Inner Local Exponent of A Two-cycle Non-Hamiltonian Two-coloured Digraph with Cycle Lengths n and $3n+1$

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***Abstract*—A digraph that has arcs of two colours is called a two-coloured digraph. In this case, the colours used are red and black. Let d and k be non-negative integers, where d represents the number of red arcs and k represents the number of black arcs. A (d,k)-walk on the two-coloured digraph is defined as a walk with d red arcs and k black arcs. The smallest integer sum of d and k such that there is a (d,k)-walk from vertex y to vertex z is called the exponent number of two-coloured digraph, whereas the smallest integer sum of d and k such that there is (d,k)-walk from each vertex to vertex v_x is called the inner local exponent of a vertex v_x. This article discusses the inner local exponent of a two-cycle non-Hamiltonian two-coloured digraph with cycle lengths n and $3n+1$. This digraph has exactly four red arcs. The four red arcs are combined consecutively or alternately when there is one allied vertex.**

***Index Terms*—primitive-digraph, two-coloured-digraph, non-Hamiltonian-digraph, inner-local-exponent.**

I. Introduction

A directed graph, or digraph, $\mathcal{D}$ consists of a finite, non-empty set $P(\mathcal{D})$ and a set $H(\mathcal{D})$, which contains ordered pairs of elements from $P(\mathcal{D})$. An element of $P(\mathcal{D})$ is called a vertex, and an element of $H(\mathcal{D})$ is referred to as an arc. A digraph for which each pair of vertices has a path in each direction is called a strongly connected digraph. A digraph that has arcs in two colours is called a two-coloured digraph and denoted as $\mathcal{D}^{(2)}$. In this case, the colours used are red and black.

Let d and k be non-negative integers, where d represents the number of red arcs, and k represents the number of black arcs. A (d,k)-walk on the two-coloured digraph is defined as a walk with d red arcs and k black arcs. For a walk M in $\mathcal{D}^{(2)}$, $a(M)$ and $b(M)$ denote the number of red and black arcs, respectively, contained in M. The vector denotes the composition of the walk M. If each pair of vertices on the two-coloured digraph $\mathcal{D}^{(2)}$ has a (d,k)-walk, then the digraph is said to be primitive. The smallest integer sum of d and k such that there is a (d,k)-walk from vertex y to vertex z is called the exponent number of two-coloured digraph, whereas the smallest integer sum of d and k such that there is (d,k)-walk from each vertex to vertex v_x is called the inner local exponent of vertex v_x and is denoted by $\text{inexp}(v_x, \mathcal{D}^{(2)})$.

Fornasini and Valcher [1] examined the relationship between the (d,k)-walk on two-coloured digraph and the specific product of the neighbour matrix. For non-negative matrices R and B and non-negative integers d and k, the Hurwitz Product (d,k) of the matrices R and B, denoted by $(R,B)^{(d,k)}$, is the sum of all matrix products consisting of the d matrix R and k matrix B. In this case, R is the neighbour matrix for the red arc, and B is the neighbour matrix for the black arc. In general, the Hurwitz Product (d,k) of the matrices R and B can be defined recursively as: $(R,B)^{(d,0)} = (R)^d$ for all $d \geq 0$, $(R,B)^{(0,k)} = (B)^k$ for all $k \geq 0$, and $(R,B)^{(d,k)} = R(R,B)^{(d-1,k)} + B(R,B)^{(d,k-1)}$ for all $d,k \geq 1$. The inner local exponent of a primitive two-coloured digraph $\mathcal{D}^{(2)}$ is obtained by using the Hurwitz Product (d,k) on the recursively defined neighbouring matrices R and B. For the local exponent, the entries of the vertices into the two-coloured digraph $\mathcal{D}^{(2)}$ are obtained by looking at the entries (i,j) of $(R,B)^{(d,k)}$ in the i^{th} column with positive values.

Research on matrix exponent was initiated by Wielandt [2]. Other matrix exponent studies were carried out by Liu et al. [3], Shen [4], Beasley [5] and Shader and Suwilo [6]. These field developed into graph exponent research, which have been done by, among others, Zhou [7], Kim et al. [8], O'Mahony and Quinlan [9] and Surbakti et al. [10]. Others exponent research on digraphs conducted by Shao [11], Shen and Neufeld [12] and Rosiak [13], among others.

The exponent study of two-cycle two-coloured digraph can be grouped into several types based on the lengths of the cycle. The first type of the study is two-cycle two-coloured digraph exponent study with a difference of t, as in the study of Gao and Shao [14]. This first type of the study was also conducted by Suwilo [15], [16] with a difference of 1. Furthermore, Shao et al. [17] and Syahmarani and Suwilo [18] conducted this type of study with a difference of 2. The second type of the study conducts exponent research on two-cycle two-coloured digraphs with differences $(m-1)n+1$, $m \geq 2$. This second type of research has been carried out by Luo [19] and Sumardi and Suwilo [20] with a difference of $n+1$. Sumardi and Suwilo [20] examined the exponent of two-cycle two-coloured digraphs that were non-Hamiltonian graphs with one allied vertex. Meanwhile, exponent studies of two-cycle two-coloured digraphs with a difference $(m-1)n+1$ for $m \geq 3$ have not discussed in the literature, as it is also true for the case of non-Hamiltonian graphs with one allied vertex. This article discusses the inner

Manuscript received January 5, 2024; revised August 16, 2024.

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local exponent of two-cycle non-Hamiltonian two-coloured digraph with cycle lengths of n and $3n+1$ and one allied vertex.

II. METHOD

The concepts of primitivity and exponent digraph were generalized into two-coloured digraphs by Fornasini and Valcher [1]. The algebraic characterization of a primitive two-coloured digraph is provided in the following theorem.

Theorem II.1. [1] *Given a strongly connected two-coloured digraph $\mathcal{D}^{(2)}$ with at least one arc of each colour and cycle M matrix, the two-coloured digraph $\mathcal{D}^{(2)}$ is said to be primitive if and only if the content of the cycle matrix is* 1.

Corollary II.1. *Given that $\mathcal{D}^{(2)}$ is a strongly connected two-coloured digraph with length n for the first cycle and length $3n+1$ for the second cycle, if $\mathcal{D}^{(2)}$ is also primitive then the cycle matrix is $L = \begin{bmatrix} 1 & 3 \\ n-1 & 3n-2 \end{bmatrix}$ or $L = \begin{bmatrix} n-1 & 3n-2 \\ 1 & 3 \end{bmatrix}$.*

Proof: Suppose $\mathcal{D}^{(2)}$ has the cycle matrix $L = \begin{bmatrix} d_1 & d_2 \\ n & 3n+1 \end{bmatrix}$, where $0 \leq d_1 \leq n$ and $0 \leq d_2 \leq 3n+1$. Since $\mathcal{D}^{(2)}$ is a primitive two-coloured digraph, the determinant of the cycle matrix is ± 1. If $\det(L) = 1$, then $(3d_1 - d_2)\,n + d_1 = 1$. Since $0 \leq d_2 \leq 3n+1$, $3d_1 - d_2 = 0$. Consequently, $d_1 = 1$ and $d_2 = 3$. Thus, $L = \begin{bmatrix} 1 & 3 \\ n-1 & 3n-2 \end{bmatrix}$. If $\det(L) = -1$, then $(d_2 - 3d_1)\,n - d_1 = 1$. Since $0 \leq d_2 \leq 3n+1$, $d_2 - 3d_1 = 1$. Consequently, $d_1 = n-1$ and $d_2 = 3n-2$. Thus, $L = \begin{bmatrix} n-1 & 3n-2 \\ 1 & 3 \end{bmatrix}$. ■

Corollary II.1's cycle matrix $L = \begin{bmatrix} 1 & 3 \\ n-1 & 3n-2 \end{bmatrix}$ results in four red arcs for the two-coloured digraph.

The proof of the lower and upper limits of the inner local exponent can be obtained using the proposition and lemmas produced by Suwilo's research [15].

Proposition II.1. [15] *Let $\mathcal{D}^{(2)}$ be a two-coloured digraph that has two cycles and v_z be any vertex on both cycles. If the system*

$$L\mathbf{w} + \begin{bmatrix} a(P_{v_y,v_z}) \\ b(P_{v_y,v_z}) \end{bmatrix} = \begin{bmatrix} d \\ k \end{bmatrix}$$

has a non-negative integer completion, then there is a (d,k)-walk from vertex y to z.

Lemma II.1. [15] *Let $\mathcal{D}^{(2)}$ be a primitive two-coloured digraph and v_y be any vertex in $\mathcal{D}^{(2)}$ with the inner local exponent $\operatorname{inexp}(v_y, \mathcal{D}^{(2)})$. For every $x = 1, 2, \ldots, 3n+1$, $\operatorname{inexp}(v_x, \mathcal{D}^{(2)}) \leq \operatorname{inexp}(v_y, \mathcal{D}^{(2)}) + \delta(v_y, v_x)$.*

Lemma II.2. [15] *Let a primitive two-coloured digraph $\mathcal{D}^{(2)}$ have two cycles, namely G_1 and G_2, and $\det(L) = 1$. If $\operatorname{inexp}(v_x, \mathcal{D}^{(2)})$ is obtained using the (d_x, k_x)-walk, then*

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} b(G_2)a(P_{v_z,v_x}) - a(G_2)b(P_{v_z,v_x}) \\ a(G_1)b(P_{v_y,v_x}) - b(G_1)a(P_{v_y,v_x}) \end{bmatrix}$$

for the paths P_{v_z,v_x} and P_{v_y,v_x}.

III. RESULTS AND DISCUSSION

This study considers a non-Hamiltonian two-coloured digraph with two cycles of lengths n and $3n+1$. There is exactly one allied vertex in this two-coloured digraph (see Figure 1).

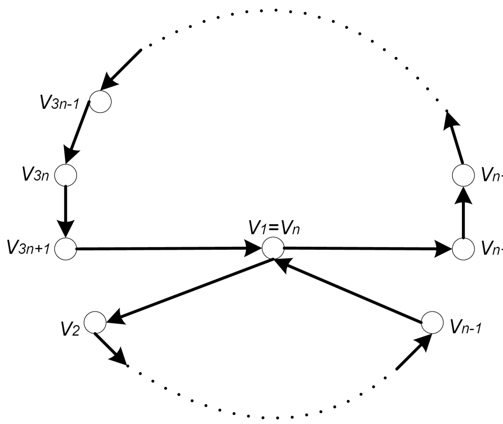

Fig. 1. Non-Hamiltonian two-coloured digraph with cycles of length n and $3n+1$ and one allied vertex.

The first cycle is $G_1 : v_1 \to v_2 \to \cdots \to v_{n-1} \to v_n = v_1$, and the second cycle is $G_2 : v_1 \to v_{n+1} \to \cdots \to v_{3n} \to v_{3n+1} \to v_1$. The red arc in the first cycle is $v_p \to v_{p+1}$, where $1 \leq p \leq n$. The other three red arcs are in the second cycle, namely $v_q \to v_{q+1}$, $v_r \to v_{r+1}$ and $v_s \to v_{s+1}$, where $1 \leq q < r < s \leq n-1$. The distance from v_{q+1} to v_1 is denoted by δ_1, the distance from v_{r+1} to v_1 is denoted by δ_2, the distance from v_{s+1} to v_1 is denoted by δ_3 and the distance from v_{p+1} to v_1 is denoted by δ_4.

Theorem III.1. *Given $\mathcal{D}^{(2)}$, a primitive two-cycle non-Hamiltonian two-coloured digraph with cycle lengths n and $3n+1$, then if the three red arcs in the second cycle are consecutive, for every $x = 1, 2, ..., 3n+1$,*
$\operatorname{inexp}(v_x, \mathcal{D}^{(2)}) =$

$$\begin{cases} 9n^2 - 6n + \delta_3 + \delta(v_1, v_x), \\ \quad \text{for } \delta_3 \geq \delta_4,\ \delta_3 - \delta_4 \leq 2n-1,\ \delta_4 = n-1 \\ 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x), \\ \quad \text{for } \delta_3 < \delta_4 \\ 3n^2 - 2n + 3n(\delta_3 - \delta_4) + \delta_3 + \delta(v_1, v_x), \\ \quad \text{for } \delta_3 > \delta_4,\ \delta_3 - \delta_4 \geq 2n-1,\ \delta_4 < n-1 \\ 9n^2 - 6n - 3n\delta_3 + \delta(v_1, v_x), \\ \quad \text{for } \delta_3 = \delta_4 = 0 \end{cases}$$

Proof: The path (d_x, k_x) is used for proving the expressions for $\operatorname{inexp}(v_x, \mathcal{D}^{(2)})$ for $x = 1, 2, ..., 3n+1$. The proof of Theorem III.1 divided into four cases.

Case 1.1 : $\delta_3 \geq \delta_4,\ \delta_3 - \delta_4 \leq 2n-1,\ \delta_4 = n-1$

First, we have to show that $\operatorname{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x)$. Select path P_{v_q,v_x} and path P_{v_{s+1},v_x} and define $h_1 = b(G_2)a(P_{v_q,v_x}) - a(G_2)b(P_{v_q,v_x})$ and $h_2 = a(G_1)b(P_{v_{s+1},v_x}) - b(G_1)a(P_{v_{s+1},v_x})$. There are four subcases that must be considered.

Subcase 1.1.1

The vertex v_x is on the path $v_1 \to v_q$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_3 + \delta(v_1, v_x))$, we get $h_1 = 9n - 6 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(0, \delta_3 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\left[\begin{array}{c} d_x \\ k_x \end{array}\right] \geq L \left[\begin{array}{c} h_1 \\ h_2 \end{array}\right]$$

$$= \left[\begin{array}{c} 9n-6 \\ 9n^2 - 15n + 6 + \delta_3 + \delta(v_1, v_x) \end{array}\right].$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta\left(v_1, v_x\right) \qquad (1)$$

for every vertex v_x on the path $v_1 \to v_q$.

Subcase 1.1.2

The vertex v_x is on the path $v_{q+1} \to v_r$. Utilizing path P_{v_q,v_x}, namely $(1, \delta_3 - 3n + 1 + \delta(v_1, v_x))$, we get $h_1 = 12n - 5 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(1, \delta_3 - 1 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 - n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\left[\begin{array}{c} d_x \\ k_x \end{array}\right] \geq L \left[\begin{array}{c} h_1 \\ h_2 \end{array}\right]$$

$$= \left[\begin{array}{c} 9n-5 \\ 9n^2 - 15n + 5 + \delta_3 + \delta(v_1, v_x) \end{array}\right].$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta\left(v_1, v_x\right) \qquad (2)$$

for every vertex v_x on the path $v_{q+1} \to v_r$.

Subcase 1.1.3

The vertex v_x is on the path $v_{r+1} \to v_s$. Utilizing path P_{v_q,v_x}, namely $(2, \delta_3 - 3n + \delta(v_1, v_x))$, we get $h_1 = 15n - 4 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(2, \delta_3 - 2 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 - 2n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\left[\begin{array}{c} d_x \\ k_x \end{array}\right] \geq L \left[\begin{array}{c} h_1 \\ h_2 \end{array}\right]$$

$$= \left[\begin{array}{c} 9n-4 \\ 9n^2 - 15n + 4 + \delta_3 + \delta(v_1, v_x) \end{array}\right].$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta\left(v_1, v_x\right) \qquad (3)$$

for every vertex v_x on the path $v_{r+1} \to v_s$.

Subcase 1.1.4

The vertex v_x is on the path $v_{s+1} \to v_{3n+1}$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_3 - 3n - 1 + \delta(v_1, v_x))$, we get $h_1 = 18n - 3 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(0, \delta_3 - 3n - 1 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 - 3n - 1 + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\left[\begin{array}{c} d_x \\ k_x \end{array}\right] \geq L \left[\begin{array}{c} h_1 \\ h_2 \end{array}\right]$$

$$= \left[\begin{array}{c} 9n-6 \\ 9n^2 - 18n + 5 + \delta_3 + \delta(v_1, v_x) \end{array}\right].$$

Let $k_1 = 9n - 6$ and $k_2 = 9n^2 - 18n + 5 + \delta_3 + \delta(v_1, v_x)$. Examining the path (k_1, k_2) from v_{s+1} to v_x, note that the path P_{v_{s+1},v_x} is $(0, \delta_3 - 3n - 1 + \delta(v_1, v_x))$ and that the completion to the system $L\mathbf{w} + \left[\begin{array}{c} a(P_{v_{s+1},v_x}) \\ b(P_{v_{s+1},v_x}) \end{array}\right] = \left[\begin{array}{c} k_1 \\ k_2 \end{array}\right]$ is $w_1 = 45n + 6$ and $w_2 = 0$. The shortest walk from v_{s+1} to v_x containing at a minimum the k_1 red arc and k_2 black arc is the $(k_1 + a(G_2), k_2 + b(G_2))$-walk. Since $a(G_2) + b(G_2) = 3n + 1$, we have

$$\left[\begin{array}{c} d_x \\ k_x \end{array}\right] \geq \left[\begin{array}{c} k_1 \\ k_2 \end{array}\right] + \left[\begin{array}{c} a(G_2) \\ b(G_2) \end{array}\right]$$

$$= \left[\begin{array}{c} 9n-3 \\ 9n^2 - 15n + 3 + \delta_3 + \delta(v_1, v_x) \end{array}\right].$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x) \qquad (4)$$

for every vertex v_x on the path $v_{s+1} \to v_{3n+1}$.

Hence, in (1), (2), (3) and (4), $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Furthermore, we need to prove that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we show that $\text{inexp}(v_1, \mathcal{D}^{(2)}) = 9n^2 - 6n + \delta_3$ since Lemma II.1 guarantees that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (1), we have $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x)$. Furthermore, it is enough to prove that $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + \delta_3$ for every $u = 1, 2, ..., 3n+1$, where the system of equations

$$L\mathbf{w} + \left[\begin{array}{c} a(P_{v_u,v_1}) \\ b(P_{v_u,v_1}) \end{array}\right]$$

$$= \left[\begin{array}{c} 9n-6 \\ 9n^2 - 15n + 6 + \delta_3 \end{array}\right] \qquad (5)$$

has a non-negative integer solution for the path P_{v_u,v_1}. From (5), we have $w_1 = 9n - 6 - 3\delta_3 - (3n - 2)a(P_{v_u,v_1}) + 3b(P_{v_u,v_1})$ and $w_2 = 6n + \delta_3 - (1 - n)a(P_{v_u,v_1}) - b(P_{v_u,v_1})$.

If v_u is on $v_1 \to v_q$, then there is a path $(3, 3n - 2 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 9n - 6 - 3(\delta_3 + \delta(v_1, v_u)) \geq 0$ since $\delta_3 + \delta(v_1, v_u) \leq 3n - 2$ and $w_2 = 6n - 1 + \delta_3 + \delta(v_1, v_u) \geq 5$ since $\delta_3 + \delta(v_1, v_u) \geq n - 1$ where $n \geq 1$. If v_u is on $v_{s+1} \to v_{3n+1}$, then there is a path $(0, 3n + 1 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 18n - 3 - 3(\delta_3 + \delta(v_1, v_u)) \geq 3$ since $\delta_3 + \delta(v_1, v_u) \leq 6n - 2$, where $n \geq 1$ and $w_2 = 3n - 1 + \delta_3 + \delta(v_1, v_u) \geq 6$ since $\delta_3 + \delta(v_1, v_u) \geq 3n + 1$, where $n \geq 1$.

Consequently, the system of equations (5) has a non-negative integer completion for every $u = 1, 2, ..., 3n + 1$. Proposition II.1 ensures that there is a path P_{v_u,v_1} with $d = 9n - 6$ and $k = 9n^2 - 15n + 6 + \delta_3$ for every $u = 1, 2, ..., 3n + 1$. So, $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + \delta_3$. Using Lemma II.1, we can conclude that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 2.1 : $\delta_3 < \delta_4$

First, we have to show that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta\left(v_1, v_x\right)$. Select path P_{v_q,v_x} and path P_{v_{p+1},v_x} and define $h_1 = b(G_2)a(P_{v_q,v_x}) - a(G_2)b(P_{v_q,v_x})$ and $h_2 = a(G_1)b(P_{v_{p+1},v_x}) - b(G_1)a(P_{v_{p+1},v_x})$. The following four subcases must be considered.

Subcase 2.1.1

The vertex v_x is on the path $v_1 \to v_q$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_3 + \delta(v_1, v_x))$, we get $h_1 = 9n - 6 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(0, \delta_4 +$

$\delta(v_1, v_x))$, we get $h_2 = \delta_4 + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9n - 6 + 3(\delta_4 - \delta_3) \\ 9n^2 - 15n + 3n(\delta_4 - \delta_3) + 6 + 3\delta_3 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x) \quad (6)$$

for every vertex v_x on the path $v_1 \to v_q$.

Subcase 2.1.2

The vertex v_x is on the path $v_{q+1} \to v_r$. Utilizing path P_{v_q,v_x}, namely $(1, \delta_3 - 3n + 1 + \delta(v_1, v_x))$, we get $h_1 = 12n - 5 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(1, \delta_4 - 1 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 - n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9n - 5 + 3(\delta_4 - \delta_3) \\ 9n^2 - 15n + 3n(\delta_4 - \delta_3) + 5 + 3\delta_3 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x) \quad (7)$$

for every vertex v_x on the path $v_{q+1} \to v_r$.

Subcase 2.1.3

The vertex v_x is on the path $v_{r+1} \to v_s$. Utilizing path P_{v_q,v_x}, namely $(2, \delta_3 - 3n + \delta(v_1, v_x))$, we get $h_1 = 15n - 4 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(2, \delta_4 - 2 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 - 2n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9n - 4 + 3(\delta_4 - \delta_3) \\ 9n^2 - 15n + 3n(\delta_4 - \delta_3) + 4 + 3\delta_3 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x) \quad (8)$$

for every vertex v_x on the path $v_{r+1} \to v_s$.

Subcase 2.1.4

The vertex v_x is on the path $v_{s+1} \to v_{3n+1}$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_3 - 3n - 1 + \delta(v_1, v_x))$, we get $h_1 = 18n - 3 - 3(\delta_3 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(3, \delta_4 - 3 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 - 3n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9n - 3 + 3(\delta_4 - \delta_3) \\ 9n^2 - 15n + 3n(\delta_4 - \delta_3) + 3 + 3\delta_3 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x) \quad (9)$$

for every vertex v_x on the path $v_{s+1} \to v_{3n+1}$.

In each of (6), (7), (8) and (9), $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Furthermore, we need to prove that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we show that $\text{inexp}(v_1, \mathcal{D}^{(2)}) = 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4$. Lemma II.1 guarantees that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x) + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (6), we have $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x)$. Furthermore, it is enough to prove that $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4$ for every $u = 1, 2, ..., 3n + 1$, when the system of equations

$$L\mathbf{w} + \begin{bmatrix} a(P_{v_u,v_1}) \\ b(P_{v_u,v_1}) \end{bmatrix} = \begin{bmatrix} 9n - 6 + 3(\delta_4 - \delta_3) \\ 9n^2 - 15n + 3n(\delta_4 - \delta_3) + 6 + 3\delta_3 - 2\delta_4 \end{bmatrix} \quad (10)$$

has a non-negative integer completion for the path P_{v_u,v_1}. From (10), we have $w_1 = 9n - 6 - 3\delta_3 - (3n-2)a(P_{v_u,v_1}) + 3b(P_{v_u,v_1})$ and $w_2 = \delta_4 - (1-n)a(P_{v_u,v_1}) - b(P_{v_u,v_1})$.

If v_u is on $v_1 \to v_q$ then there is a path $(3, 3n - 2 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 9n - 6 - 3(\delta_3 + \delta(v_1, v_u)) \geq 0$ since $\delta_3 + \delta(v_1, v_u) \leq 3n - 2$ and $w_2 = \delta_4 - 1 + \delta(v_1, v_u) \geq 1$ since $\delta_4 + \delta(v_1, v_u) \geq n$ where $n \geq 1$. If v_u is on $v_{s+1} \to v_{3n+1}$, then there is a path $(0, 3n + 1 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 18n - 3 - 3(\delta_3 + \delta(v_1, v_u)) \geq 3$ since $\delta_3 + \delta(v_1, v_u) \leq 3n + 1$, where $n \geq 1$ and $w_2 = \delta_4 - 3n - 1 + \delta(v_1, v_u) \geq 1$ since $\delta_4 + \delta(v_1, v_u) \geq 3n + 2$, where $n \geq 1$.

Consequently, the system of equations (10) has a non-negative integer solution for every $u = 1, 2, ..., 3n + 1$. Proposition II.1 ensures that there is a path P_{v_u,v_1} with $d = 9n - 6 + 3(\delta_4 - \delta_3)$ and $k = 9n^2 - 15n + 3n(\delta_4 - \delta_3) + 6 + 3\delta_3 - 2\delta_4$ for every $u = 1, 2, ..., 3n + 1$. So, $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4$. Using Lemma II.1, we can conclude that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 - 6n + 3n(\delta_4 - \delta_3) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 3.1 : $\delta_3 > \delta_4,\ \delta_3 - \delta_4 \geq 2n - 1,\ \delta_4 < n - 1$

First, we have to show that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 3n^2 - 2n + 3n(\delta_3 - \delta_4) + \delta_3 + \delta(v_1, v_x)$. Select path P_{v_p,v_x} and path P_{v_{s+1},v_x} and define $h_1 = b(G_2)a(P_{v_p,v_x}) - a(G_2)b(P_{v_p,v_x})$ and $h_2 = a(G_1)b(P_{v_{s+1},v_x}) - b(G_1)a(P_{v_{s+1},v_x})$. The following four subcases must be considered.

Subcase 3.1.1

The vertex v_x is on the path $v_1 \to v_q$. Utilizing path P_{v_p,v_x}, namely $(1, \delta_4 + \delta(v_1, v_x))$, we get $h_1 = 3n - 2 - 3(\delta_4 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(0, \delta_3 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 3n - 2 + 3(\delta_3 - \delta_4) \\ 3n^2 - 5n + 3n(\delta_3 - \delta_4) + 2 - 2\delta_3 + 3\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 3n^2 - 2n + 3n(\delta_3 - \delta_4) + \delta_3 + \delta(v_1, v_x) \quad (11)$$

for every vertex v_x on the path $v_1 \to v_q$.

Subcase 3.1.2

The vertex v_x is on the path $v_{q+1} \to v_r$. Utilizing path P_{v_p,v_x}, namely $(2, \delta_4 - 1 + \delta(v_1, v_x))$, we get $h_1 = 6n - 1 -$

$3(\delta_4+\delta(v_1,v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(1,\delta_3-1+\delta(v_1,v_x))$, we get $h_2=\delta_3-n+\delta(v_1,v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 3n-1+3(\delta_3-\delta_4) \\ 3n^2-5n+3n(\delta_3-\delta_4)+1-2\delta_3+3\delta_4+\delta(v_1,v_x) \end{bmatrix}.$$

Thus,

$$\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right) \quad (12)$$

for every vertex v_x on the path $v_{q+1}\to v_r$.

Subcase 3.1.3

The vertex v_x is on the path $v_{r+1}\to v_s$. Utilizing path P_{v_p,v_x}, namely $(3,\delta_4-2+\delta(v_1,v_x))$, we get $h_1=9n-3(\delta_4+\delta(v_1,v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(2,\delta_3-2+\delta(v_1,v_x))$, we get $h_2=\delta_3-2n+\delta(v_1,v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 3n+3(\delta_3-\delta_4) \\ 3n^2-5n+3n(\delta_3-\delta_4)-2\delta_3+3\delta_4+\delta(v_1,v_x) \end{bmatrix}.$$

Thus,

$$\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right) \quad (13)$$

for every vertex v_x on the path $v_{r+1}\to v_s$.

Subcase 3.1.4

The vertex v_x is on the path $v_{s+1}\to v_{3n+1}$. Utilizing path P_{v_p,v_x}, namely $(4,\delta_4-3+\delta(v_1,v_x))$, we get $h_1=12n+1-3(\delta_4+\delta(v_1,v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(0,\delta_3-3n-1+\delta(v_1,v_x))$, we get $h_2=\delta_3-3n-1+\delta(v_1,v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 3n-1+3(\delta_3-\delta_4) \\ 3n^2-5n+3n(\delta_3-\delta_4)+1-2\delta_3+3\delta_4+\delta(v_1,v_x) \end{bmatrix}.$$

Thus,

$$\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right) \quad (14)$$

for every vertex v_x on the path $v_{s+1}\to v_{3n+1}$.

For each of (11), (12), (13) and (14), $\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right)$ for every $x=1,2,...,3n+1$.

Furthermore, we need to prove that $\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \leq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right)$ for every $x=1,2,...,3n+1$. First, we show that $\mathrm{inexp}(v_1,\mathcal{D}^{(2)}) = 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3$. Lemma II.1 guarantees that $\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \leq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right)$ for every $x=1,2,...,3n+1$.

From (11), we have that $\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right)$. Furthermore, it is enough to prove that $\mathrm{inexp}(v_1,\mathcal{D}^{(2)}) \leq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3$ for every $u=1,2,...,3n+1$, when the system of equations

$$L\mathbf{w}+\begin{bmatrix} a(P_{v_u,v_1}) \\ b(P_{v_u,v_1}) \end{bmatrix} = \begin{bmatrix} 3n-2+3(\delta_3-\delta_4) \\ 3n^2-5n+3n(\delta_3-\delta_4)+2-2\delta_3+3\delta_4 \end{bmatrix} \quad (15)$$

has a non-negative integer solution for the path P_{v_u,v_1}. From (15), we have $w_1=3n-2-3\delta_4-(3n-2)a(P_{v_u,v_1})+3b(P_{v_u,v_1})$ and $w_2=\delta_3-(1-n)a(P_{v_u,v_1})-b(P_{v_u,v_1})$.

If v_u is on $v_1\to v_q$ then there is a path $(3,3n-2-\delta(v_1,v_u))$. Utilizing this path, we obtain $w_1=3n-2-3(\delta_4+\delta(v_1,v_u))\geq 1$ since $\delta_4+\delta(v_1,v_u)\leq n-1$ and $w_2=\delta_3-1+\delta(v_1,v_u)\geq 3$ since $\delta_3+\delta(v_1,v_u)\geq 3n+1$ where $n\geq 1$. If v_u is on $v_{s+1}\to v_{3n+1}$, then there is a path $(0,3n+1-\delta(v_1,v_u))$. Utilizing this path, we obtain $w_1=12n+1-3(\delta_4+\delta(v_1,v_u))\geq 1$ since $\delta_4+\delta(v_1,v_u)\leq 3n+1$, where $n\geq 1$ and $w_2=\delta_3-3n-1+\delta(v_1,v_u)\geq 0$ since $\delta_3+\delta(v_1,v_u)\geq 3n+1$.

Consequently, the system of equations (15) has a non-negative integer solution for every $u=1,2,...,3n+1$. Proposition II.1 ensures that there is a path P_{v_u,v_1} with $d=3n-2+3(\delta_3-\delta_4)$ and $k=3n^2-5n+3n(\delta_3-\delta_4)+2-2\delta_3+3\delta_4$ for every $u=1,2,...,3n+1$. So, $\mathrm{inexp}(v_1,\mathcal{D}^{(2)}) \leq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3$. Using Lemma II.1, we can conclude that $\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \leq 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right)$ for every $x=1,2,...,3n+1$.

∎

Theorem III.2. *Let $\mathcal{D}^{(2)}$ be a primitive two-cycle non-Hamiltonian two-coloured digraph with cycle lengths n and $3n+1$. If the three red arcs in the second cycle alternate, then for every $x=1,2,...,3n+1$,*
$\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) =$

$$\begin{cases} 9n^2+3n(\delta_3-\delta_1)+\delta_3+\delta\left(v_1,v_x\right), \\ \quad \text{for } \delta_3\geq\delta_4,\ \delta_3-\delta_4\leq n,\ \delta_4=n{-}1 \\ 9n^2+3n(\delta_4-\delta_1)+\delta_4+\delta\left(v_1,v_x\right), \\ \quad \text{for } \delta_3<\delta_4 \\ 3n^2-2n+3n(\delta_3-\delta_4)+\delta_3+\delta\left(v_1,v_x\right), \\ \quad \text{for } \delta_3>\delta_4,\ \delta_3-\delta_4\geq n,\ \delta_4<n{-}1 \\ 9n^2-3n\delta_1+\delta\left(v_1,v_x\right), \\ \quad \text{for } \delta_3=\delta_4=0 \end{cases}$$

Proof: The path (d_x,k_x) will be used for proving the expressions used in $\mathrm{inexp}(v_x,\mathcal{D}^{(2)})$ for $x=1,2,...,3n+1$. The proof of Theorem III.2 divided into four cases.

Case 1.2 : $\delta_3\geq\delta_4,\ \delta_3-\delta_4\leq n,\ \delta_4=n{-}1$

First, we have to show that $\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 9n^2+3n(\delta_3-\delta_1)+\delta_3+\delta\left(v_1,v_x\right)$. Select path P_{v_q,v_x} and path P_{v_{s+1},v_x} and define $h_1=b(G_2)a(P_{v_q,v_x})-a(G_2)b(P_{v_q,v_x})$ and $h_2=a(G_1)b(P_{v_{s+1},v_x})-b(G_1)a(P_{v_{s+1},v_x})$. The following four subcases must be considered.

Subcase 1.2.1

The vertex v_x is on the path $v_1\to v_q$. Utilizing path P_{v_q,v_x}, namely $(3,\delta_1-2+\delta(v_1,v_x))$, we get $h_1=9n-3(\delta_1+\delta(v_1,v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(0,\delta_3+\delta(v_1,v_x))$, we get $h_2=\delta_3+\delta(v_1,v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9n-3\delta_1+3\delta_3 \\ 9n^2+3n(\delta_3-\delta_1)-9n+3\delta_1-2\delta_3+\delta(v_1,v_x) \end{bmatrix}.$$

Thus,

$$\mathrm{inexp}(v_x,\mathcal{D}^{(2)}) \geq 9n^2+3n(\delta_3-\delta_1)+\delta_3+\delta\left(v_1,v_x\right) \quad (16)$$

for every vertex v_x on the path $v_1\to v_q$.

Subcase 1.2.2

The vertex v_x is on the path $v_{q+1} \to v_r$. Utilizing path P_{v_q,v_x}, namely $(1, \delta_1 - 3n - 1 + \delta(v_1, v_x))$, we get $h_1 = 12n + 1 - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(1, \delta_3 - 1 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 - n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$
$$= \begin{bmatrix} 9n - 3\delta_1 + 3\delta_3 \\ 9n^2 + 3n(\delta_3 - \delta_1) - 9n + 3\delta_1 - 2\delta_3 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x) \quad (17)$$

for every vertex v_x on the path $v_{q+1} \to v_r$.

Subcase 1.2.3

The vertex v_x is on the path $v_{r+1} \to v_s$. Utilizing path P_{v_q,v_x}, namely $(2, \delta_1 - 3n - 2 + \delta(v_1, v_x))$, we get $h_1 = 15n + 2 - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(2, \delta_3 - 2 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 - 2n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$
$$= \begin{bmatrix} 9n - 3\delta_1 + 3\delta_3 \\ 9n^2 + 3n(\delta_3 - \delta_1) - 9n + 3\delta_1 - 2\delta_3 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x) \quad (18)$$

for every vertex v_x on the path $v_{r+1} \to v_s$.

Subcase 1.2.4

The vertex v_x is on the path $v_{s+1} \to v_{3n+1}$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_1 - 3n - 3 + \delta(v_1, v_x))$, we get $h_1 = 18n + 3 - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{s+1},v_x}, namely $(0, \delta_3 - 3n - 1 + \delta(v_1, v_x))$, we get $h_2 = \delta_3 - 3n - 1 + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$
$$= \begin{bmatrix} 9n + 3\delta_3 \\ 9n^2 + 3n(\delta_3 - \delta_1) - 12n - 1 - 2\delta_3 + \delta(v_1, v_x) \end{bmatrix}.$$

Let $k_1 = 9n + 3\delta_3$ and $k_2 = 9n^2 + 3n(\delta_3 - \delta_1) - 12n - 1 - 2\delta_3 + \delta(v_1, v_x)$. Examining the path (k_1, k_2) from v_{s+1} to v_x, note that the path P_{v_{s+1},v_x} is $(0, \delta_3 - 3n - 1 + \delta(v_1, v_x))$ and that the completion to the system $L\mathbf{w} + \begin{bmatrix} a(P_{v_{s+1},v_x}) \\ b(P_{v_{s+1},v_x}) \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ is $w_1 = 9n - 3\delta_1 + 3\delta_3$ and $w_2 = 0$. The shortest walk from v_{s+1} to v_x containing at a minimum the k_1 red arc and k_2 black arc is the $(k_1 + a(G_2), k_2 + b(G_2))$-walk. Since $a(G_2) + b(G_2) = 3n + 1$, we obtain

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} a(G_2) \\ b(G_2) \end{bmatrix}$$
$$= \begin{bmatrix} 9n - 3\delta_1 + 3\delta_3 \\ 9n^2 + 3n(\delta_3 - \delta_1) - 9n + 3\delta_1 - 2\delta_3 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x) \quad (19)$$

for every vertex v_x on the path $v_{s+1} \to v_{3n+1}$.

In each of (16), (17), (18) and (19), $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Furthermore, we need to prove that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we show that $\text{inexp}(v_1, \mathcal{D}^{(2)}) = 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3$. Lemma II.1 guarantees that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (16), we have that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x)$. Furthermore, it is enough to prove that $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3$ for every $u = 1, 2, ..., 3n + 1$, when the system of equations

$$L\mathbf{w} + \begin{bmatrix} a(P_{v_u,v_1}) \\ b(P_{v_u,v_1}) \end{bmatrix}$$
$$= \begin{bmatrix} 9n - 3\delta_1 + 3\delta_3 \\ 9n^2 + 3n(\delta_3 - \delta_1) - 9n + 3\delta_1 - 2\delta_3 + \delta(v_1, v_x) \end{bmatrix} \quad (20)$$

has a non-negative integer completion for the path P_{v_u,v_1}. From (20), we have that $w_1 = 9n - 3\delta_1 - (3n-2)a(P_{v_u,v_1}) + 3b(P_{v_u,v_1})$ and $w_2 = \delta_3 - (1-n)a(P_{v_u,v_1}) - b(P_{v_u,v_1})$.

If v_u is on $v_1 \to v_q$ then there is a path $(3, 3n - 2 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 9n - 3(\delta_1 + \delta(v_1, v_u)) \geq 0$ since $\delta_1 + \delta(v_1, v_u) \leq 3n$ and $w_2 = \delta_3 - 1 + \delta(v_1, v_u) \geq 0$ since $\delta_3 + \delta(v_1, v_u) \geq n$ where $n \geq 1$. If v_u is on $v_{s+1} \to v_{3n+1}$, then there is a path $(0, 3n + 1 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 18n + 3 - 3(\delta_1 + \delta(v_1, v_u)) \geq 6$ since $\delta_1 + \delta(v_1, v_u) \leq 6n - 1$, and $w_2 = \delta_3 - 3n - 1 + \delta(v_1, v_u) \geq 0$ since $\delta_3 + \delta(v_1, v_u) \geq 3n + 1$.

Consequently, the system of equations (20) has a non-negative integer solution for every $u = 1, 2, ..., 3n + 1$. Proposition II.1 ensures that there is a path P_{v_u,v_1} with $d = 9n + 3(\delta_3 - \delta_1)$ and $k = 9n^2 - 9n + 3n(\delta_3 - \delta_1) + 3\delta_1 - 2\delta_3$ for every $u = 1, 2, ..., 3n + 1$. So, $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3$. Using Lemma II.1, we can conclude that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_3 - \delta_1) + \delta_3 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 2.2 : $\delta_3 < \delta_4$

First, we have to show that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x)$. Select path P_{v_q,v_x} and path P_{v_{p+1},v_x} and define $h_1 = b(G_2)a(P_{v_q,v_x}) - a(G_2)b(P_{v_q,v_x})$ and $h_2 = a(G_1)b(P_{v_{p+1},v_x}) - b(G_1)a(P_{v_{p+1},v_x})$. The following four subcases must be considered.

Subcase 2.2.1

The vertex v_x is on the path $v_1 \to v_q$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_1 - 2 + \delta(v_1, v_x))$, we get $h_1 = 9n - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(0, \delta_4 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =$$
$$\begin{bmatrix} 9n + 3(\delta_4 - \delta_1) \\ 9n^2 - 9n + 3n(\delta_4 - \delta_1) + 3\delta_1 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x) \quad (21)$$

for every vertex v_x on the path $v_1 \to v_q$.

Subcase 2.2.2
The vertex v_x is on the path $v_{q+1} \to v_r$. Utilizing path P_{v_q,v_x}, namely $(1, \delta_1 - 3n - 1 + \delta(v_1, v_x))$, we get $h_1 = 12n + 1 - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(1, \delta_4 - 1 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 - n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 1 + 3(\delta_4 - \delta_1) \\ 9n^2 - 9n - 1 + 3n(\delta_4 - \delta_1) + 3\delta_1 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x) \quad (22)$$

for every vertex v_x on the path $v_{q+1} \to v_r$.

Subcase 2.2.3
The vertex v_x is on the path $v_{r+1} \to v_s$. Utilizing path P_{v_q,v_x}, namely $(2, \delta_1 - 3n - 2 + \delta(v_1, v_x))$, we get $h_1 = 15n + 2 - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(2, \delta_4 - 2 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 - 2n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 2 + 3(\delta_4 - \delta_1) \\ 9n^2 - 9n - 2 + 3n(\delta_4 - \delta_1) + 3\delta_1 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x) \quad (23)$$

for every vertex v_x on the path $v_{r+1} \to v_s$.

Subcase 2.2.4
The vertex v_x is on the path $v_{s+1} \to v_{3n+1}$. Utilizing path P_{v_q,v_x}, namely $(3, \delta_1 - 3n - 3 + \delta(v_1, v_x))$, we get $h_1 = 18n + 3 - 3(\delta_1 + \delta(v_1, v_x))$. Utilizing path P_{v_{p+1},v_x}, namely $(3, \delta_4 - 3 + \delta(v_1, v_x))$, we get $h_2 = \delta_4 - 3n + \delta(v_1, v_x)$. By Lemma II.2, we get

$$\begin{bmatrix} d_x \\ k_x \end{bmatrix} \geq L \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =$$

$$\begin{bmatrix} 9n + 3 + 3(\delta_4 - \delta_1) \\ 9n^2 - 9n - 3 + 3n(\delta_4 - \delta_1) + 3\delta_1 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix}.$$

Thus,

$$\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x) \quad (24)$$

for every vertex v_x on the path $v_{s+1} \to v_{3n+1}$.

In each of (21), (22), (23) and (24), $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Furthermore, we need to prove that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$. First, we show that $\text{inexp}(v_1, \mathcal{D}^{(2)}) = 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4$. Lemma II.1 guarantees that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

From (21), we have that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \geq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x)$. Furthermore, it is enough to prove that $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4$ for every $u = 1, 2, ..., 3n + 1$, when the system of equations

$$L\mathbf{w} + \begin{bmatrix} a(P_{v_u,v_1}) \\ b(P_{v_u,v_1}) \end{bmatrix}$$

$$= \begin{bmatrix} 9n - 3\delta_1 + 3\delta_4 \\ 9n^2 + 3n(\delta_4 - \delta_1) - 9n + 3\delta_1 - 2\delta_4 + \delta(v_1, v_x) \end{bmatrix} \quad (25)$$

has a non-negative integer completion for the path P_{v_u,v_1}. From (25), we have $w_1 = 9n - 3\delta_1 - (3n-2)a(P_{v_u,v_1}) + 3b(P_{v_u,v_1})$ and $w_2 = \delta_4 - (1-n)a(P_{v_u,v_1}) - b(P_{v_u,v_1})$.

If v_u is on $v_1 \to v_q$, then there is a path $(3, 3n - 2 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 9n - 3(\delta_1 + \delta(v_1, v_u)) \geq 0$ since $\delta_1 + \delta(v_1, v_u) \leq 3n$ and $w_2 = \delta_4 - 1 + \delta(v_1, v_u) \geq 0$ since $\delta_4 + \delta(v_1, v_u) \geq 1$. If v_u is on $v_{s+1} \to v_{3n+1}$, then there is a path $(0, 3n + 1 - \delta(v_1, v_u))$. Utilizing this path, we obtain $w_1 = 18n + 3 - 3(\delta_1 + \delta(v_1, v_u)) \geq 9$ since $\delta_1 + \delta(v_1, v_u) \leq 5n - 1$, where $n \geq 1$, and $w_2 = \delta_4 - 3n - 1 + \delta(v_1, v_u) \geq 1$ since $\delta_4 + \delta(v_1, v_u) \geq 3n + 2$.

Consequently, the system of equations (25) has a non-negative integer solution for every $u = 1, 2, ..., 3n + 1$. Proposition II.1 ensures that there is a path P_{v_u,v_1} with $d = 9n + 3(\delta_4 - \delta_1)$ and $k = 9n^2 - 9n + 3n(\delta_4 - \delta_1) + 3\delta_1 - 2\delta_4$ for every $u = 1, 2, ..., 3n + 1$. So, $\text{inexp}(v_1, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4$. Using Lemma II.1, we can conclude that $\text{inexp}(v_x, \mathcal{D}^{(2)}) \leq 9n^2 + 3n(\delta_4 - \delta_1) + \delta_4 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 3n + 1$.

Case 3 : $\delta_3 > \delta_4,\ \delta_3 - \delta_4 \geq n,\ \delta_4 < n - 1$
Case 3 in the same in Theorems III.1 and III.2, so that the proofs are identical (see the proof in Theorem III.1).

■

IV. Conclusion

The inner local exponent a two-cycle non-Hamiltonian two-coloured digraph with cycle lengths n and $3n + 1$ is generally obtained by determining the inner local exponent at point one, then adding the distance from point one to the point where the value is to be determined. In this case, the number of red arcs is exactly four arcs. In further research, we plan to determine inner local exponents for other classes, for example, when $k > 3$ or for more than one allied point.

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