Global Dynamics of An Exponential Type 3-species Difference System

Changyou Wang, Tao Yang, Qiyu Wang, and Lili Jia

Abstract—**In this study, we focus on a novel exponential type 3-species biological difference system. To begin with, we verify that each positive solution of the aforementioned system is persistent and bounded. Next, we provide sufficient conditions under which the unique positive balance point of the system is globally asymptotically stable. Subsequently, we explore the convergence rate of the positive solutions for the biological system. Eventually, two numerical examples are presented to validate the efficacy of the outcomes gained in this work. Our method for addressing the problem is founded on the Lyapunov stability theorem and Poincaré theorem for the more generalized nonlinear difference equations along with the linearization theory.**

Index Terms—**Biology system, Boundedness, Persistence, Balance point, Stability**

I. INTRODUCTION

Mathematical modellings have been extensively investigated as they are capable of depicting a multitude of real-life issues in fields such as biology, ecology, economics, physics, and the like. (see [1-7]). In the past few years, accompanied by the swift advancement of biotechnological progress, biological mathematical models have received more and more attention from mathematical and biological worker (see, [8-14]). The difference equation and system, as an important biological mathematical model, can be used to describe real-life issues in population biology (see, [15-16]). Recently, significant endeavors have been made in the analysis of the qualitative behavior of the exponential-type difference system (as seen in [17-21]). Though the form of the difference system is rather straightforward, it is exceptionally hard to have a thorough comprehension of the dynamical behavior of its solutions (refer to [22-32]).

There are many works relational to the exponential type biology difference model, for example, El-Metwally et al. in [33] investigated the following single species difference biology system

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}, n = 0,1,2,\cdots, \quad (1)$$

here the coefficients $\alpha, \beta \in R^+$, and the initial conditions $x_{-1}, x_0 \in R^+ \bigcup \{0\}$. In model (1), α and β respectively stand for the movement rate and the growth rate of the population. The authors researched the boundedness, persistence and periodicity of solution, along with the global stability of positive balance point for the difference system (1). Moreover, Wang and Feng [34] changed model (1) to a new version as

$$x_{n+1} = \alpha + \beta x_n e^{-x_{n-1}}, n = 0,1,2,\cdots, \quad (2)$$

here the coefficients and initial conditions have the same meanings as those in model (1). Authors dissused the dynamics of the positive solution for the biology difference system (2). In [35], Ozturk et al. investigated the periodicity, convergence and boundedness of the solutions for the following difference system

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, n = 0,1,2,\cdots, \quad (3)$$

here the parameters $\alpha, \beta, \gamma \in R^+$, and the initial values $y_{-1}, y_0 \in R^+ \bigcup \{0\}$. In model (3), α, β and γ denote the movement rate, the growth rate and the carrying capacity of the species, respectively. Authors analyzed the boundedness, the convergence and the periodicity of the solutions for the model (3). In 2012, Papaschinopoulos et al. [36] extended the equation (3) to the following 2-species exponential type difference biology systems

$$\begin{aligned} &x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, y_{n+1} = \frac{\delta + \varepsilon e^{-x_n}}{\zeta + x_{n-1}}, n = 0,1,2,\cdots, \\ &x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, y_{n+1} = \frac{\delta + \varepsilon e^{-x_n}}{\zeta + y_{n-1}}, n = 0,1,2,\cdots, \quad (4) \\ &x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, y_{n+1} = \frac{\delta + \varepsilon e^{-y_n}}{\zeta + x_{n-1}}, n = 0,1,2,\cdots, \end{aligned}$$

and explored the persistence of biological system (4) and the dynamic behavior of its equilibrium point with the help of the linearization theory.

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Changyou Wang is a professor of College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225 China. (e-mail: wangchangyou417@163.com).

Tao Yang is a postgraduate student of College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225 China. (e-mail: 811247770@qq.com).

Qiyu Wang is a postgraduate student of College of Applied Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225 China. (e-mail: 958751002@qq.com).

Lili Jia is a professor of Department of Basic Teaching, Dianchi College of Yunnan University, Kunming, Yunnan 650228 China. (Corresponding author, phone: 13759190665; e-mail: lilijiadianchi@163.com).

More recently, Thai et al. [37] examined the dynamical properties of the 2-species exponential-type biology systems as follows

$$x_{n+1}=\frac{\alpha_1+\beta_1 e^{-x_{n-1}}}{\gamma_1+y_n},\ y_{n+1}=\frac{\alpha_2+\beta_2 e^{-y_{n-1}}}{\gamma_2+x_n},\ n=0,1,2,\cdots,$$
$$x_{n+1}=\frac{\alpha_1+\beta_1 e^{-y_{n-1}}}{\gamma_1+x_n},\ y_{n+1}=\frac{\alpha_2+\beta_2 e^{-x_{n-1}}}{\gamma_2+y_n},\ n=0,1,2,\cdots, \tag{5}$$

where the parameters $\alpha_i,\beta_i,\gamma_i\in R^+$, $i=1,2,$ and the initial conditions $x_j,y_j\in R^+\bigcup\{0\}$, $j=-1,0$.

Inspired by the aforementioned works, our objective in this work is to explore the characteristics of solution for the exponential-type 3-species biology systems

$$x_{n+1}=\frac{A_1+B_1e^{-y_{n-1}}}{C_1+x_n},\ y_{n+1}=\frac{A_2+B_2e^{-z_{n-1}}}{C_2+y_n},$$
$$z_{n+1}=\frac{A_3+B_3e^{-x_{n-1}}}{C_3+z_n},\ n=0,1,\cdots, \tag{6}$$

here the coefficients A_i, B_i, C_i, $i=1,2,3$ and the initial values x_{-j}, y_{-j}, z_{-j}, $j=0,1$ are positive constants. In model (6), A_i, B_i and C_i denote the movement rates, the growth rates and the carrying capacity of the x_n, y_n and z_n species, respectively. The primary objective of this article is to explore the persistence, the boundedness, and the dynamical behavior of the solution for systems (6). To start with, by employing inequality techniques, we demonstrate that each positive solution of systems (6) is bounded and persistent. Next, in light of the Lyapunov stability theorem, we provide sufficient conditions under which the unique positive balance point of model (6) is globally asymptotically stable. Then, through the utilization of the Poincaré theorem and the linearization technique, we examine the convergence rate of the solution of the model (6). Eventually, two numerical examples are presented to validate the feasibility of the theoretical outcomes achieved in this article.

Remark 1. As we all know, the complexity of ecosystems increases with the increase of population size, which leads to difficulties in studying multiple species dynamics systems. Additionally, it is extremely challenging to formulate certain conditions under which the balance point of multiple-species model is globally asymptotically stable, and this demands numerous tests. The contributions and innovations of this paper are stated as follows: (1) To describe the interaction among multi-species more precisely, an exponential-type 3-species biological difference system is established based on the known 2-species biological difference system. (2) By making use of the Lyapunov stability theorem and Poincaré theorem, and inventing some novel analysis approaches, certain criteria for the global asymptotically stable and convergence rate of the unique positive equilibrium for exponential 3-species biological difference system have been obtained. (3) Based on the fact that the system studied in this article has cyclic symmetry, the approaches achieved in this work can be improved and expanded to research higher dimensional exponential-type biological difference system. (4) Compared with the results in [31-33], the findings achieved in this paper are more generalized, which will considerably expand the application domain of the exponential biological difference system.

II. Preliminaries and Notations

For proving the main outcomes in this article, we first put forward some definitions and lemmas [4, 15, 16, 38, 39, 40] that are utilized throughout this work.

Lemma 1. Set I_x, I_y, I_z be certain intervals of real numbers and $f:I_x^2\times I_y^2\times I_z^2\to I_x$, $g:I_x^2\times I_y^2\times I_z^2\to I_y$, $h:I_x^2\times I_y^2\times I_z^2\to I_z$ be continuously differentiable functions. Thus, for any initial values $(x_i,y_i,z_i)\in I_x\times I_y\times I_z, (i=-1,0)$ the difference systems type such as

$$\begin{cases} x_{n+1}=f(x_n,x_{n-1},y_n,y_{n-1},z_n,z_{n-1}),\\ y_{n+1}=g(x_n,x_{n-1},y_n,y_{n-1},z_n,z_{n-1}), \quad n=0,1,2,\cdots,\\ z_{n+1}=h(x_n,x_{n-1},y_n,y_{n-1},z_n,z_{n-1}),\end{cases} \tag{7}$$

has a unique solution $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$.

Definition 1. For $(\bar{x},\bar{y},\bar{z})\in I_x\times I_y\times I_z$, if $\bar{x}=f(\bar{x},\bar{x},\bar{y},\bar{y},\bar{z},\bar{z})$, $\bar{y}=g(\bar{x},\bar{x},\bar{y},\bar{y},\bar{z},\bar{z})$, $\bar{z}=h(\bar{x},\bar{x},\bar{y},\bar{y},\bar{z},\bar{z})$, i. e., $(x_n,y_n,z_n)=(\bar{x},\bar{y},\bar{z})$ for all $n\geq 0$, then $(\bar{x},\bar{y},\bar{z})$ is called an equilibrium point of system (7).

Definition 2. Interval $I_x\times I_y\times I_z$ is called invariant about equations (7) if, $x_n\in I_x, y_n\in I_y, z_n\in I_y$, for all $n\geq 0$, when the initial values $x_{-1},x_0\in I_x$, $y_{-1},y_0\in I_y$, $z_{-1},z_0\in I_z$.

Definition 3. Assume that $(\bar{x},\bar{y},\bar{z})$ is an equilibrium point of system (7). Then

(i) point $(\bar{x},\bar{y},\bar{z})$ is called to be stable relative to $I_x\times I_y\times I_z$ if for every $\varepsilon>0$, there exits $\delta>0$ such that for any initial values $(x_i,y_i,z_i)\in I_x\times I_y\times I_z$, with

$$\sum_{i=-1}^{0}|x_i-\bar{x}|<\delta,\ \sum_{i=-1}^{0}|y_i-\bar{y}|<\delta,\ \sum_{i=-1}^{0}|y_i-\bar{y}|<\delta,$$

can imply $|x_n-\bar{x}|<\varepsilon, |y_n-\bar{y}|<\varepsilon, |z_n-\bar{z}|<\varepsilon$.

(ii) point $(\bar{x},\bar{y},\bar{z})$ is called an attractor relative to $I_x\times I_y\times I_z$ if for all $(x_i,y_i,z_i)\in I_x\times I_y\times I_z, (i=-1,0)$, $\lim_{n\to\infty}x_n=\bar{x}, \lim_{n\to\infty}y_n=\bar{y}, \lim_{n\to\infty}z_n=\bar{z}$.

(iii) point $(\bar{x},\bar{y},\bar{z})$ is called asymptotically stable relative to $I_x \times I_y \times I_z$ if it is stable and an attractor.

(iv) point $(\bar{x},\bar{y},\bar{z})$ is called unstable if it is not stable.

Definition 4. If $(\bar{x},\bar{x},\bar{y},\bar{y},\bar{z},\bar{z})$ is an equilibrium point of the vector valued mapping $F=(f,x_n,g,y_n,h,z_n)$ where f,g and h are continuous differentiable functions at point $(\bar{x},\bar{y},\bar{z})$. Then the linearized equations of (7) about the balance point $(\bar{x},\bar{y},\bar{z})$ is

$$\Psi_{n+1} = J_F \Psi_n ,$$

where $\Psi_n=(x_n,x_{n-1},y_n,y_{n-1},z_n,z_{n-1})^T$ and J_F is the Jacobian matrix of system (7) about the balance point $(\bar{x},\bar{y},\bar{z})$.

Definition 5. If there exist positive constants m, M and an integer $N \geq -1$ such that $m \leq x_n, y_n, z_n \leq M$, the positive solution $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ of system (7) is called to be persists and bounded

Definition 6. A scalar continuous function $V: R^n \to R$ is called to be a positive definite Lyapunov function relative to an equilibrium point $\bar{\Psi}$ if $V(\bar{\Psi})=0$ and $V(\Psi_n)>0$ for $\Psi_n \neq \bar{\Psi}$ in some open ball $B_r(\bar{\Psi})$.

Lemma 2. (The Linearized Stability Theorem) Suppose that $\Psi_{n+1} = F(\Psi_n), n \in N,$ is a system of difference equations and $\bar{\Psi}$ is the balance point of the system, i.e. $F(\bar{\Psi}) = \bar{\Psi}$.

(i) If all eigenvalues of the Jacobian matrix J_F about $\bar{\Psi}$ lie inside the open unit disk $|\lambda|<1$, then $\bar{\Psi}$ is locally asymptotically stable.

(ii) If at least one of eigenvalues of the Jacobian matrix J_F about $\bar{\Psi}$ has modulus greater than one, then $\bar{\Psi}$ is unstable.

With the aid of the Rouche Theorem, the following results are achieved.

Lemma 3. Consider the following polynomial equation

$$\lambda^{v+1} - \sum_{i=0}^{v} p_i \lambda^{v-i} = 0 . \tag{8}$$

Assume that $p_i \in R, i=0,1,2,\cdots, v \in \{0,1,2,\cdots\}$, then all roots of equation (8) lie inside the open unit disk when $\sum_{i=0}^{v} |p_i| < 1$.

Proof. Suppose that C is a circular curve with the coordinate origin as the center and 1 as the radius, $f(z)=z^{v+1}$ and $\varphi(z) = -\sum_{i=0}^{v} p_i z^{v-i}$. It is not difficult to prove that functions $f(z)$ and $\varphi(z)$ meet with all the conditions of Rouche Theorem. Moreover, function $f(z)+\varphi(z)$ and $f(z)$ have the same number of zeros in curve C. Obviously, function $f(z)$ has $v+1$ zeros in curve C, so function $f(z)+\varphi(z)$ also has $v+1$ zeros in curve C. That is, the function

$$f(z)+\varphi(z) = z^{v+1} - \sum_{i=0}^{v} p_i z^{v-i}$$

has $v+1$ zeros in curve C. Therefore, the modulus of all characteristic roots of equation (8) is less than 1. □

Lemma 4. (Lyapunov Stability Theorem) Let V be a positive definite Lyapunov function relative to an isolated equilibrium point $\bar{\Psi}$ of F. If $-\Delta V = V(\Psi_n) - V(\Psi_{n+1})$ is also positive definite relative to $\bar{\Psi}$, then $\bar{\Psi}$ is asymptotically stable. Furthermore, if V and $-\Delta V$ are positive definite on the invariant $I_x \times I_y \times I_z$ of F, $r<\infty$, and either: (i) $r=\infty$ and $V(\Psi) \to \infty$ as $\|\Psi\| \to \infty$, or (ii) $r<\infty$ and $F(B_r(\bar{\Psi})) \subset B_r(\bar{\Psi})$. Then, $\bar{\Psi}$ is globally asymptotically stable relative to $B_r(\bar{\Psi})$.

Lemma 5. (Poincaré Theorem) Consider the following Poincaré difference system

$$X(n+1) = [A+B(n)]X(n), \tag{9}$$

where $A \in C^{k\times k}$ is a constant matrix and $B: Z^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ as } \|B(n)\| \to 0 \tag{10}$$

If $X(n)$ is a solution of (9), then either $X(n)=0$ for all large n or

$$\sigma = \lim_{n\to\infty} \sqrt[n]{\|X_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

III. DYNAMICAL BEHAVIOR OF POSITIVE SOLUTION

Next, we research the dynamical behavior of positive solutions of the systems (6).

Theorem 1. Each positive solution of system (6) are persist and bounded.

Proof. Suppose that $\{(x_n, y_n, z_n)\}_{n=-1}^{\infty}$ is an arbitrary solution of model (6), we can obtain

$$x_{n+1} \leq \frac{A_1+B_1}{C_1} = M_1, \ y_{n+1} \leq \frac{A_2+B_2}{C_2} = M_2, \\ z_{n+1} \leq \frac{A_3+B_3}{C_3} = M_3, n \in N. \tag{11}$$

Then, from system (6) and equations (11), one has

$$x_{n+1} \geq \frac{A_1 + B_1 e^{-M_2}}{C_1 + M_1} = m_1, y_{n+1} \geq \frac{A_2 + B_2 e^{-M_3}}{C_2 + M_2} = m_2,$$
$$z_{n+1} \geq \frac{A_3 + B_3 e^{-M_1}}{C_3 + M_3} = m_3, n \in N. \quad (12)$$

Therefore, by means of equations (11) and (12), the proof of Theorem 1 has been finished. □

Theorem 2. Assume that

$$(\overline{x}, \overline{y}, \overline{z}) \in [m_1, M_1] \times [m_2, M_2] \times [m_3, M_3]$$

is the balance point of model (6), then $(\overline{x}, \overline{y}, \overline{z})$ is local asymptotic stable when $\Theta < 1$, where

$$\Theta = \frac{1}{C_1 C_2 C_3}[C_2 M_1 (C_3 + 2M_3) + C_3 M_2 (C_1 + 2M_1) + C_1 M_3 (C_2 + 2M_2) + M_1 M_2 M_3 + B_1 B_2 B_3]. \quad (13)$$

Proof. From system (6), it holds that

$$\overline{x} = \frac{A_1 + B_1 e^{-\overline{y}}}{C_1 + \overline{x}}, \overline{y} = \frac{A_2 + B_2 e^{-\overline{z}}}{C_2 + \overline{y}}, \overline{z} = \frac{A_3 + B_3 e^{-\overline{x}}}{C_3 + \overline{z}}. \quad (14)$$

To construct the corresponding linearization form of system (6), we contemplate the transformations such as

$$(x_{n+1}, x_n, y_{n+1}, y_n, z_{n+1}, z_n) \to F = (f, f_1, g, g_1, h, h_1) \quad (15)$$

here

$$f = \frac{A_1 + B_1 e^{-y_{n-1}}}{C_1 + y_n}, f_1 = x_n, g = \frac{A_2 + B_2 e^{-z_{n-1}}}{C_2 + z_n},$$
$$g_1 = y_n, h = \frac{A_3 + B_3 e^{-x_{n-1}}}{C_3 + x_n}, h_1 = z_n.$$

By using transformation (15), we have

$$F_J(\overline{x}, \overline{y}, \overline{z}) = \begin{pmatrix} s_1 & 0 & 0 & t_1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_2 & 0 & 0 & t_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & s_3 & 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (16)$$

where

$$s_1 = \frac{A_1 + B_1 e^{-\overline{y}}}{(C_1 + \overline{x})^2}, t_1 = -\frac{B_1 e^{-\overline{y}}}{C_1 + \overline{x}}, s_2 = \frac{A_2 + B_2 e^{-\overline{z}}}{(C_2 + \overline{y})^2},$$
$$t_2 = -\frac{B_2 e^{-\overline{z}}}{C_2 + \overline{y}}, s_3 = -\frac{B_3 e^{-\overline{x}}}{C_3 + \overline{z}}, t_3 = \frac{A_3 + B_3 e^{-\overline{x}}}{(C_3 + \overline{z})^2}.$$

The characteristic equation of $F_J(\overline{x}, \overline{y}, \overline{z})$ is as follow

$$\lambda^6 + a_1 \lambda^5 + a_2 \lambda^4 + a_3 \lambda^3 + a_4 = 0, \quad (17)$$

where

$$a_1 = -(s_1 + s_2 + t_3), a_2 = s_1 s_2 + s_1 t_3 + s_2 t_3,$$
$$a_3 = -s_1 s_2 t_3, a_4 = -s_3 t_1 t_2.$$

Then we cancompute $\sum_{i=1}^{4} |a_i|$ as follows

$$\begin{aligned}
\sum_{i=1}^{4} |a_i| &= \frac{A_1 + B_1 e^{-\overline{y}}}{(C_1 + \overline{x})^2} + \frac{A_2 + B_2 e^{-\overline{z}}}{(C_2 + \overline{y})^2} + \frac{A_3 + B_3 e^{-\overline{x}}}{(C_3 + \overline{z})^2} \\
&+ \frac{(A_1 + B_1 e^{-\overline{y}})(A_2 + B_2 e^{-\overline{z}})}{(C_1 + \overline{x})^2 (C_2 + \overline{y})^2} \\
&+ \frac{(A_1 + B_1 e^{-\overline{y}})(A_3 + B_3 e^{-\overline{x}})}{(C_1 + \overline{x})^2 (C_3 + \overline{z})^2} \\
&+ \frac{(A_2 + B_2 e^{-\overline{z}})(A_3 + B_3 e^{-\overline{x}})}{(C_2 + \overline{y})^2 (C_3 + \overline{z})^2} \\
&+ \frac{(A_1 + B_1 e^{-\overline{y}})(A_2 + B_2 e^{-\overline{z}})(A_3 + B_3 e^{-\overline{x}})}{(C_1 + \overline{x})^2 (C_2 + \overline{y})^2 (C_3 + \overline{z})^2} \\
&+ \frac{B_1 B_2 B_3 e^{-\overline{x} - \overline{y} - \overline{z}}}{(C_1 + \overline{x})(C_2 + \overline{y})(C_3 + \overline{z})} \\
&= \frac{\overline{x}}{C_1 + \overline{x}} + \frac{\overline{y}}{C_2 + \overline{y}} + \frac{\overline{z}}{C_3 + \overline{z}} + \frac{\overline{x}\,\overline{y}}{(C_1 + \overline{x})(C_2 + \overline{y})} \\
&+ \frac{\overline{x}\,\overline{z}}{(C_1 + \overline{x})(C_3 + \overline{z})} + \frac{\overline{y}\,\overline{z}}{(C_2 + \overline{y})(C_3 + \overline{z})} \\
&+ \frac{\overline{x}\,\overline{y}\,\overline{z}}{(C_1 + \overline{x})(C_2 + \overline{y})(C_3 + \overline{z})} \\
&+ \frac{B_1 B_2 B_3 e^{-\overline{x} - \overline{y} - \overline{z}}}{(C_1 + \overline{x})(C_2 + \overline{y})(C_3 + \overline{z})} \\
&= \frac{\overline{x}}{(C_1 + \overline{x})(C_3 + \overline{z})}(C_3 + 2\overline{z}) \\
&+ \frac{\overline{y}}{(C_1 + \overline{x})(C_2 + \overline{y})}(C_1 + 2\overline{x}) \\
&+ \frac{\overline{z}}{(C_2 + \overline{y})(C_3 + \overline{z})}(C_2 + 2\overline{y}) \\
&+ \frac{\overline{x}\,\overline{y}\,\overline{z} + B_1 B_2 B_3 e^{-\overline{x} - \overline{y} - \overline{z}}}{(C_1 + \overline{x})(C_2 + \overline{y})(C_3 + \overline{z})} \\
&< \frac{\overline{x}}{C_1 C_3}(C_3 + 2\overline{z}) + \frac{\overline{y}}{C_1 C_2}(C_1 + 2\overline{x}) \\
&+ \frac{\overline{z}}{C_2 C_3}(C_2 + 2\overline{y}) + \frac{\overline{x}\,\overline{y}\,\overline{z} + B_1 B_2 B_3}{C_1 C_2 C_3} \\
&= \frac{1}{C_1 C_2 C_3}[C_2 \overline{x}(C_3 + 2\overline{z}) + C_3 \overline{y}(C_1 + 2\overline{x}) \\
&+ C_1 \overline{z}(C_2 + 2\overline{y}) + \overline{x}\,\overline{y}\,\overline{z} + B_1 B_2 B_3] \\
&< \frac{1}{C_1 C_2 C_3}[C_2 M_1 (C_3 + 2M_3) + C_3 M_2 (C_1 + 2M_1) \\
&+ C_1 M_3 (C_2 + 2M_2) + M_1 M_2 M_3 + B_1 B_2 B_3] = \Theta.
\end{aligned} \quad (18)$$

From equation (18) and supposing condition $\Theta < 1$, it holds that $\sum_{i=1}^{4}|a_i| < 1$. Therefore, by using **Lemma** 2 and 3, the positive balance point $(\bar{x}, \bar{y}, \bar{z})$ of model (6) is local asymptotic stable. □

Theorem 3. The balance point $(\bar{x}, \bar{y}, \bar{z})$ of model (6) is global asymptotic stable if

$$\max_{1\le i\le 3} \frac{A_i + B_i}{C_i} < 1, \tag{19}$$

and

$$\begin{aligned} &A_1 + B_1 e^{-m_1} < \bar{x}(C_1 + m_2), \\ &A_2 + B_2 e^{-m_2} < \bar{y}(C_2 + m_3), \\ &A_3 + B_3 e^{-m_3} < \bar{z}(C_3 + m_1). \end{aligned} \tag{20}$$

Proof. We take into consideration the discrete time analogue of the Lyapunov function such as

$$\begin{aligned} V(x_n, y_n, z_n) = \bar{x}(\frac{x_n}{\bar{x}} - 1 - \ln\frac{x_n}{\bar{x}}) + \\ \bar{y}(\frac{y_n}{\bar{y}} - 1 - \ln\frac{y_n}{\bar{y}}) + \bar{z}(\frac{z_n}{\bar{z}} - 1 - \ln\frac{z_n}{\bar{z}}). \end{aligned} \tag{21}$$

It is very easy to prove that $V(x_n, y_n, z_n)$ is a positive definite Lyapunov function relative to an balance point $(\bar{x}, \bar{y}, \bar{z})$ by using the following inequality

$$x - 1 - \ln x \ge 0, \forall x > 0. \tag{22}$$

From (22), it follows that

$$\begin{aligned} -\ln\frac{x_{n+1}}{x_n} &\le -\frac{x_{n+1} - x_n}{x_{n+1}}, \\ -\ln\frac{y_{n+1}}{y_n} &\le -\frac{y_{n+1} - y_n}{y_{n+1}}, \\ -\ln\frac{z_{n+1}}{z_n} &\le -\frac{z_{n+1} - z_n}{z_{n+1}}. \end{aligned} \tag{23}$$

Furthermore, from (6), (20), (21) and (23), we have

$$\begin{aligned} &\Delta V = V(x_{n+1}, y_{n+1}, z_{n+1}) - V(x_n, y_n, z_n) \\ &= \bar{x}(\frac{x_{n+1}}{\bar{x}} - 1 - \ln\frac{x_{n+1}}{\bar{x}}) + \bar{y}(\frac{y_{n+1}}{\bar{y}} - 1 - \ln\frac{y_{n+1}}{\bar{y}}) \\ &+ \bar{z}(\frac{z_{n+1}}{\bar{z}} - 1 - \ln\frac{z_{n+1}}{\bar{z}}) - \bar{x}(\frac{x_n}{\bar{x}} - 1 - \ln\frac{x_n}{\bar{x}}) \\ &- \bar{y}(\frac{y_n}{\bar{y}} - 1 - \ln\frac{y_n}{\bar{y}}) - \bar{z}(\frac{z_n}{\bar{z}} - 1 - \ln\frac{z_n}{\bar{z}}) \\ &= (x_{n+1} - x_n) + (y_{n+1} - y_n) + (z_{n+1} - z_n) \\ &- \bar{x}\ln\frac{x_{n+1}}{x_n} - \bar{y}\ln\frac{y_{n+1}}{y_n} - \bar{z}\ln\frac{z_{n+1}}{z_n} \\ &\le (x_{n+1} - x_n) + (y_{n+1} - y_n) + (z_{n+1} - z_n) \\ &- \bar{x}(\frac{x_{n+1} - x_n}{x_{n+1}}) - \bar{y}(\frac{y_{n+1} - y_n}{y_{n+1}}) - \bar{z}(\frac{z_{n+1} - z_n}{z_{n+1}}) \\ &= (x_{n+1} - x_n)(1 - \frac{\bar{x}}{x_{n+1}}) + (y_{n+1} - y_n) \\ &(1 - \frac{\bar{y}}{y_{n+1}}) + (z_{n+1} - z_n)(1 - \frac{\bar{z}}{z_{n+1}}) \\ &= (x_{n+1} - x_n)(1 - \frac{\bar{x}(C_1 + y_n)}{A_1 + B_1 e^{-x_{n-1}}}) \\ &+ (y_{n+1} - y_n)(1 - \frac{\bar{y}(C_2 + z_n)}{A_2 + B_2 e^{-y_{n-1}}}) \\ &+ (z_{n+1} - z_n)(1 - \frac{\bar{z}(C_3 + x_n)}{A_3 + B_3 e^{-z_{n-1}}}) \\ &\le (M_1 - m_1)[\frac{A_1 + B_1 e^{-m_1} - \bar{x}(C_1 + m_2)}{A_1 + B_1 e^{-m_1}}] \\ &+ (M_2 - m_2)[\frac{A_2 + B_2 e^{-m_2} - \bar{y}(C_2 + m_3)}{A_2 + B_2 e^{-m_2}}] \\ &+ (M_3 - m_3)[\frac{A_3 + B_3 e^{-m_3} - \bar{z}(C_3 + m_1)}{A_3 + B_3 e^{-m_3}}] < 0. \end{aligned} \tag{24}$$

Therefore, by means of (19), (24) and **Lemma 4**, the balance point of $(\bar{x}, \bar{y}, \bar{z})$ model (6) is global asymptotic stable to $[m_1, M_1] \times [m_2, M_2] \times [m_3, M_3]$. □

Next, we investigate the convergence rate of the positive solution of the difference model (6). Set the error vector is as follow

$$R_n = (R_n^{(1)}, R_n^{(2)}, R_n^{(3)}) = (x_n - \bar{x}, y_n - \bar{y}, z_n - \bar{z}),$$

then we can obtain the following result..

Theorem 4. If (x_n, y_n, z_n) is a positive solution of model (6) with

$$\lim_{n\to\infty}(x_n, y_n, z_n) = (\bar{x}, \bar{y}, \bar{z}),$$

where $(\bar{x}, \bar{y}, \bar{z}) \in [m_1, M_1] \times [m_2, M_2] \times [m_3, M_3]$ is the equilibrium point of system (6), then the error vector R_n of each solution of system (6) supplies the asymptotic relation as follows

$$\lim_{n\to\infty}(x_n, y_n, z_n) = (\bar{x}, \bar{y}, \bar{z}), \tag{25}$$

here σ is the modulus of one of the eigenvalues of matrix $F_J(\bar{x}, \bar{y}, \bar{z})$.

Proof. From the system (6), it holds that

$$\begin{aligned} R_{n+1}^{(1)} = x_{n+1} - \bar{x} &= \frac{A_1 + B_1 e^{-y_{n+1}}}{C_1 + x_n} - \frac{A_1 + B_1 e^{-\bar{y}}}{C_1 + \bar{x}} \\ &= -\frac{A_1 + B_1 e^{-\bar{y}}}{(C_1 + x_n)(C_1 + \bar{x})}(x_n - \bar{x}) \end{aligned}$$

$$+\frac{B_1(e^{-y_{n-1}}-e^{-\bar{y}})}{(C_1+x_n)(y_{n-1}-\bar{y})}(y_{n-1}-\bar{y}). \tag{26}$$

Similarly, it can be concluded that

$$\begin{aligned} R_{n+1}^{(2)} &= y_{n+1}-\bar{y} \\ &= -\frac{A_2+B_2e^{-\bar{z}}}{(C_2+y_n)(C_2+\bar{y})}(y_n-\bar{y}) \\ &\quad+\frac{B_2(e^{-z_{n-1}}-e^{-\bar{z}})}{(C_2+y_n)(z_{n-1}-\bar{z})}(z_{n-1}-\bar{z}), \end{aligned} \tag{27}$$

and

$$\begin{aligned} R_{n+1}^{(3)} &= z_{n+1}-\bar{z} \\ &= \frac{A_3+B_3e^{-\bar{x}}}{(C_3+z_n)(C_3+\bar{z})}(z_n-\bar{z}) \\ &\quad+\frac{B_3(e^{-x_{n-1}}-e^{-\bar{x}})}{(C_3+z_n)(x_{n-1}-\bar{x})}(x_{n-1}-\bar{x}). \end{aligned} \tag{28}$$

Set

$$\begin{aligned} d_n &= -\frac{A_1+B_1e^{-\bar{y}}}{(C_1+x_n)(C_1+\bar{x})}, e_n=\frac{B_1(e^{-y_{n-1}}-e^{-\bar{y}})}{(C_1+x_n)(y_{n-1}-\bar{y})}, \\ f_n &= -\frac{A_2+B_2e^{-\bar{z}}}{(C_2+y_n)(C_2+\bar{y})}, g_n=\frac{B_2(e^{-z_{n-1}}-e^{-\bar{z}})}{(C_2+y_n)(z_{n-1}-\bar{z})}, \\ h_n &= -\frac{A_3+B_3e^{-\bar{x}}}{(C_3+z_n)(C_3+\bar{z})}, j_n=\frac{B_3(e^{-x_{n-1}}-e^{-\bar{x}})}{(C_3+z_n)(x_{n-1}-\bar{x})}, \end{aligned} \tag{29}$$

then, from (26)-(28), we have

$$\begin{aligned} R_{n+1}^{(1)} &= d_nR_n^{(1)}+e_nR_{n-1}^{(2)}, \\ R_{n+1}^{(2)} &= f_nR_n^{(2)}+g_nR_{n-1}^{(3)}, \\ R_{n+1}^{(3)} &= h_nR_n^{(3)}+j_nR_{n-1}^{(1)}. \end{aligned} \tag{30}$$

Moreover, taking the limits of (29), we obtain

$$\begin{aligned} &\lim_{n\to\infty} d_n=-\frac{A_1+B_1e^{-\bar{y}}}{(C_1+\bar{x})^2}, \lim_{n\to\infty} e_n=-\frac{B_1e^{-\bar{y}}}{C_1+\bar{x}}, \\ &\lim_{n\to\infty} f_n=-\frac{A_2+B_2e^{-\bar{z}}}{(C_2+\bar{y})^2}, \lim_{n\to\infty} g_n=-\frac{B_2e^{-\bar{z}}}{C_2+\bar{y}}, \\ &\lim_{n\to\infty} h_n=-\frac{A_3+B_3e^{-\bar{x}}}{(C_3+\bar{z})^2}, \lim_{n\to\infty} j_n=-\frac{B_3e^{-\bar{x}}}{C_3+\bar{z}}, \end{aligned} \tag{31}$$

According to the theorem of the relationship between limits and infinitesimals, it can be obtained that

$$\begin{aligned} d_n &= -\frac{A_1+B_1e^{-\bar{y}}}{(C_1+\bar{x})^2}+\varepsilon_n^{(1)}, e_n=-\frac{B_1e^{-\bar{y}}}{C_1+\bar{x}}+\varepsilon_{n-1}^{(2)}, \\ f_n &= -\frac{A_2+B_2e^{-\bar{z}}}{(C_2+\bar{y})^2}+\varepsilon_n^{(2)}, g_n=-\frac{B_2e^{-\bar{z}}}{C_2+\bar{y}}+\varepsilon_{n-1}^{(3)}, \\ h_n &= -\frac{A_3+B_3e^{-\bar{x}}}{(C_3+\bar{z})^2}+\varepsilon_n^{(3)}, j_n=-\frac{B_3e^{-\bar{x}}}{C_3+\bar{z}}+\varepsilon_{n-1}^{(1)}, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \lim_{n\to\infty}\varepsilon_n^{(1)} &= \lim_{n\to\infty}\varepsilon_{n-1}^{(2)}=\lim_{n\to\infty}\varepsilon_n^{(2)} \\ &= \lim_{n\to\infty}\varepsilon_{n-1}^{(3)}=\lim_{n\to\infty}\varepsilon_n^{(3)} \\ &= \lim_{n\to\infty}\varepsilon_{n-1}^{(1)}=0. \end{aligned} \tag{33}$$

Set $X_n=(R_n^{(1)},R_{n-1}^{(2)},R_n^{(2)},R_{n-1}^{(2)},R_n^{(3)},R_{n-1}^{(3)})$, from (30), (32) and (33), we can get the following Poincaré difference system

$$X(n+1)=[A+B(n)]X(n),$$

where

$$A=\begin{pmatrix} -\frac{A_1+B_1e^{-\bar{y}}}{(C_1+\bar{x})^2} & 0 & 0 & -\frac{B_1e^{-\bar{y}}}{C_1+\bar{x}} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{A_2+B_2e^{-\bar{z}}}{(C_2+\bar{y})^2} & 0 & 0 & -\frac{B_2e^{-\bar{z}}}{C_2+\bar{y}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{A_3+B_3e^{-\bar{x}}}{(C_3+\bar{z})^2} & 0 & 0 & -\frac{B_3e^{-\bar{x}}}{C_3+\bar{z}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$=F_J(\bar{x},\bar{y},\bar{z}),$$

and

$$B(n)=\begin{pmatrix} \varepsilon_n^{(1)} & 0 & 0 & \varepsilon_{n-1}^{(2)} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_n^{(2)} & 0 & 0 & \varepsilon_{n-1}^{(3)} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \varepsilon_n^{(3)} & 0 & 0 & \varepsilon_{n-1}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

Therefore, by means of **Lemmas 5**, the error vector R_n of each solution of model (6) fulfils the asymptotic relation

$$\sigma=\lim_{n\to\infty}\sqrt[n]{\|R_n\|}. \qquad \square$$

Remark 2. From the Remark 2 of [36], the Theorem 4 remains valid if (25) is replaced with

$$\sigma=\lim_{n\to\infty}\frac{\|X_{n+1}\|}{\|X_n\|}.$$

IV. Numerical examples

In this part, two numerical examples are provided to exhibit the validity of the theoretical findings achieved previously.

Example 1. Research the exponential type 3-species biology difference system

$$x_{n+1}=\frac{0.6+0.2e^{-y_{n-1}}}{3.2+x_n},\ y_{n+1}=\frac{0.5+0.1e^{-z_{n-1}}}{3+y_n},$$
$$z_{n+1}=\frac{0.7+0.3e^{-x_{n-1}}}{4+z_n},\ n=0,1,\cdots. \quad (34)$$

It is not difficult to see that system (34) is the result of system (6) taking the following parameters

$$A_1=0.6,\ B_1=0.2,\ C_1=3.2,\ A_2=0.5,\ B_2=0.1,$$
$$C_2=3.0,\ \ A_3=0.7,\ B_3=0.3,\ C_3=4. \quad (35)$$

From equations (34) and (35), through simple calculations, we can obtain

$$m_1=0.2182\,,\ M_1=0.25,\ m_2=0.1806\,,$$
$$M_2=0.2,\ m_3=0.2197\,,\ M_3=0.25\,. \quad (36)$$

Figure 1 displays the change course of the solutions for system (34) with the following initial values

$$x_{-1}=0.2182,\ x_0=0.231,\ y_{-1}=0.1806,$$
$$y_0=0.192,\ z_{-1}=0.2197,\ z_0=0.227. \quad (37)$$

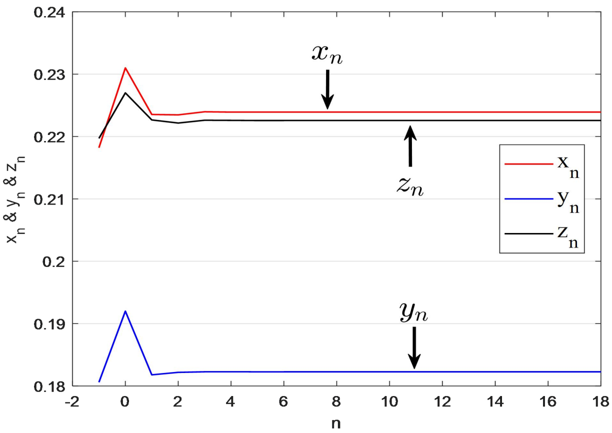

Figure 1. The change course of the solution for system (34) with initial values (37)

Through computing, we are able to obtain that the balance point of equation (34) is

$$(\overline{x},\overline{y},\overline{z})=(0.2239,0.1823,0.2226). \quad (38)$$

From (35), (36) and (38), it is very easy to prove that equation (34) gratifies the conditions of **Theorem 3.** Therefore, from **Theorem 3**, the positive balance point $(\overline{x},\overline{y},\overline{z})$ of the model (34) is global asymptotically stability for any initial conditions

$$(x_i,y_i,z_i)\in(0.2182\,,0.25)\times$$
$$(0.1806\,,0.2)\times(0.2197\,,0.25),\ i=0,1,$$

which is shown in **Figures 2-4.**

Example 2. Consider the following exponential 3-species biology difference system

$$x_{n+1}=\frac{1+0.4e^{-y_{n-1}}}{2.8+x_n},\ y_{n+1}=\frac{0.8+0.2e^{-z_{n-1}}}{3+y_n},$$
$$z_{n+1}=\frac{1.2+0.4e^{-x_{n-1}}}{3.2+z_n},\ n=0,1,\cdots. \quad (39)$$

It is not difficult to see that system (39) is the result of system (6) taking the following parameters

$$A_1=1.0,\ B_1=0.4,\ C_1=2.8,\ A_2=0.8,\ B_2=0.2,$$
$$C_2=3.0,\ A_3=1.2,\ B_3=0.4,\ C_3=3.2. \quad (40)$$

From (39) and (40), through simple calculations, we can obtain

$$m_1=0.3899,\ M_1=0.5,\ m_2=0.2764,$$
$$M_2=0.3333,\ m_3=0.3899,\ M_3=0.5. \quad (41)$$

Figure 5 shows the change course of the solutions for model (39) with the following initial values

$$x_{-1}=0.4,\ x_0=0.45,\ y_{-1}=0.31,$$
$$y_0=0.29,\ z_{-1}=0.43,\ z_0=0.39, \quad (42)$$

Through computing, we are able to obtain that the balance point of equation (39) is

$$(\overline{x},\overline{y},\overline{z})=(0.4058,0.2841,0.4066). \quad (43)$$

From (39), (40) and (43), it is very easy to prove that model (40) gratifies the conditions of **Theorem 3**. Thus, from **Theorem 3**, the positive balance point $(\overline{x},\overline{y},\overline{z})$ of the model (39) is global asymptotically stability for any initial conditions

$$(x_i,y_i,z_i)\in(0.3899,\ 0.5)\times$$
$$(0.2764,\ 0.3333)\times(0.3899,\ 0.5),\ i=0,1,$$

which is shown in **Figures 6-8**.

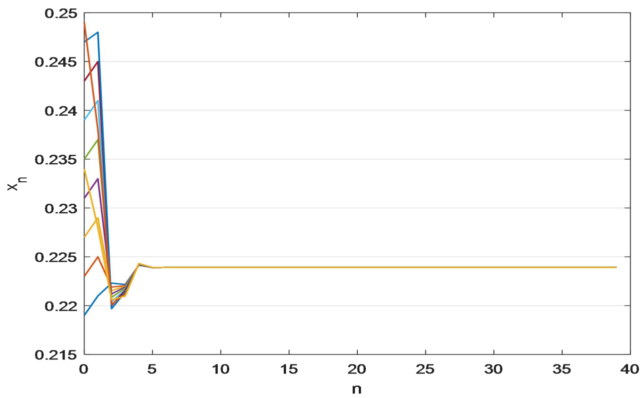

Figure 2. The change course of the solution x_n for equation (34) with different initial values

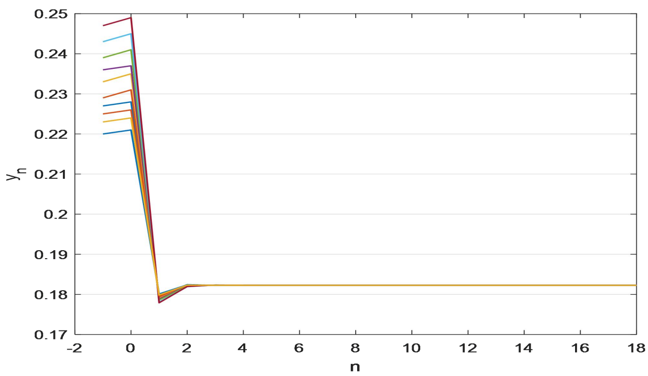

Figure 3. The change process of the solution y_n of system (34) with different initial conditions

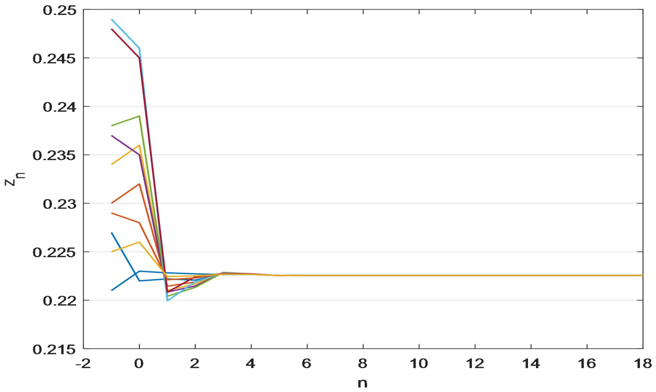

Figure 4. The change process of the solution z_n of system (34) with different initial conditions

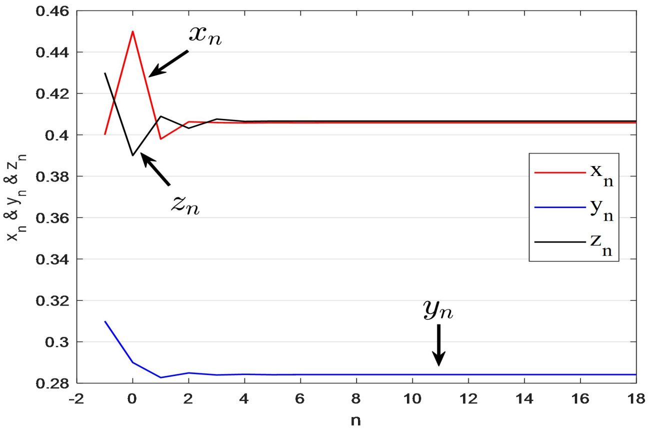

Figure 5. The change process of the solution of system (39) with initial values (42)

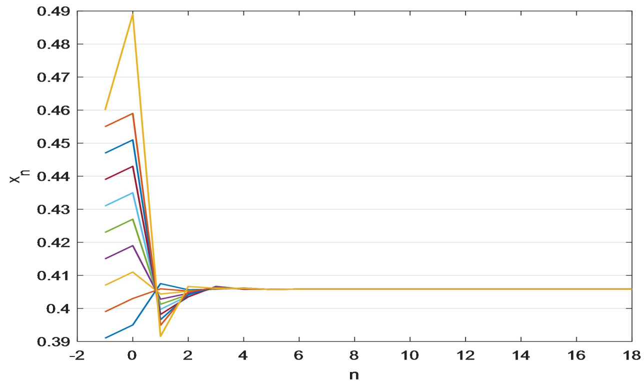

Figure 6. The change course of the solution x_n for model (39) with different initial conditions

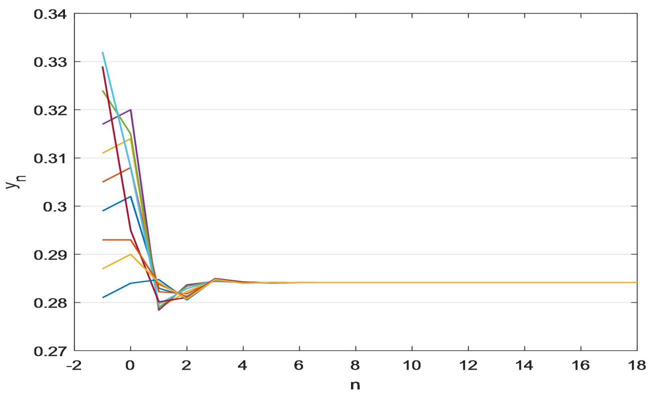

Figure 7. The change process for the solution y_n of equation (39) with different initial values

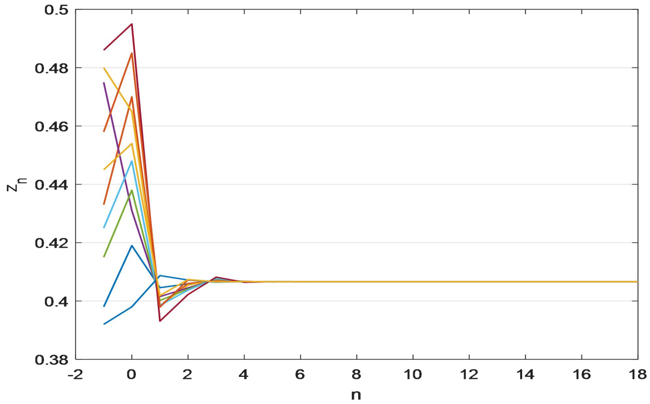

Figure 8. The change process for the solution z_n of model (39) with different initial conditions

V. CONCLUSIONS

This work introduces the Lyapunov stability approach for more general nonlinear difference equation. The method constitutes a potent instrument for resolving all kinds of difference equation. Particularly, this approach can also be utilized in other nonlinear ODE and PDE in mathematical physics. In this article, the dynamics of an exponential-type 3-species difference system are examined. The principal findings are as follows:

(a) It is shown that the positive solutions of the difference biological model (6) are persistent and bounded with the help of inequality techniques.

(b) It is demonstrated that the unique positive balance point of model (6) is locally asymptotically stable in terms of the linearization theory.

(c) It is verified that the positive balance points of biological model (6) are globally asymptotically stable while the conditions in Theorem 3 are hold by making use of the Lyapunov stability approach for the more general nonlinear difference model, together with inequality techniques and the linearization method.

(d) The convergence rate of the unique positive balance for the exponential-type 3-species difference model (6) is obtained by utilizing the Poincaré theorem and inequality techniques.

Based on the fact that the system studied in this article has cyclic symmetry, the method obtained in this article can be improved and expanded to study the following exponential type m-species biology difference systems

$$x_{n+1}^{(1)}=\frac{A_1+B_1e^{-x_{n-1}^{(2)}}}{C_1+x_n^{(1)}},\ x_{n+1}^{(2)}=\frac{A_2+B_2e^{-x_{n-1}^{(3)}}}{C_2+x_n^{(2)}},\cdots,$$

$$x_{n+1}^{(m-1)}=\frac{A_{m-1}+B_{m-1}e^{-x_{n-1}^{(m)}}}{C_{m-1}+x_n^{(m-1)}},\ x_{n+1}^{(m)}=\frac{A_m+B_me^{-x_{n-1}^{(1)}}}{C_m+x_n^{(m)}},\ n=0,1,\cdots,$$

where A_i, B_i and $C_i, i\in\{1,2,\cdots,m\}$ stand for the movement rate, the population growth rate and the carrying capacity of the $x_n^{(i)}$ species, respectively.

AUTHORS' CONTRIBUTIONS

C. Wang, T. Yang, Q. Wang and L. Jia contributed equally to every part of this article.

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Changyou Wang is Professor of College of Applied Mathematics, Chengdu University of Information Technology, Sichuan, China, since September 2017, and is Professor and Director of Institute of Applied Mathematics, Chongqing University of Posts and Telecommunications, Chongqing, China, from November 2011 to August 2017. He acts as a reviewer for Mathematical Reviews for American Mathematical Society, since 2014. His research area include delay reaction-diffusion equation, functional differential equation, fractional-order differential equation, difference equation, biomathematics, control theory and control engineering, neural network, and digital image processing. His education background are as follows: (1) Ph.D. degree from College of Applied Science, Beijing University of Technology, Beijing, China, in September 2008-June 2012, majoring in applied mathematics; (2) M.S. degree from College of Mathematics and Software, Sichuan Normal University, Chengdu, China, in September 2001-June 2004, majoring in applied mathematics; (3) B.S. degree from College of Mathematics, East China University of Science and Technology, Nanchang, China, majoring in basic mathematics, in September 1987-June 1989.

Tao Yang is a postgraduate student of School of Applied Mathematics, Chengdu University of Information Technology, Sichuan, China. His research interests include fuzzy difference equations, stability theory of nonlinear systems and biomathematics.

Qiyu Wang is a postgraduate student of School of Applied Mathematics, Chengdu University of Information Technology, Sichuan, China. Her research interests include fuzzy difference equations, stability theory of nonlinear systems and biomathematics.

Lili Jia is a professor and director of Department of Basic Teaching, Dianchi College of Yunnan University, Yunnan, China, since December 2020. Her research interests include difference equations, differential equation and biomathematics.