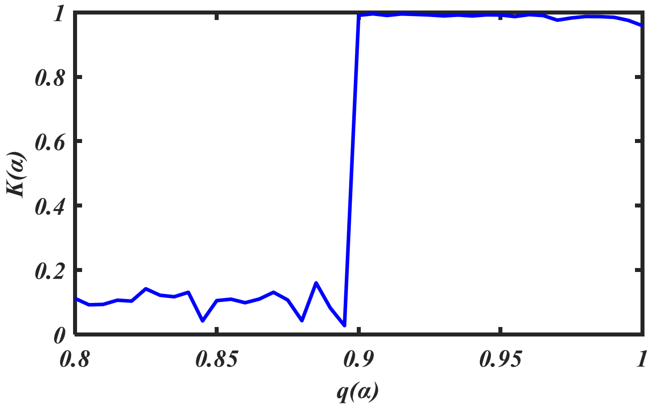

Fig. 8. The order K value diagram in relation to the fractional-order chaotic system: $q(\alpha)$-$K(\alpha)$

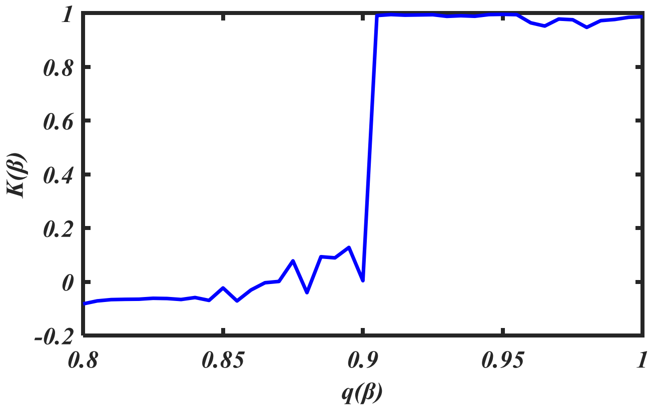

Fig. 9. The order K value diagram in accordance with the fractional-order chaotic system : $q(\beta)$-$K(\beta)$

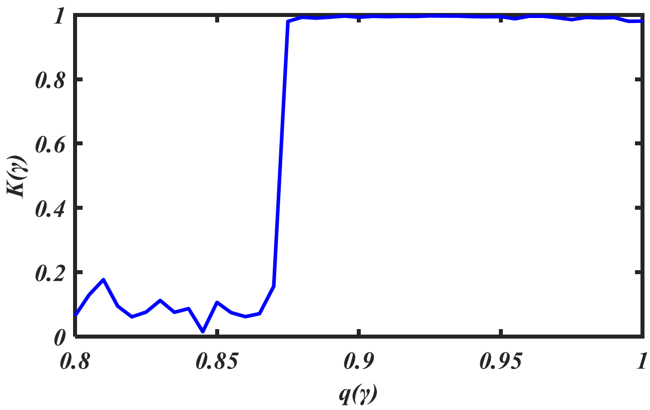

Fig. 10. The order K value diagram matching with the fractional-order chaotic system : $q(\gamma)-K(\gamma)$

When the orders α,β,γ are in the range of [0.9,1) and the K value is near 1, the system presents a chaotic state, with violent fluctuations, which needs to be properly adjusted to hinder long-term violent fluctuations. In the range of [0.8,0.9), the value of $K(\gamma)$ reaches 1, as a whole, whose part is still transitioning from a stable state to chaos.

B. Determination of the coefficient variation range

When $\alpha = 0.92$, $\beta = 0.92$, $\gamma = 0.95$, use the coefficient a of the system as the test parameter. Generate 200 corresponding iterative sequences for $a \in [-15, 5]$ .At the same time, use the change parameter a as the abscissa, and the interval is 0.1. A discrete sequence with a data span of 3000 is created for parameter a, the noise-free 0-1 test is used to calculate the K value corresponding to the three dimensions of x, y, z as the ordinate. The K diagram of the iterative sequence of which is shown in Figure 11-13.

They can be seen from Figs. 11-13 that when the K value is near 0 at $a \in [-15, -13.1)$, the system presents a transitioning from a stable state to chaos. When $a \in [-13.1, -6.5]$, the K value rapidly tends to 1, indicating that the fractional-order system may enter a chaotic state. When $a \in (-6.5, -0.9)$, the fractional-order chaotic system undergoes a transition from chaotic to non-chaotic or periodic behavior. As a result, some parts of the system's K value start to decrease from 1 to 0. When the $a \in [-0.9, 5]$, the sequence is a non-chaotic or periodic state, the corresponding 3-dimensional K values tend to 0.

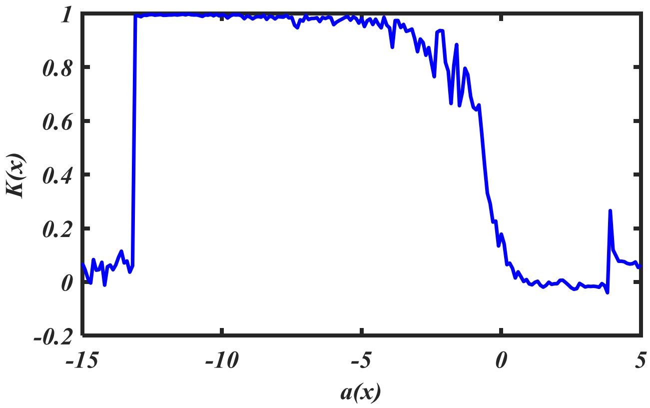

Fig. 11. The K diagram corresponding to the coefficients of the fractional chaotic system: $x - K(x)$

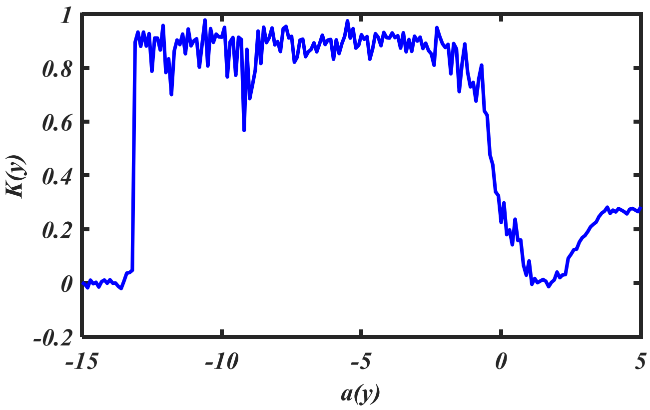

Fig. 12. The K diagram in relation to the coefficients of the fractional chaotic system: $y - K(y)$

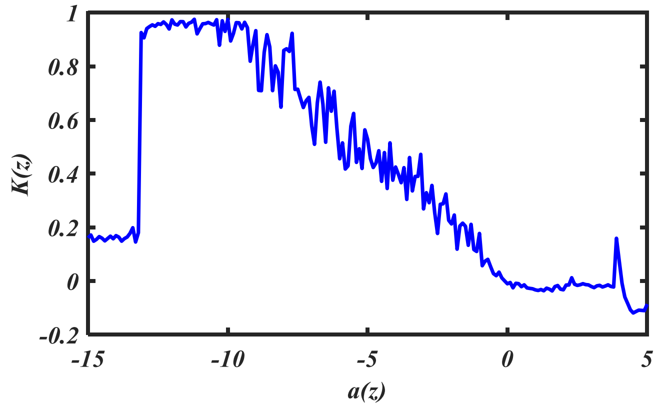

Fig. 13. The K diagram in relation to the coefficients of the fractional chaotic system: z-K(z)

C. Dynamics analysis within the specified range

This paper is primarily aimed at on examining the dynamics of the financial system by exploring variations in savings values a within a given range. The analysis of changes in investment costs b and commodity demand elasticity c, which similarly impact the financial system, is not included in this discussion.

(1) When $a = -14$, the financial system (2.1) performs on 0-1 test, the average of the three-dimensional K value of the financial system is 0-0.0595 (judged as "0"), 0.0429 (judged as "0"), 0.1647 (judged as "0"), respectively. As shown in Fig. 5 is the *s-p* trajectory diagram in accordance with the phase diagram of the financial system. The scatter plots of $M(n)$-n and $K(c)$-c provide evidence for this.

The graph for x dimensional data is only given in this paper. From Figs. 14-16 the trajectory of $p(t)$ is a periodic bounded motion. $M(n)$ generally decreases as n grows. The values of K is mostly concentrated near 0, indicating that the system is in an asymptotically stable equilibrium state. The entire financial system can operate stably and orderly. The long-term behavior of the system can be predicted.

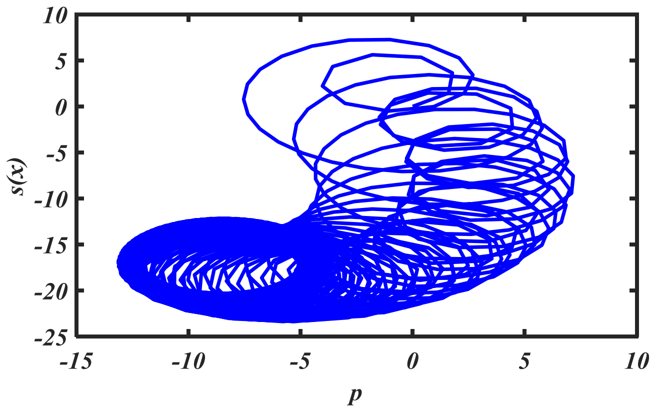

Fig. 14. The x-th dimension data of the system when a=-14 (a) p-s(x) trajectory graph

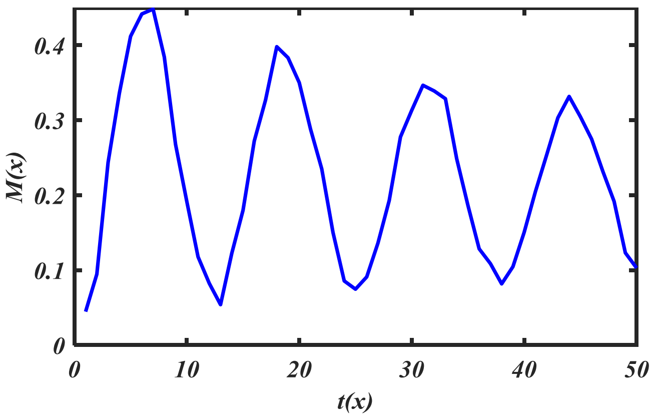

Fig. 15. The *x*-th dimension data of the system when a=-14 : *x*-M(*x*) graph

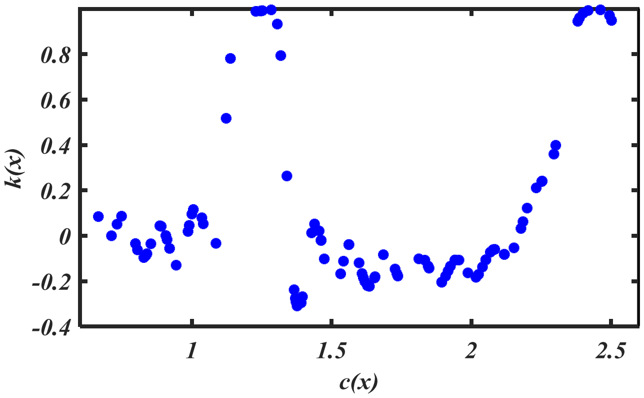

Fig. 16. The *x*-th dimension data of the system when a=-14:c(*x*)-K(*x*) scatter plot

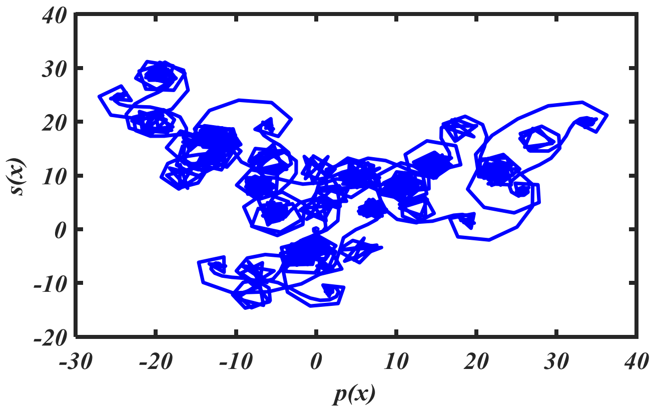

Fig. 17. The *x*-th dimension data of the system when a=-11 : p-s(*x*) trajectory graph

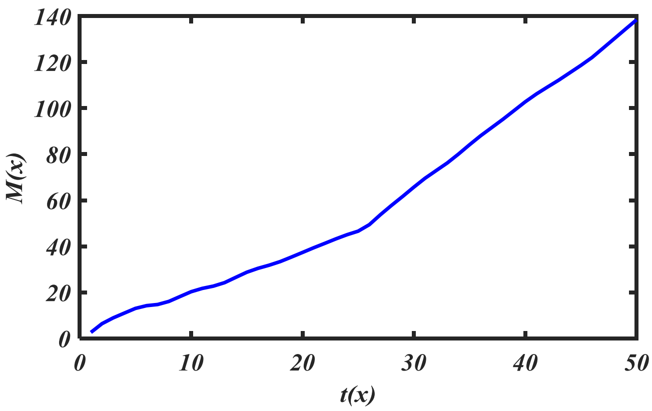

Fig. 18. The *x*-th dimension data of the system when a=-11 : $x-M\left(x\right)$ diagram

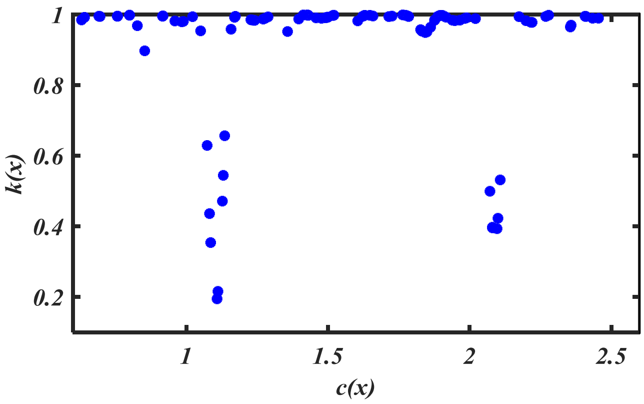

Fig. 19. The *x*-th dimension data of the system when a=-11 : c(*x*)-K(*x*) scatter plot

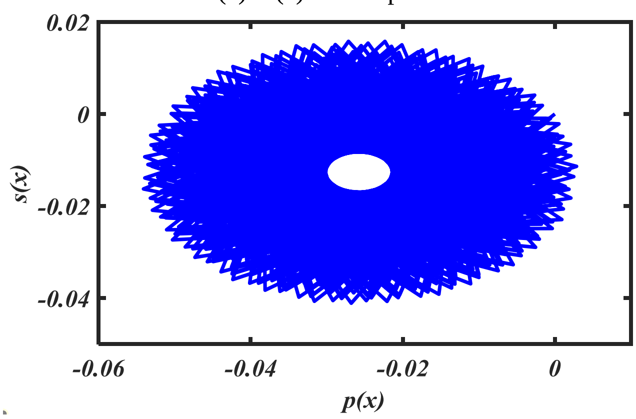

Fig. 20. The *x*-th dimension is data of the system when a=4:p-s(x) trajectory graph

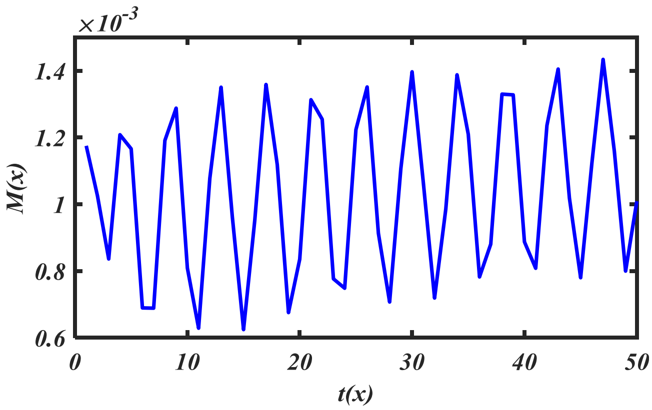

Fig. 21. The *x*-th dimension is data of the system when a=4: $x-M\left(x\right)$ diagram

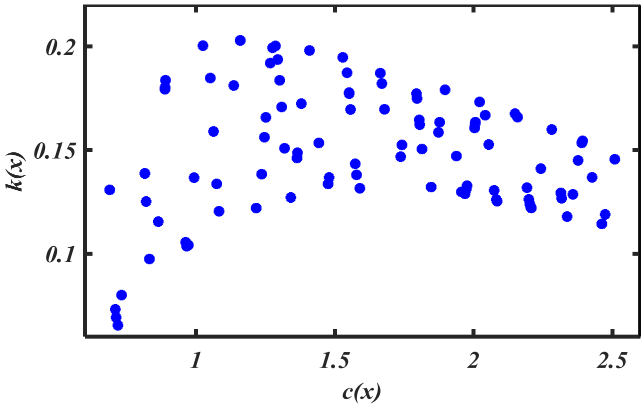

Fig. 22. The x-th dimension is data of the system when a=4 : c(x)-K(x) scatter plot

(2) When $a=-11$, perform 0-1 test on the financial system (2.1), the median of the three-dimensional K of the financial system is 0.9932 (judged as "1") and 0.9093 (judged as "1"), 0.9684 (decided as "1"). Figs. 17-19 show the s-p trajectory diagram. The scatter plots of $M(n)$-n and $c(x)-K(x)$ is corresponded to the phase diagram of the financial system. It is clear that the movement of $p(t)$ exhibits unbounded motion, similar to Brownian motion. $M(n)$ generally increases as n grows. The values of K are mostly concentrated around 1. It can be observed that the economic and financial systems have experienced significant and turbulent changes, indicating a state of chaos. Hence, making long-term predictions about the forthcoming state the economic and financial system becomes an arduous task due to its inherent chaotic nature. The system is in a state of disarray, posing a substantial threat to economic development. Given the detrimental effects it can have, it becomes imperative to implement timely macro-control measures in order to mitigate the long-term violent fluctuations and restore stability to the system.

(3) When $a=4$, Figs. 20-22 is similar to Figs. 14-16. The trajectory of $p(t)$ is a periodic and bounded motion. $M(n)$ shows irregular and bounded oscillation as n increased. $K(x)$ is mostly concentrated around 0, indicating that the system is non-chaotic at this time.

The above analysis highlights that the financial system can achieve stable operation and development when there is an appropriate balance among savings, investment costs, and elasticity of commodity demand. By maintaining the right combination of these key factors, the financial system can function smoothly. When excessive economic behavior occurs, it can lead to improper combination of key financial indicators. The emergence of chaotic phenomena is caused financial crises.

IV. New Sliding Mode Controller

The above research shows that the financial market is a complex dynamic system with significant chaotic effects. The objective of employing chaos control techniques in the financial market is to attenuate or eliminate chaotic behavior, thereby promoting stability and exerting successful regulation over the inconsistent periodic state within the chaotic attractor. By actively guiding the transformation of market chaos towards a desired direction, it is possible to reshape the dynamics of the underlying economic system and establish a new spatiotemporal order structure that aligns with predefined expectations. Therefore, a newly proposed sliding mode controller designed specifically for fractional-order financial chaotic system models can effectively control stability and produce positive outcomes in curtailing or preventing financial crises.

A. New sliding mode controller

Combined Eq (1) with sliding mode control theory, Eq (2) can be obtained

$$\begin{cases} \dfrac{d^{\alpha}x}{dt^{\alpha}}=z+(y-a)x+u_1 \\ \dfrac{d^{\beta}y}{dt^{\beta}}=1-by-x^2+u_2 \\ \dfrac{d^{\gamma}z}{dt^{\lambda}}=-x-cz+u_3 \end{cases} \tag{2}$$

Where u_0 represents the rate of sliding mode control, $u_0=[u_1 \quad u_2 \quad u_3]^T$ and u_1,u_2,u_3 represent the rates of control corresponding to each dimension, respectively. The fundamental operation of fractional calculus used in this paper is D_t^{∂}, which is defined as:

$$D_t^{\partial}=\begin{cases} \mathrm{d}^{\partial}/\mathrm{dt}^{\partial} & \mathrm{Re}(\partial)>0 \\ 1 & \mathrm{Re}(\partial)=0 \\ \int_{\partial}^{t}(d\tau)^{-\partial} & \mathrm{Re}(\partial)<0 \end{cases}$$

Given that the sliding mode switched function of this system is represented as:

$$s=c_1\cdot e+D_t^{-\sigma}e \tag{3}$$

Where c_1 is the sliding mode parameter, $D_t^{-\sigma}e$ represents the $-\sigma$ derivative of e,

$D_t^{-\sigma}e=\left[D_t^{-\sigma_1}e \quad D_t^{-\sigma_2}e \quad D_t^{-\sigma_3}e\right]^T$, $s=[s_1 \quad s_2 \quad s_3]^T$

$e=[e_1 \quad e_2 \quad e_3]=[x-x_e \quad y-y_e \quad z-z_e]^T$,and x_e,y_e,z_e are the reference values of state variables.

Eq (3) can be derived from Eq (4).

$$\dot{s}=c_1\cdot\dot{e}+D_t^{1-\sigma}e \tag{4}$$

When the parameters σ_1=0.08, σ_2=0.08, σ_3=0.05, the sliding mode approach rate is set to $\dot{s}=-hs-\varepsilon\,\mathrm{sgn}(s)$, the control rate can be obtained by combining Eq (3):

$$\begin{cases} u_1=-hs-\varepsilon\,\mathrm{sgn}(s)-c_1\cdot D_t^1e_1+D_t^{0.92}x_e-z-(y-a) \\ u_2=-hs-\varepsilon\,\mathrm{sgn}(s)-c_1\cdot D_t^1e_2+D_t^{0.92}y_e-1+by+x^2 \\ u_3=-hs-\varepsilon\,\mathrm{sgn}(s)-c_1\cdot D_t^1e_3+D_t^{0.95}z_e+x+cz \end{cases}$$

According to the structure of the observation system and the unique characteristics in the system, this paper sets a new sliding mode approach rate, as shown in Eq (5):

$$\dot{s}=\begin{bmatrix}-\left|\frac{e_1}{e_2+e_3}\right|\cdot s-\left(e_1-e_2\cdot e_3\right)^2\cdot\operatorname{sgn}(s)\\-\left|\frac{e_2}{e_1+e_3}\right|\cdot s-\left(e_2-e_1\cdot e_3\right)^2\cdot\operatorname{sgn}(s)\\-\left|\frac{e_3}{e_1+e_2}\right|\cdot s-\left(e_3-e_1\cdot e_2\right)^2\cdot\operatorname{sgn}(s)\end{bmatrix} \quad (5)$$

The control rate is:

$$\begin{cases}u_1=-\left|\frac{e_1}{e_2+e_3}\right|\cdot s-\left(e_1-e_2\cdot e_3\right)^2\cdot\operatorname{sgn}(s)-c_1\cdot D_t^1e_1\\ \quad +D_t^{0.92}x_e-z-(y-a)\\ u_2=-\left|\frac{e_2}{e_1+e_3}\right|\cdot s-\left(e_2-e_1\cdot e_3\right)^2\cdot\operatorname{sgn}(s)-c_1\cdot D_t^1e_2\\ \quad +D_t^{0.92}y_e-1+by+x^2\\ u_3=-\left|\frac{e_3}{e_1+e_2}\right|\cdot s-\left(e_3-e_1\cdot e_2\right)^2\cdot\operatorname{sgn}(s)-c_1\cdot D_t^1e_3\\ \quad +D_t^{0.95}z_e+x+cz\end{cases} \quad (6)$$

The new sliding mode approach rate design obviates the need for explicit parameter design, leading to a substantially simplified structure. The replacement of parameter design with the system error enables more variability in system control, leading to better realization of variable system control.

B. Stability analysis

The paper demonstrates the stability of the proposed approach rate using the Lyapunov stability theorem. According to this theorem, the system's state variable motion trajectory is guaranteed to converge to the sliding mode surface within a finite time. The stability of the system can be achieved. Hence, the aim is to demonstrate the stability of a new approach, which is a crucial aspect of our research.

Theorem 1: The switched function for sliding mode variable structure control systems is represented by Eq (3) in the paper. The Lyapunov function is defined as follows:

$$V=s^T\cdot s$$

if

$$\dot{V}=\dot{s}^Ts+s^T\dot{s}\le 0$$

then the control system is gradually stable.

Proof:

$$\dot{V}=\dot{s}^Ts+s^T\dot{s}=2s^T\cdot\dot{s}=2s^T\cdot\left(c_1\cdot\dot{e}+D_t^{1-\sigma}e\right)$$

$$=2s^T\cdot\begin{bmatrix}c_1\cdot D_t^1e_1+z+(y-a)+u_1-D_t^{0.92}x_e\\c_1\cdot D_t^1e_2+1-by-x^2+u_2-D_t^{0.92}y_e\\c_1\cdot D_t^1e_3-x-cz+u_3-D_t^{0.95}z_e\end{bmatrix}$$

$$=2s^T\cdot\left(-\begin{bmatrix}\left|\frac{e_1}{e_2+e_3}\right|\cdot s_1+\left(e_1-e_2\cdot e_3\right)^2\cdot\operatorname{sgn}(s_1)\\\left|\frac{e_2}{e_1+e_3}\right|\cdot s_2+\left(e_2-e_1\cdot e_3\right)^2\cdot\operatorname{sgn}(s_2)\\\left|\frac{e_3}{e_1+e_2}\right|\cdot s_3+\left(e_3-e_1\cdot e_2\right)^2\cdot\operatorname{sgn}(s_3)\end{bmatrix}\right)$$

$$=-2\left(\left|\frac{e_1}{e_2+e_3}\right|\cdot s_1^2+\left(e_1-e_2\cdot e_3\right)^2\cdot|s_1|+\left|\frac{e_2}{e_1+e_3}\right|\cdot s_2^2+\left(e_2-e_1\cdot e_3\right)^2\cdot|s_2|+\left|\frac{e_3}{e_1+e_2}\right|\cdot s_3^3+\left(e_3-e_1\cdot e_2\right)^2\cdot|s_3|\right)$$

$$<0$$

Based on the analysis and the evidence provided, we can indeed conclude that this system exhibits gradual stability.

V. Experimental simulation

The paper only conducts theoretical research on financial chaos. Set the control time at 1000s, keep $\alpha=0.92$, $\beta=0.92$, $\gamma=0.95, a=-4$, $b=0.2$, $c=0.9$, set parameters $x_e=\sin(0.1\cdot t)$, $h=0.0001$, $\varepsilon=0.0001$, $c_1=100$. $y_e=0.8\sin(0.1\cdot t)$, $z_e=0.5\sin(0.1\cdot t+2)$ are reference values, respectively, to display the effectiveness of the proposed method,

From Fig. 23, it is evident that the system experiences high-amplitude oscillations before 1000s. The new approach rate comparison is tracking response diagram after added sliding mode control. The system exhibits instability and chaotic motion in its current state. When the sliding mode control is introduced with a time span of 1000s, the system begins to track the desired reference value x_e. As a result, the system exhibits periodic motion state and achieves stability. In the paper, the traditional approach rate and the new approach rate are simulated and compared simultaneously, as shown in the enlarged diagram at 1000s. The new approach rate *x*-dimensional curve completely is overlaped with the reference value x_e. The results show that the new approach rate can achieve a better tracking effect compared to the traditional approach rate. The error response diagram in Fig. 24 illustrates the difference between the *x*-dimensional traditional approach rate and the new approach rate after applying sliding mode control. This diagram showcases the effectiveness of the new approach in reducing errors and improving performance. Indeed, it is evident from the figure that at around 1000s, the addition of sliding mode control significantly reduces the error in the new approach rate compared to the traditional approach rate. Both the traditional approach rate and the new approach rate exhibit stable behavior after the incorporation of sliding mode control. Definitely, it is notable that the error value of the new approach rate is considerably smaller compared to the traditional approach rate. The result indeed demonstrates that the tracking performance of the new approach rate is superior to that of the traditional approach rate. Notably, this improvement is achieved without the need for additional redundant parameters. This finding highlights the effectiveness and efficiency of the new approach, suggesting its potential for enhancing system performance and reducing errors.

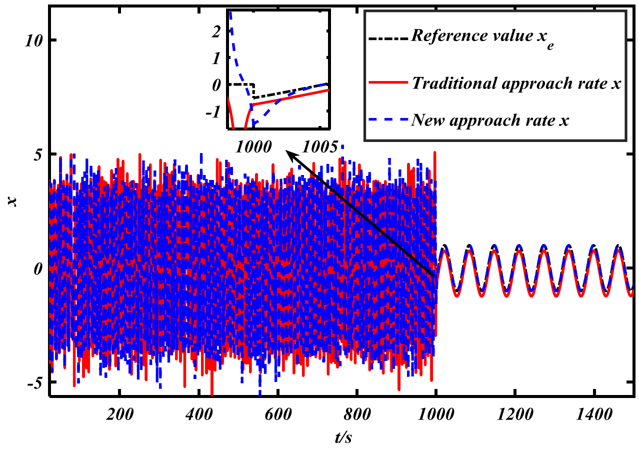

Fig. 23 Contrast track in gresponsesgraph about x

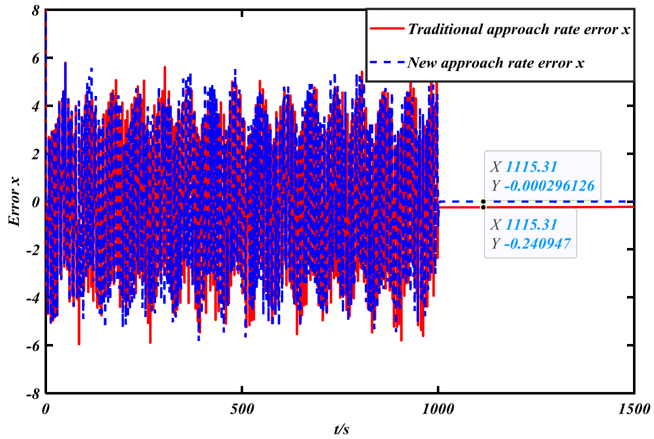

Fig. 24 Error responses graph about x

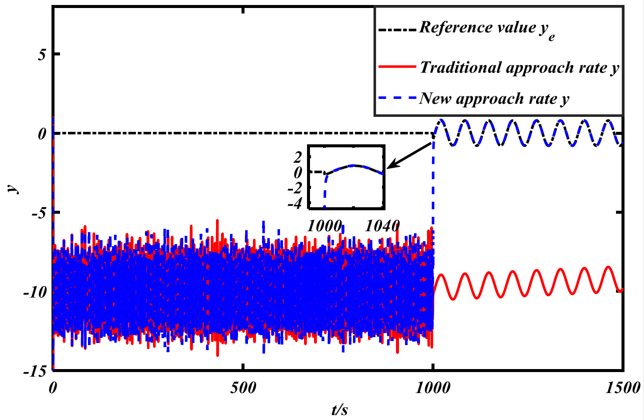

Fig. 25 Contrast tracking responses graph about y

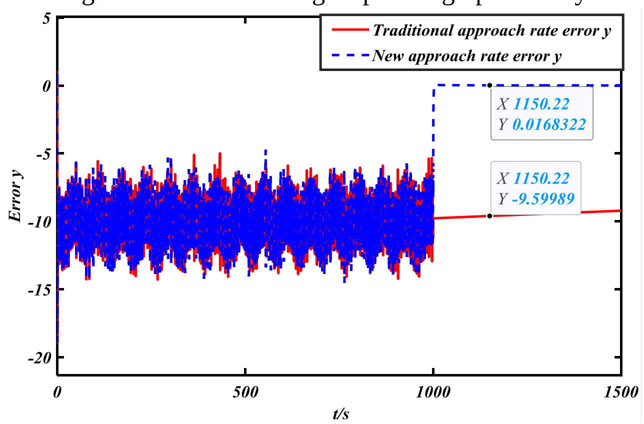

Fig. 26 Error responses graph about y

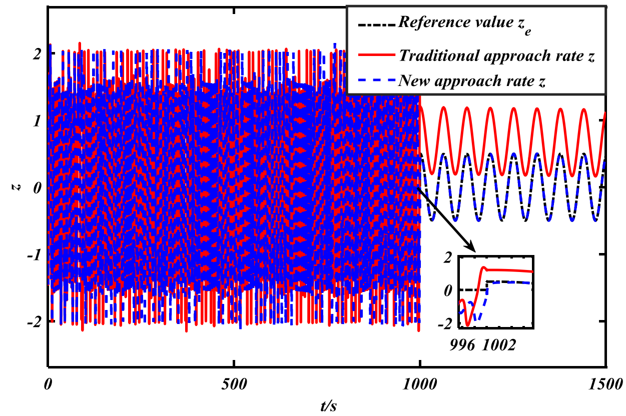

Fig. 27 Contrast tracking responses graph about z

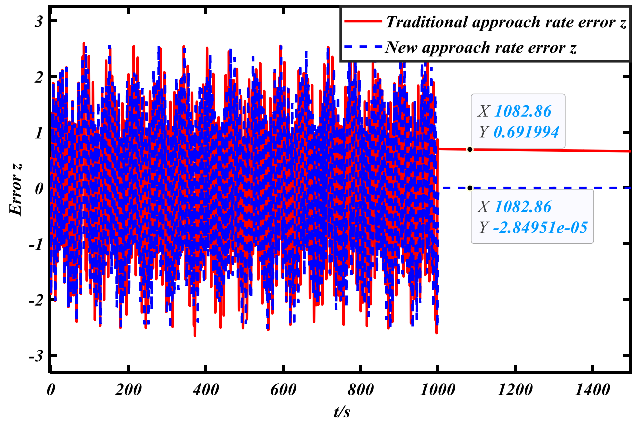

Fig.28 Error responses graph about z

The tracking response diagrams for the y and z dimensions are illustrated in Figure 25 and Figure 27, respectively. These figures depict a comparison between the traditional approach rate and the new approach rate after incorporating sliding mode control. The tracking response diagrams comparing the traditional approach rate and the new approach rate after adding sliding mode control for the y and z dimensions are actually presented in Figures 26 and 28, respectively. The rate error response graphs demonstrate that the introduction of the sliding mode control system at 1000s results in a transition from chaotic motions to a periodic state. This indicates that the new method effectively stabilizes the system and brings it into a more controlled and regular behavior. This conversion indicates that the system has achieved stability. Once it is established that the sliding mode controller achieves a desirable control effect, it can be observed that a new type of sliding mode controller exhibits superior tracking performance compared to the traditional sliding mode controller. The advantage in tracking performance can be attributed to advancements in control algorithms, system modeling, or the incorporation of innovative control techniques.

The above numerical results show that when chaos occurs in the financial system, the paper highlights the potential of a new sliding mode controller in achieving financial system stabilization. So, the numerical results are fully congruent with the theoretical derivation.

VI. Conclusion

Merging our understanding of financial chaos dynamics with the fundamental theories and methods of dynamic systems, a novel sliding mode control approach for chaos in fractional financial systems is presented. This study deeply explores the intrinsic characteristics of financial systems by establishing a fractional-order financial mathematical model. Through this analysis, the chaotic states hidden within the financial system and discern underlying patterns amidst the complexity of economic phenomena are uncovered. Constructed a sliding mode controller to govern and dictate chaotic states, with the ultimate objective of restoring normalcy to the financial system, holds immense practical significance. By employing this control mechanism, the aim is to bring order to a chaotic financial system, thereby ensuring stable and predictable functioning. Overall, this research endeavors to unravel the dynamics of financial chaos and provides avenues to mitigate its impact through the application of the sliding mode control approach. By providing an in-depth analysis of the intricacies of the financial system, the order and stability can be navigated towards being restored, therefore promoted the sustainable development of financial markets.

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