Composite Anti-disturbance Synchronization Control for Lur'e Systems: An Event-triggered Disturbance Observer-based Design

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***Abstract*—This paper investigates anti-disturbance synchronization for Lur'e systems using a hierarchical composite control mechanism. This mechanism integrates two control approaches: disturbance observer-based control and switched-gain event-triggered control, ensuring the $\mathcal{H}_\infty$ exponential stability of the synchronization-error system in the presence of multiple disturbances. A condition on the $\mathcal{H}_\infty$ exponential stability is derived utilizing a piecewise-defined and time-dependent Lyapunov function and several inequalities. Based on the condition, a co-design is proposed for the gains of the event-triggered controller and disturbance observer. For comparison, the hierarchical composite control with a fixed gain is also investigated, and the corresponding design approach is presented. Finally, the effectiveness of the proposed composite anti-disturbance synchronization control mechanism is validated through an example involving master-slave Chua's circuits.**

***Index Terms*—Lur'e system, event-triggered control, disturbance observer, synchronization**

I. Introduction

LUR'E systems (LSs) are common nonlinear systems characterized by the coexistence of linear dynamic systems and feedback nonlinearity that is constrained by sector-bounded conditions. Numerous dynamic systems, such as Chua's circuits [1] and Hopfield network [2], can be effectively modeled within the framework of LSs. LSs can manifest complex and chaotic behavior. Over the past two decades, chaos synchronization of LSs has emerged as a prominent research hotspot, garnering significant attention due to its widespread applications in secure communications, image processing, and various other fields [3–5].

There are numerous factors influencing the chaos synchronization of LSs, among which the effect of disturbances stands out as a key concern. Real-world dynamic systems are susceptible to exogenous disturbances originating from the environment or unidentified factors. These disturbances can significantly impact system performance, leading to deviations from expected behavior [6–8]. To reduce the influence of external disturbances on the chaos synchronization of LSs, various effective control methods have been proposed, including, but not limited to, sliding-mode control [9], adaptive control [10], intuitionistic fuzzy control [11], fractional-order control [12], impulsive pinning control [13], $\mathcal{H}_\infty$ control [14], and sampled-data control [15].

It should be noted that most existing control methods primarily address norm-bounded disturbances. However, in practical applications, disturbances often exhibit diverse characteristics, including multiple disturbances with both harmonic and norm-bounded attributes. Dealing with such complex scenarios with a singular control method is often infeasible, necessitating unconventional control strategies. In 2004, a composite control mechanism integrating disturbance observer-based control (DOBC) with traditional control methods was proposed [16]. Subsequently, this composite control idea has been successfully applied in many different dynamic systems [17–21].

Event-triggered control (ETC) has garnered growing attention with the advancement of control theory. Instead of continuously updating and transmitting control signals, as in traditional time-triggered control architectures, ETC schemes only update and transmit control signals when certain predefined events occur or specific conditions are met [22–25]. In this way, ETC optimizes the utilization of computational resources and network bandwidth while ensuring the desired control performance. Thus, the question arises: Can ETC and DOBC be combined to address the composite anti-disturbance synchronization control of LSs with multiple disturbances? This issue, to our knowledge, remains open and challenging, warranting further investigation.

Based on the aforementioned discussion, this paper aims to investigate the hierarchical composite anti-disturbance synchronization control of LSs with multiple disturbances. These disturbances comprise two types: one in the form of a norm-bounded vector, and the other described by an exogenous system. The hierarchical composite anti-disturbance synchronization control mechanism integrates two control approaches: DOBC and switched-gain ETC, ensuring the $\mathcal{H}_\infty$ exponential stability of the synchronization-error system (SS) in the presence of multiple disturbances. By selecting an appropriate Lyapunov function, a condition for $\mathcal{H}_\infty$ exponential stability of the SS is derived. Subsequently, based on this condition, a co-design approach is proposed for determining the gains of the event-triggered controller and disturbance observers. For comparison, the composite anti-disturbance synchronization control with a fixed gain is also considered, and the corresponding design approach is presented. Finally, the effectiveness of the proposed composite anti-disturbance synchronization control mechanism is verified through a

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numerical example.

Notation: Throughout, $\mathbb{R}^{n_1\times n_2}$ and $\mathbb{R}^n$ represent the sets of $n_1\times n_2$-real matrices and n-dimensional real vectors. For a matrix Z, $Z>0$ ($Z<0$) represents that Z is positive (negative) definite, and $\mathscr{S}(Z)$ represents the expression $Z+Z^T$. diag$\{\cdot\}$ represents a diagonal matrix and col$\{\cdot\}$ denotes a column vector. In block symmetric matrices, the symbol "$*$" indicates a block derived by symmetry. Furthermore, the dimensions of matrices, if not explicitly stated, are assumed to be compatible.

II. Preliminaries

Consider the following LS with master-slave synchronization mechanism, the master system is

$$\begin{cases}\dot{\delta}_m(t)=A\delta_m(t)+\mathcal{H}f(D\delta_m(t))+D_md_m(t)\\ \sigma_m(t)=C\delta_m(t)\end{cases}\tag{1}$$

and the slave system is

$$\begin{cases}\dot{\delta}_s(t)=A\delta_s(t)+\mathcal{H}f(D\delta_s(t))+D_sd_s(t)\\ \qquad\quad +B_0\omega(t)+u(t)\\ \sigma_s(t)=C\delta_s(t)\end{cases}\tag{2}$$

where $\delta_m(t)\in\mathbb{R}^m$ and $\delta_s(t)\in\mathbb{R}^m$ are the state vectors of master system and slave system, $\sigma_m(t)\in\mathbb{R}^n$ and $\sigma_s(t)\in\mathbb{R}^n$ are the corresponding output vectors, $u(t)\in\mathbb{R}^m$ is the control input vector, $d_m(t)\in\mathbb{R}^{d_m}$ and $d_s(t)\in\mathbb{R}^{d_s}$ are the unknown disturbances. $\omega(t)\in\mathbb{R}^{\omega_s}$ is another additional disturbance in $\mathcal{L}_2[0,\infty)$. $A\in\mathbb{R}^{m\times m}$, $\mathcal{H}\in\mathbb{R}^{m\times f}$, $D_m\in\mathbb{R}^{m\times d_m}$, $D_s\in\mathbb{R}^{m\times d_s}$, $C\in\mathbb{R}^{n\times m}$, and $B_0\in\mathbb{R}^{m\times\omega_s}$ are all known constant matrices. $f(\cdot)=\text{col}\{f_1(\cdot),f_2(\cdot),\ldots,f_f(\cdot)\}\in\mathbb{R}^f$ is a nonlinear function vector with $f(0)=0$, and for any $\varsigma_1,\varsigma_2\in\mathbb{R}$, $i\in\{1,2,\ldots,f\}$, it satisfies the following sector-bounded constraint:

$$(f_i(\varsigma_1)-f_i(\varsigma_2))(f_i(\varsigma_1)-f_i(\varsigma_2)-\mathcal{E}_i(\varsigma_1-\varsigma_2))\le 0.$$

Define $\delta_e(t)=\delta_m(t)-\delta_s(t)$, $\sigma_e(t)=\sigma_m(t)-\sigma_s(t)$. Then, combining systems (1) and (2), the SS can be expressed as

$$\begin{cases}\dot{\delta}_e(t)=A\delta_e(t)+\mathcal{HF}(D\delta_e(t),\delta_s(t))+D_md_m(t)\\ \qquad\quad -D_sd_s(t)-B_0\omega(t)-u(t)\\ \sigma_e(t)=C\delta_e(t)\end{cases}\tag{3}$$

where $\mathcal{F}(D\delta_e(t),\delta_s(t))=f(D\delta_e(t)+D\delta_s(t))-f(D\delta_s(t))$. It is assumed that $\mathcal{F}(D\delta_e(t),\delta_s(t))$ belongs to sector $[0,\mathcal{E}]$ [26–28], which means that

$$\mathcal{F}^T(D\delta_e(t),\delta_s(t))(\mathcal{F}(D\delta_e(t),\delta_s(t))-\mathcal{E}D\delta_e(t))\le 0\tag{4}$$

with $\mathcal{E}=\text{diag}\{\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_f\}$.

Assumption 1. The unknown disturbances $d_m(t)$ and $d_s(t)$ can be generated by the following exogenous systems:

$$\begin{cases}\dot{\xi}_m(t)=W_m\xi_m(t)\\ d_m(t)=V_m\xi_m(t)\end{cases}\tag{5}$$

$$\begin{cases}\dot{\xi}_s(t)=W_s\xi_s(t)\\ d_s(t)=V_s\xi_s(t)\end{cases}\tag{6}$$

where $W_m\in\mathbb{R}^{W_{\xi_m}\times W_{\xi_m}}$, $V_m\in\mathbb{R}^{d_m\times W_{\xi_m}}$, $W_s\in\mathbb{R}^{W_{\xi_s}\times W_{\xi_s}}$, and $V_s\in\mathbb{R}^{d_s\times W_{\xi_s}}$ are all known real constant matrices.

In order to get the estimations of unknown disturbances $\hat{d}_m(t)$ and $\hat{d}_s(t)$, the corresponding disturbance observers can be designed as

$$\begin{cases}\hat{d}_m(t)=V_m\hat{\xi}_m(t)\\ \hat{\xi}_m(t)=\rho_m(t)-L_m\delta_m(t)\\ \dot{\rho}_m(t)=(W_m+L_mD_mV_m)(\rho_m(t)-L_m\delta_m(t))\\ \qquad\quad +L_m(A\delta_m(t)+\mathcal{H}f(D\delta_m(t)))\end{cases}\tag{7}$$

$$\begin{cases}\hat{d}_s(t)=V_s\hat{\xi}_s(t)\\ \hat{\xi}_s(t)=\rho_s(t)-L_s\delta_s(t)\\ \dot{\rho}_s(t)=(W_s+L_sD_sV_s)(\rho_s(t)-L_s\delta_s(t))\\ \qquad\quad +L_s(A\delta_s(t)+\mathcal{H}f(D\delta_s(t))+u(t))\end{cases}\tag{8}$$

where L_m and L_s are the observer gains that need to be obtained. The disturbance estimation errors can be expressed as

$$e_{\xi_m}(t)=\xi_m(t)-\hat{\xi}_m(t)\tag{9}$$

$$e_{\xi_s}(t)=\xi_s(t)-\hat{\xi}_s(t).\tag{10}$$

Combined with (1), (2), and (5)-(10), the error dynamics are represented as

$$\dot{e}_{\xi_m}(t)=(W_m+L_mD_mV_m)e_{\xi_m}(t)\tag{11}$$

$$\dot{e}_{\xi_s}(t)=(W_s+L_sD_sV_s)e_{\xi_s}(t)+L_sB_0\omega(t).\tag{12}$$

In this paper, the controller within the DOBC scheme is designed to be event-triggered. The following trigger rule is defined to obtain triggering instants:

$$t_{k+1}=\min\{t\ge t_k+h\,|(\sigma_e(t)-\sigma_e(t_k))^T\Lambda(\sigma_e(t)-\sigma_e(t_k))>\alpha\sigma_e^T(t)\Lambda\sigma_e^T(t)\}$$

where $\Lambda\ge 0$ represents the trigger matrix and needs to be determined, $\alpha\ge 0$ is known threshold parameter. Let $e_k(t)=\sigma_e(t)-\sigma_e(t_k)$ be the error between the current sampling time and the latest transmission time. Then, the trigger rule can be rewritten as:

$$t_{k+1}=\min\{t\ge t_k+h\,|\,e_k^T(t)\Lambda e_k(t)>\alpha\sigma_e^T(t)\Lambda\sigma_e^T(t)\}.\tag{13}$$

Based on this rule, the imposed anti-disturbance controller is given by

$$u(t)=D_m\hat{d}_m(t)-D_s\hat{d}_s(t)-\mathcal{K}_{\zeta(t)}\sigma_e(t_k)\tag{14}$$

where $\zeta(t)$ is a switched signal defined as

$$\zeta(t)=\begin{cases}1, & t\in[t_k,t_k+h)\\ 2, & t\in[t_k+h,t_{k+1})\end{cases}$$

and $\mathcal{K}_{\zeta(t)}$ represents the corresponding gain matrices that need to be ascertained. Thus, the SS (3) can be rewritten as

$$\begin{aligned}\dot{\delta}_e(t)=&(A+\mathcal{K}_{\zeta(t)}C)\delta_e(t)+\mathcal{HF}(D\delta_e(t),\delta_s(t))-B_0\omega(t)\\ &+D_mV_me_{\xi_m}(t)-D_sV_se_{\xi_s}(t)-\mathcal{K}_{\zeta(t)}Ce_k(t).\end{aligned}\tag{15}$$

Defining the reference output as $z(t)=\mathcal{Z}_1\delta_e(t)+\mathcal{Z}_2e_{\xi_s}(t)$, for SS (15), the system is $\mathcal{H}_\infty$ exponentially stable, in the sense that

1) It is exponentially stable when $\omega(t)\equiv 0$;
2) It has a specified $\mathcal{H}_\infty$ disturbance-rejection performance level γ, meaning that under the zero initial condition,

$$\int_0^\infty(z^T(t)z(t)-\gamma^2\omega^T(t)\omega(t))\,dt<0.$$

III. Main results

A. $\mathcal{H}_\infty$ Stability Analysis

This subsection is indicated to the $\mathcal{H}_\infty$ stability analysis of the SS (15). The following condition can be obtained through using a Lyapunov function.

Theorem 1. For given constants $\alpha > 0, \eta > 0, h > 0, \gamma > 0$, and arbitrary matrices $\mathcal{K}_1, \mathcal{K}_2$, SS (15) is $\mathcal{H}_\infty$ exponentially stable if there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $P_4 > 0$, $P_5 > 0$, $P_6 > 0$, $\Lambda \geq 0$, Y_1, Y_2, Q_1, Q_2, Q_3, X, X_1, and diagonal matrix $U > 0$ such that

$$\Phi_1^1 = \begin{bmatrix} P_1 + h\mathscr{S}(\frac{X}{2}) & h(-X + X_1) \\ * & h\mathscr{S}(-X_1 + \frac{X}{2}) \end{bmatrix} > 0 \tag{16}$$

$$\Phi_1^2 = \begin{bmatrix} \Sigma(h) & \nu_1 \\ * & \Omega \end{bmatrix} < 0 \tag{17}$$

$$\Phi_1^3 = \begin{bmatrix} \Theta(h) & \nu_2 \\ * & \Omega \end{bmatrix} < 0 \tag{18}$$

$$\Phi_1^4 = \begin{bmatrix} \Gamma(h) & \nu_3 \\ * & \Omega \end{bmatrix} < 0 \tag{19}$$

hold, where $\Sigma(h) = [\Sigma_{i_1 j_1}]$, $i_1, j_1 = 1, 2, \ldots, 9$, $\Theta(h) = [\Theta_{i_2 j_2}]$, $i_2, j_2 = 1, 2, \ldots, 7$, $\Gamma(h) = [\Gamma_{i_3 j_3}]$, $i_3, j_3 = 1, 2, \ldots, 6$,

$$\Sigma_{11} = 2\eta P_1 + \mathscr{S}(Y_1^T(A + \mathcal{K}_1 C) - \frac{X}{2} - Q_1)$$
$$\Sigma_{12} = P_1 - Q_2 - Y_1^T + (A + \mathcal{K}_1 C)^T Y_2$$
$$\Sigma_{13} = Y_1^T D_m V_m, \ \Sigma_{14} = -Y_1^T D_s V_s$$
$$\Sigma_{15} = X - X_1 + Q_1^T - Q_3$$
$$\Sigma_{16} = Y_1^T \mathcal{H} + \mathcal{E} D^T U, \ \Sigma_{17} = -Y_1^T \mathcal{K}_1 C$$
$$\Sigma_{18} = \Sigma_{19} = hQ_1^T, \ \Sigma_{22} = -\mathscr{S}(Y_2)$$
$$\Sigma_{23} = Y_2^T D_m V_m, \ \Sigma_{24} = -Y_2^T D_s V_s$$
$$\Sigma_{25} = Q_2, \ \Sigma_{26} = Y_2^T \mathcal{H}, \ \Sigma_{27} = -Y_2^T \mathcal{K}_1 C$$
$$\Sigma_{28} = \Sigma_{29} = hQ_2^T$$
$$\Sigma_{33} = 2\eta P_2 + \mathscr{S}(P_2(W_m + L_m D_m V_m))$$
$$\Sigma_{44} = 2\eta P_3 + \mathscr{S}(P_3(W_s + L_s D_s V_s))$$
$$\Sigma_{55} = \mathscr{S}(Q_3 + X_1 - \frac{X}{2}), \ \Sigma_{58} = \Sigma_{59} = hQ_3^T$$
$$\Sigma_{66} = -2U, \ \Sigma_{77} = -P_6, \ \Sigma_{88} = -hP_4$$
$$\Sigma_{99} = -hP_5, \ \Theta_{11} = \Sigma_{11} + \eta^2 h P_5 + \eta h \mathscr{S}(X)$$
$$\Theta_{12} = \Sigma_{12} + h\frac{X}{2}, \ \Theta_{13} = \Sigma_{13}, \ \Theta_{14} = \Sigma_{14}$$
$$\Theta_{15} = \Sigma_{15} + h\eta(-X + X_1), \ \Theta_{16} = \Sigma_{16}$$
$$\Theta_{17} = \Sigma_{17}, \ \Theta_{22} = \Sigma_{22} + hP_4, \ \Theta_{23} = \Sigma_{23}$$
$$\Theta_{24} = \Sigma_{24}, \ \Theta_{25} = \Sigma_{25} + h(-X + X_1)$$
$$\Theta_{26} = \Sigma_{26}, \ \Theta_{27} = \Sigma_{27} + hC^T P_6^T, \ \Theta_{33} = \Sigma_{33}$$
$$\Theta_{44} = \Sigma_{44}, \ \Theta_{55} = \Sigma_{55}, \ \Theta_{66} = \Sigma_{66}, \ \Theta_{77} = \Sigma_{77} + 2\eta h P_6$$
$$\Gamma_{11} = 2\eta P_1 + \mathscr{S}(Y_1^T(A + \mathcal{K}_2 C)) + \alpha C^T \Lambda C$$
$$\Gamma_{12} = P_1 - Y_1^T + (A + \mathcal{K}_2 C)^T Y_2, \ \Gamma_{13} = \Sigma_{13}$$
$$\Gamma_{14} = \Sigma_{14}, \ \Gamma_{15} = \Sigma_{16}, \ \Gamma_{16} = -Y_1^T \mathcal{K}_2 C$$
$$\Gamma_{22} = \Sigma_{22}, \ \Gamma_{23} = \Sigma_{23}, \ \Gamma_{24} = \Sigma_{24}, \ \Gamma_{25} = \Sigma_{26}$$
$$\Gamma_{26} = -Y_2^T \mathcal{K}_2 C, \ \Gamma_{33} = \Sigma_{33}, \ \Gamma_{44} = \Sigma_{44}$$
$$\Gamma_{55} = \Sigma_{55}, \ \Gamma_{66} = -\Lambda$$
$$\nu_1^T = \begin{bmatrix} -B_0^T Y_1 & -B_0^T Y_2 & 0 & B_0^T L_s^T & 0 & 0 & 0 & 0 & 0 \\ \mathcal{Z}_1 & 0 & 0 & \mathcal{Z}_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\nu_2^T = \begin{bmatrix} -B_0^T Y_1 & -B_0^T Y_2 & 0 & B_0^T L_s^T & 0 & 0 & 0 \\ \mathcal{Z}_1 & 0 & 0 & \mathcal{Z}_2 & 0 & 0 & 0 \end{bmatrix}$$
$$\nu_3^T = \begin{bmatrix} -B_0^T Y_1 & -B_0^T Y_2 & 0 & B_0^T L_s^T & 0 & 0 \\ \mathcal{Z}_1 & 0 & 0 & \mathcal{Z}_2 & 0 & 0 \end{bmatrix}$$
$$\Omega = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$$

and other blocks unspecified are zero matrices.

Proof: Choose the following piecewise-defined and time-dependent Lyapunov function:

$$V(t) = \begin{cases} V_a(t) = \sum_{i=1}^{5} V_i(t), \ t \in [t_k, t_k + h) \\ V_b(t) = V_1(t), \ t \in [t_k + h, t_{k+1}) \end{cases}$$

where

$$V_1(t) = \delta_e^T(t) P_1 \delta_e(t) + e_{\xi_m}^T(t) P_2 e_{\xi_m}(t) + e_{\xi_s}^T(t) P_3 e_{\xi_s}(t)$$
$$V_2(t) = (t_k + h - t) \int_{t_k}^{t} e^{2\eta(s-t)} \dot{\delta}_e^T(s) P_4 \dot{\delta}_e(s)\, ds$$
$$V_3(t) = \eta^2 (t_k + h - t) \int_{t_k}^{t} e^{2\eta(s-t)} \delta_e^T(s) P_5 \delta_e(s)\, ds$$
$$V_4(t) = (t_k + h - t) e_k^T(t) P_6 e_k(t)$$
$$V_5(t) = (t_k + h - t) \varpi^T(t) \mathcal{M} \varpi(t)$$

with

$$\mathcal{M} = \begin{bmatrix} \mathscr{S}(\frac{X}{2}) & -X + X_1 \\ * & \mathscr{S}(-X_1 + \frac{X}{2}) \end{bmatrix}$$
$$\varpi(t) = \mathrm{col}\{\delta_e(t), e^{-\eta(t-t_k)} \delta_e(t_k)\}.$$

Evidently, $V_1(t)$, $V_2(t)$, $V_3(t)$, and $V_4(t)$ are positive definite. From inequality (16), it can be seen that

$$\begin{aligned} & V_1(t) + V_5(t) \\ =& \varpi^T(t) \left(\begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} + (t_k + h - t)\mathcal{M} \right) \varpi(t) \\ & + e_{\xi_m}^T(t) P_2 e_{\xi_m}(t) + e_{\xi_s}^T(t) P_3 e_{\xi_s}(t) \\ =& \varpi^T(t) \left(\frac{t - t_k}{h} \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{t_k + h - t}{h} \left(\begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} + h\mathcal{M} \right) \right) \\ & \times \varpi(t) + e_{\xi_m}^T(t) P_2 e_{\xi_m}(t) + e_{\xi_s}^T(t) P_3 e_{\xi_s}(t) \\ =& \frac{t - t_k}{h} \varpi^T(t) \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \varpi(t) + \frac{t_k + h - t}{h} \varpi^T(t) \Phi_1^1 \varpi(t) \\ & + e_{\xi_m}^T(t) P_2 e_{\xi_m}(t) + e_{\xi_s}^T(t) P_3 e_{\xi_s}(t) \end{aligned}$$

$P_1 > 0$, $P_2 > 0$, $P_3 > 0$, and $\Phi_1^1 > 0$. Therefore, $V_1(t) + V_5(t)$ is positive definite, $V_a(t)$ and $V_b(t)$ are confirmed to be both positive definite.

Furthermore, for $t \in [t_k, t_{k+1})$, from the expression of $V(t)$, it can be seen that

$$V_2(t_k) = V_3(t_k) = V_4(t_k) = V_5(t_k) = 0$$
$$\lim_{t \to (t_k+h)^-} V_2(t) = \lim_{t \to (t_k+h)^-} V_3(t) = \lim_{t \to (t_k+h)^-} V_4(t) = \lim_{t \to (t_k+h)^-} V_5(t) = 0$$

which indicates

$$\lim_{t \to t_k} V(t) = V(t_k), \ \lim_{t \to (t_k+h)^-} V(t) = V(t_k + h).$$

Thus, $V(t)$ is continuous at instants t_k and $t_k + h$. This, along with the arbitrariness of the time instants t_k, implies that $V(t)$ is continuous on $[0, +\infty)$.

Calculating the derivatives of $V_i(t)(i = 1, \ldots, 5)$ along the trajectories of system (15), it yields

$$\begin{aligned}\dot{V}_1(t) =& 2\delta_e^T(t)P_1\dot{\delta}_e(t)+2e_{\xi_m}^T(t)P_2\dot{e}_{\xi_m}(t) + 2e_{\xi_s}^T(t)P_3\dot{e}_{\xi_s}(t)\\ \dot{V}_2(t) =& -\int_{t_k}^{t} e^{2\eta(s-t)}\dot{\delta}_e^T(s)P_4\dot{\delta}_e(s)\,ds - 2\eta V_2(t)\\ &+ (t_k + h - t)\dot{\delta}_e^T(t)P_4\dot{\delta}_e(t)\\ \dot{V}_3(t) =& -\eta^2\int_{t_k}^{t} e^{2\eta(s-t)}{\delta_e}^T(s)P_5 r(s)\,ds - 2\eta V_3(t)\\ &+ \eta^2(t_k + h - t)\delta_e^T(t)P_5\delta_e(t)\\ \dot{V}_4(t) =& -e_k^T(t)P_6e_k(t) + 2(t_k + h - t)e_k^T(t)P_6C\dot{\delta}_e(t)\\ \dot{V}_5(t) =& -\varpi^T(t)\mathcal{M}\varpi(t) + 2(t_k + h - t)\dot{\varpi}^T(t)\mathcal{M}\varpi(t).\end{aligned}$$

Then the discussion of stability analysis can be divided into two different intervals based on the switched event-triggered rule.

Case 1: $t \in [t_k, t_k + h)$

In this case, $\zeta(t) = 1$, $u(t) = D_m\hat{d}_m(t) - D_s\hat{d}_s(t) - \mathcal{K}_1\sigma_e(t_k)$, the Lyapunov function $V_a(t)$ can be employed. It follows that there is a small positive scalar $\varrho_a = \min\{\lambda_{min}(P_1), \lambda_{min}(P_2), \lambda_{min}(P_3), \lambda_{min}(\Phi_1^1)\}$, which makes

$$V_a(t) \geq \varrho_a\,|\delta_e(t)|^2. \tag{20}$$

Through some calculations, it can obtain

$$\begin{aligned}&\dot{V}_a(t) + 2\eta V_a(t)\\ =\;& 2\delta_e^T(t)P_1\dot{\delta}_e(t) + 2e_{\xi_m}^T(t)P_2\dot{e}_{\xi_m}(t) + 2e_{\xi_s}^T(t)P_3\dot{e}_{\xi_s}(t)\\ &+ 2\eta(\delta_e^T(t)P_1\delta_e(t) + e_{\xi_m}^T(t)P_2e_{\xi_m}(t) + e_{\xi_s}^T(t)P_3e_{\xi_s}(t))\\ &+(t_k + h - t)\dot{\delta}_e^T(t)P_4\dot{\delta}_e(t)+\eta^2(t_k + h - t)\delta_e^T(t)P_5\delta_e(t)\\ &+2\eta(t_k+h-t)e_k^T(t)P_6e_k(t)+2\eta(t_k+h-t)\varpi^T(t)\mathcal{M}\\ &\times\varpi(t) - e_k^T(t)P_6e_k(t) + 2(t_k + h - t)e_k^T(t)P_6C\dot{\delta}_e(t)\\ &-\varpi^T(t)\mathcal{M}\varpi(t) + 2(t_k + h - t)\dot{\varpi}^T(t)\mathcal{M}\varpi(t)\\ &-\int_{t_k}^{t} e^{2\eta(s-t)}\dot{\delta}_e^T(s)P_4\dot{\delta}_e(s)\,ds\\ &-\eta^2\int_{t_k}^{t} e^{2\eta(s-t)}{\delta_e}^T(s)P_5\delta_e(s)ds.\end{aligned} \tag{21}$$

Define

$$\begin{aligned}W_\alpha(t) &= \frac{1}{t - t_k}\int_{t_k}^{t} e^{\eta(s-t)}\dot{\delta}_e(s)\,ds\\ W_\beta(t) &= \frac{\eta}{t - t_k}\int_{t_k}^{t} e^{\eta(s-t)}\delta_e(s)\,ds.\end{aligned}$$

Then, the integral terms can be estimated by using Jensen's inequality. The estimated results are as follows:

$$-\int_{t_k}^{t} e^{2\eta(s-t)}\dot{\delta}_e^T(s)P_4\dot{\delta}_e(s)\,ds \leq -(t-t_k)W_\alpha^T(t)P_4W_\alpha(t) \tag{22}$$

$$-\eta^2\int_{t_k}^{t} e^{2\eta(s-t)}\delta_e^T(s)P_5\delta_e(s)\,ds \leq -(t-t_k)W_\beta^T(t)P_5W_\beta(t). \tag{23}$$

Moreover, according to the basic theorem of calculus and SS (15), the following two equations hold true for arbitrary matrices Y_1, Y_2, Q_1, Q_2, Q_3 with appropriate dimensions:

$$\begin{aligned}0 =& 2[\delta_e^T(t)Q_1^T + \dot{\delta}_e^T(t)Q_2^T + e^{-\eta(t-t_k)}\delta_e^T(t_k)Q_3^T][-\delta_e(t)\\ &+e^{-\eta(t-t_k)}\delta_e(t_k)+(t-t_k)W_\alpha(t)+(t-t_k)W_\beta(t)]\end{aligned} \tag{24}$$

$$\begin{aligned}0 =& 2[\delta_e^T(t)Y_1^T + \dot{\delta}_e^T(t)Y_2^T][-\dot{\delta}_e(t) + (A + \mathcal{K}_1)\delta_e\\ &+ \mathcal{H}\mathcal{F}(D\delta_e(t), \delta_s(t)) + D_mV_me_{\xi_m}(t) - D_sV_se_{\xi_s}(t)\\ &- \mathcal{K}_1Ce_k(t) - B_0\omega(t)].\end{aligned} \tag{25}$$

According to sector condition (4), there exists a diagonal matrix $U = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_f\} > 0$ such that

$$\begin{aligned}0 \leq& -2\sum_{i=1}^{f}\mu_i\mathcal{F}_i(d_i^T\delta_e(t), \delta_s(t))(\mathcal{F}_i(d_i^T\delta_e(t), \delta_s(t))\\ &-\mathcal{E}_id_i^T\delta_e(t))\\ =& -2\mathcal{F}^T(D\delta_e(t), \delta_s(t))U\mathcal{F}(D\delta_e(t), \delta_s(t))\\ &+ 2\mathcal{E}\delta_e^T(t)D^TU\mathcal{F}(D\delta_e(t), \delta_s(t))\end{aligned} \tag{26}$$

holds.

When $\omega(t) = 0$, it can be derived from (21)-(26) that

$$\begin{aligned}&\dot{V}_a(t) + 2\eta V_a(t)\\ \leq& \frac{t - t_k}{h}\varphi_1^T(t)\Sigma(h)\varphi_1(t)+\frac{t_k + h - t}{h}\varphi_2^T(t)\Theta(h)\varphi_2(t)\end{aligned} \tag{27}$$

where

$$\begin{aligned}\varphi_1(t) =& \text{col}\{\delta_e(t), \dot{\delta}_e(t), e_{\xi_m}(t), e_{\xi_s}(t), e^{-\eta(t-t_k)}\delta_e(t_k),\\ &\mathcal{F}(D\delta_e(t), \delta_s(t)), e_k(t), W_\alpha(t), W_\beta(t)\}\\ \varphi_2(t) =& \text{col}\{\delta_e(t), \dot{\delta}_e(t), e_{\xi_m}(t), e_{\xi_s}(t), e^{-\eta(t-t_k)}\delta_e(t_k),\\ &\mathcal{F}(D\delta_e(t), \delta_s(t)), e_k(t)\}.\end{aligned}$$

It is evident that $\Sigma(h)$ and $\Theta(h)$ are the sub-matrices of Φ_1^2 and Φ_1^3, respectively. Thus, (17) and (18) imply that $\Sigma(h) < 0$ and $\Theta(h) < 0$. Consequently, from (27) we have

$$\dot{V}_a(t) + 2\eta V_a(t) \leq 0.$$

Case 2: $t \in [t_k + h, t_{k+1})$

In this case, $\zeta(t) = 2$, $u(t) = D_m\hat{d}_m(t) - D_s\hat{d}_s(t) - \mathcal{K}_2\sigma_e(t_k)$, the Lyapunov function $V_b(t)$ can be employed. It follows that there is a small positive scalar $\varrho_b = \min\{\lambda_{min}(P_1), \lambda_{min}(P_2), \lambda_{min}(P_3)\}$, which makes

$$V_b(t) \geq \varrho_b\,|\delta_e(t)|^2. \tag{28}$$

By performing certain calculations, we can derive the result:

$$\begin{aligned}&\dot{V}_b(t) + 2\eta V_b(t)\\ =\;& 2\delta_e^T(t)P_1\dot{\delta}_e(t) + 2e_{\xi_m}^T(t)P_2\dot{e}_{\xi_m}(t) + 2e_{\xi_s}^T(t)P_3\dot{e}_{\xi_s}(t)\\ &+2\eta(\delta_e^T(t)P_1\delta_e(t)+e_{\xi_m}^T(t)P_2e_{\xi_m}(t)+e_{\xi_s}^T(t)P_3e_{\xi_s}(t)).\end{aligned} \tag{29}$$

In addition, from trigger rule (13), we have

$$0 \leq -e_k^T(t)\Lambda e_k(t) + \alpha\sigma_e^T(t)\sigma_e(t). \tag{30}$$

And for arbitrary matrices Y_1, Y_2, such that

$$\begin{aligned}0 =& 2[\delta_e^T(t)Y_1^T + \dot{\delta}_e^T(t)Y_2^T][-\dot{\delta}_e(t) + (A + \mathcal{K}_2)\delta_e\\ &+ \mathcal{H}\mathcal{F}(D\delta_e(t), \delta_s(t)) + D_mV_me_{\xi_m}(t) - D_sV_se_{\xi_s}(t)\\ &- \mathcal{K}_2Ce_k(t) - B_0\omega(t)]\end{aligned} \tag{31}$$

holds true. Therefore, combining (26), (29)-(31), when $\omega(t) = 0$, it can infer that

$$\dot{V}_b(t) + 2\eta V_b(t) \leq \varphi_3^T(t)\Gamma(h)\varphi_3(t) \tag{32}$$

where

$$\varphi_3(t)=\text{col}\{\delta_e(t),\dot{\delta}_e(t), e_{\xi_m}(t), e_{\xi_s}(t),\mathcal{F}(D\delta_e(t), \delta_s(t)), e_k(t)\}.$$

Since $\Gamma(h)$ is a sub-matrix of Φ_1^4, (19) implies that $\Gamma(h) < 0$. Therefore, we have from (32) that

$$\dot{V}_b(t) + 2\eta V_b(t) \le 0.$$

Besides, $V(t)$ is continuous at instants t_k and $t_k + h$, so for any $t \in [t_k, t_{k+1})$

$$\dot{V}(t) + 2\eta V(t) \le 0$$
$$\varrho\,|\delta_e(t)|^2 \le V(t)$$

where $\varrho = \min\{\varrho_a, \varrho_b\}$. Note this, it can deduce that

$$\begin{aligned} V(t) &\le V(t_k)e^{-2\eta(t-t_k)} \\ &\le V(t_{k-1})e^{-2\eta(t-t_{k-1})} \\ &\vdots \\ &\le V(0)e^{-2\eta t}. \end{aligned}$$

Further,

$$|\delta_e(t)| \le \sqrt{\frac{V(0)}{\varrho}}e^{-\eta t}$$

which means that SS (15) is exponentially stable with a decay rate η in the absence of disturbance.

Next, we will define an index function $\mathcal{J}(t)$ to evaluate the $\mathcal{H}_\infty$ disturbance-rejection performance of the system when $\omega(t) \neq 0$,

$$\mathcal{J}(t) = z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t).$$

Adding $\mathcal{J}(t)$ to both sides of inequalities (27) and (32), we can get

$$\begin{aligned} &\dot{V}_a(t) + 2\eta V_a(t) + \mathcal{J}(t) \\ &\le \frac{t-t_k}{h}\bar{\varphi}_1^T(t)\Phi_1^2\bar{\varphi}_1(t) + \frac{t_k+h-t}{h}\bar{\varphi}_2^T(t)\Phi_1^3\bar{\varphi}_2(t) \\ &\dot{V}_b(t) + 2\eta V_b(t) + \mathcal{J}(t) \le \bar{\varphi}_3^T(t)\Phi_1^4\bar{\varphi}_3(t) \end{aligned}$$

where

$$\begin{aligned} \bar{\varphi}_1(t) =& \mathrm{col}\{\delta_e(t), \dot{\delta}_e(t), e_{\xi_m}(t), e_{\xi_s}(t), e^{-\eta(t-t_k)}\delta_e(t_k), \\ & \mathcal{F}(D\delta_e(t), \delta_s(t)), e_k(t), W_\alpha(t), W_\beta(t), \omega(t), z(t)\} \\ \bar{\varphi}_2(t) =& \mathrm{col}\{\delta_e(t), \dot{\delta}_e(t), e_{\xi_m}(t), e_{\xi_s}(t), e^{-\eta(t-t_k)}\delta_e(t_k), \\ & \mathcal{F}(D\delta_e(t), \delta_s(t)), e_k(t), \omega(t), z(t)\} \\ \bar{\varphi}_3(t) =& \mathrm{col}\{\delta_e(t), \dot{\delta}_e(t), e_{\xi_m}(t), e_{\xi_s}(t), \mathcal{F}(D\delta_e(t), \delta_s(t)), \\ & e_k(t), \omega(t), z(t)\}. \end{aligned}$$

From (17)-(19), it can be seen that $\Phi_1^2 < 0$, $\Phi_1^3 < 0$, $\Phi_1^4 < 0$, and $V(t)$ is continuous at instants t_k and $t_k + h$, therefore, for any $t \in [t_k, t_{k+1})$

$$\dot{V}(t) + 2\eta V(t) + \mathcal{J}(t) \le 0$$

because $\eta \ge 0$ and $V(t) \ge 0$, such that

$$\dot{V}(t) + \mathcal{J}(t) \le 0. \tag{33}$$

For any $t \in [0, t_g]$, integrating both sides of (33) from 0 to t_g, then we can get

$$\begin{aligned} &\int_0^{t_g} \dot{V}(t) + \mathcal{J}(t)\,dt \\ &= V(t_g) - V(t_{g-1}) + V(t_{g-1}^-) - \cdots - V(0) + \int_0^{t_g}\mathcal{J}(t)\,dt \\ &\le 0. \end{aligned}$$

Since $V(t_g) \ge 0$, $V(0) = 0$, and $V(t_{m-1}^-) - V(t_{m-1}) = 0$ for $m = 2, 3, \cdots, g$, it can obtained that

$$\int_0^{t_g} \mathcal{J}(t)\,dt \le 0$$

so when $t_g \to \infty$,

$$\int_0^\infty z^T(t)z(t)\,dt \le \gamma^2\int_0^\infty \omega^T(t)\omega(t)\,dt$$

which indicates that SS (15) is exponentially stable when $\omega(t) = 0$ and has the prescribed $\mathcal{H}_\infty$ disturbance-rejection performance. Thus, the proof is completed. ■

B. Controller Synthesis

From Theorem 1, it is easy to write the following results:

Theorem 2. For given constants $\alpha > 0$, $\eta > 0$, $h > 0$, $\gamma > 0$, and $\theta > 0$, suppose there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $P_4 > 0$, $P_5 > 0$, $P_6 > 0$, $\Lambda \ge 0$, Y_1, Q_1, Q_2, Q_3, X, X_1, $\hat{\mathcal{K}}_1$, $\hat{\mathcal{K}}_2$, $\hat{L}_m$, $\hat{L}_s$, and diagonal matrix $U > 0$ such that

$$\Phi_1^1 = \begin{bmatrix} P_1 + h\mathscr{S}(\frac{X}{2}) & h(-X + X_1) \\ * & h\mathscr{S}(-X_1 + \frac{X}{2}) \end{bmatrix} > 0 \tag{34}$$

$$\Phi_2^2 = \begin{bmatrix} \hat{\Sigma}(h) & \nu_1 \\ * & \Omega \end{bmatrix} < 0 \tag{35}$$

$$\Phi_2^3 = \begin{bmatrix} \hat{\Theta}(h) & \nu_2 \\ * & \Omega \end{bmatrix} < 0 \tag{36}$$

$$\Phi_2^4 = \begin{bmatrix} \hat{\Gamma}(h) & \nu_3 \\ * & \Omega \end{bmatrix} < 0 \tag{37}$$

hold, where $\hat{\Sigma}(h) = [\hat{\Sigma}_{i_1j_1}]$, $i_1, j_1 = 1, 2, \ldots, 9$, $\hat{\Theta}(h) = [\hat{\Theta}_{i_2j_2}]$, $i_2, j_2 = 1, 2, \ldots, 7$, $\hat{\Gamma}(h) = [\hat{\Gamma}_{i_3j_3}]$, $i_3, j_3 = 1, 2, \ldots, 6$,

$$\begin{aligned} &\hat{\Sigma}_{11} = 2\eta P_1 + \mathscr{S}(Y_1^TA + \hat{\mathcal{K}}_1C - \frac{X}{2} - Q_1) \\ &\hat{\Sigma}_{12} = P_1 - Q_2 - Y_1^T + \theta(A^TY_1 + C^T\hat{\mathcal{K}}_1^T) \\ &\hat{\Sigma}_{17} = -\hat{\mathcal{K}}_1C,\ \hat{\Sigma}_{22} = -\theta\mathscr{S}(Y_1),\ \hat{\Sigma}_{23} = \theta Y_1^TD_mV_m \\ &\hat{\Sigma}_{24} = -\theta Y_1^TD_sV_s,\ \hat{\Sigma}_{26} = \theta Y_1^T\mathcal{H},\ \hat{\Sigma}_{27} = -\theta\hat{\mathcal{K}}_1C \\ &\hat{\Sigma}_{33} = 2\eta P_2 + \mathscr{S}(P_2W_m + \hat{L}_mD_mV_m) \\ &\hat{\Sigma}_{44} = 2\eta P_3 + \mathscr{S}(P_3W_s + \hat{L}_sD_sV_s) \\ &\hat{\Theta}_{11} = \hat{\Sigma}_{11} + \eta^2hP_5 + \eta h\mathscr{S}(X),\ \hat{\Theta}_{12} = \hat{\Sigma}_{12} + h\frac{X}{2} \\ &\hat{\Theta}_{17} = \hat{\Sigma}_{17},\ \hat{\Theta}_{22} = \hat{\Sigma}_{22} + hP_4,\ \hat{\Theta}_{23} = \hat{\Sigma}_{23},\ \hat{\Theta}_{24} = \hat{\Sigma}_{24} \\ &\hat{\Theta}_{26} = \hat{\Sigma}_{26},\ \hat{\Theta}_{27} = \hat{\Sigma}_{27} + hC^TP_6^T,\ \hat{\Theta}_{33} = \hat{\Sigma}_{33} \\ &\hat{\Theta}_{44} = \hat{\Sigma}_{44},\ \hat{\Gamma}_{11} = 2\eta P_1 + \mathscr{S}(Y_1^TA + \hat{\mathcal{K}}_2C) + \alpha C^T\Lambda C \\ &\hat{\Gamma}_{12} = P_1 - Y_1^T + \theta(A^TY_1 + C^T\hat{\mathcal{K}}_2),\ \hat{\Gamma}_{16} = -\hat{\mathcal{K}}_2C \\ &\hat{\Gamma}_{22} = \hat{\Sigma}_{22},\ \hat{\Gamma}_{23} = \hat{\Sigma}_{23},\ \hat{\Gamma}_{24} = \hat{\Sigma}_{24},\ \hat{\Gamma}_{25} = \hat{\Sigma}_{26} \end{aligned}$$

and other blocks unspecified are the same as Theorem 1. Then, under the controller (14) with gains $\mathcal{K}_1 = (Y_1^T)^{-1}\hat{\mathcal{K}}_1$, $\mathcal{K}_2 = (Y_1^T)^{-1}\hat{\mathcal{K}}_2$, disturbance observer gains $L_m = P_2^{-1}\hat{L}_m$, $L_s = P_3^{-1}\hat{L}_s$, and trigger matrix Λ, the SS (15) is exponentially stable when $\omega(t) = 0$ and has the prescribed $\mathcal{H}_\infty$ disturbance-rejection performance.

Proof: Set $Y_2 = \theta Y_1$, $\hat{\mathcal{K}}_1 = Y_1^T\mathcal{K}_1$, $\hat{\mathcal{K}}_2 = Y_1^T\mathcal{K}_2$, $\hat{L}_m = P_2L_m$, $\hat{L}_s = P_3L_s$. Then, the inequalities (35)-(37)

in Theorem 2 can be rewritten as the inequalities (17)-(19) in Theorem 1, respectively. This completes the proof. ■

In Theorem 2, the controller gain is allowed to be switched between $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$. When $\hat{\mathcal{K}}_1 = \hat{\mathcal{K}}_2 = \hat{\mathcal{K}}$, the controller simplifies to the following form:

$$u(t) = D_m\hat{d}_m(t) - D_s\hat{d}_s(t) - \mathcal{K}\sigma_e(t_k) \quad (38)$$

and the SS becomes

$$\begin{aligned}\dot{\delta}_e(t) =&(A+\mathcal{K}C)\delta_e(t) + \mathcal{H}\mathcal{F}(D\delta_e(t),\delta_s(t)) - B_0\omega(t)\\ &+ D_mV_me_{\xi_m}(t) - D_sV_se_{\xi_s}(t) - \mathcal{K}Ce_k(t). \quad (39)\end{aligned}$$

Then, the following corollary can be obtained:

Corollary 1. For given constants $\alpha > 0$, $\eta > 0$, $h > 0$, $\gamma > 0$, and $\theta > 0$, if there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $P_4 > 0$, $P_5 > 0$, $P_6 > 0$, $\Lambda \geq 0$, Y_1, Q_1, Q_2, Q_3, X, X_1, $\hat{\mathcal{K}}$, $\hat{L}_m$, $\hat{L}_s$, and diagonal matrix $U > 0$ such that the following inequalities hold

$$\Phi_1^1 = \begin{bmatrix} P_1 + h\mathscr{S}(\frac{X}{2}) & h(-X+X_1) \\ * & h\mathscr{S}(-X_1+\frac{X}{2}) \end{bmatrix} > 0 \quad (40)$$

$$\Phi_3^2 = \begin{bmatrix} \hat{\Sigma}(h) & \nu_1 \\ * & \Omega \end{bmatrix} < 0 \quad (41)$$

$$\Phi_3^3 = \begin{bmatrix} \hat{\Theta}(h) & \nu_2 \\ * & \Omega \end{bmatrix} < 0 \quad (42)$$

$$\Phi_3^4 = \begin{bmatrix} \hat{\Gamma}(h) & \nu_3 \\ * & \Omega \end{bmatrix} < 0 \quad (43)$$

where $\hat{\Sigma}(h) = [\hat{\Sigma}_{i_1j_1}]$, $i_1, j_1 = 1,2,\ldots,9$, $\hat{\Theta}(h) = [\hat{\Theta}_{i_2j_2}]$, $i_2, j_2 = 1,2,\ldots,7$, $\hat{\Gamma}(h) = [\hat{\Gamma}_{i_3j_3}]$, $i_3, j_3 = 1,2,\ldots,6$,

$$\begin{aligned}
&\hat{\Sigma}_{11} = 2\eta P_1 + \mathscr{S}(Y_1^TA + \hat{\mathcal{K}}C - \frac{X}{2} - Q_1)\\
&\hat{\Sigma}_{12} = P_1 - Q_2 - Y_1^T + \theta(A^TY_1 + C^T\hat{\mathcal{K}}^T)\\
&\hat{\Sigma}_{17} = -\hat{\mathcal{K}}C,\ \hat{\Sigma}_{22} = -\theta\mathscr{S}(Y_1),\ \hat{\Sigma}_{23} = \theta Y_1^TD_mV_m\\
&\hat{\Sigma}_{24} = -\theta Y_1^TD_sV_s,\ \hat{\Sigma}_{26} = \theta Y_1^T\mathcal{H},\ \hat{\Sigma}_{27} = -\theta\hat{\mathcal{K}}C\\
&\hat{\Sigma}_{33} = 2\eta P_2 + \mathscr{S}(P_2W_m + \hat{L}_mD_mV_m)\\
&\hat{\Sigma}_{44} = 2\eta P_3 + \mathscr{S}(P_3W_s + \hat{L}_sD_sV_s)\\
&\hat{\Theta}_{11} = \hat{\Sigma}_{11} + \eta^2hP_5 + \eta h\mathscr{S}(X),\ \hat{\Theta}_{12} = \hat{\Sigma}_{12} + h\frac{X}{2}\\
&\hat{\Theta}_{17} = \hat{\Sigma}_{17},\ \hat{\Theta}_{22} = \hat{\Sigma}_{22} + hP_4,\ \hat{\Theta}_{23} = \hat{\Sigma}_{23},\ \hat{\Theta}_{24} = \hat{\Sigma}_{24}\\
&\hat{\Theta}_{26} = \hat{\Sigma}_{26},\ \hat{\Theta}_{27} = \hat{\Sigma}_{27} + hC^TP_6^T,\ \hat{\Theta}_{33} = \hat{\Sigma}_{33}\\
&\hat{\Theta}_{44} = \hat{\Sigma}_{44},\ \hat{\Gamma}_{11} = 2\eta P_1 + \mathscr{S}(Y_1^TA + \hat{\mathcal{K}}C) + \alpha C^T\Lambda C\\
&\hat{\Gamma}_{12} = P_1 - Y_1^T + \theta(A^TY_1 + C^T\hat{\mathcal{K}}),\ \hat{\Gamma}_{16} = -\hat{\mathcal{K}}C\\
&\hat{\Gamma}_{22} = \hat{\Sigma}_{22},\ \hat{\Gamma}_{23} = \hat{\Sigma}_{23},\ \hat{\Gamma}_{24} = \hat{\Sigma}_{24},\ \hat{\Gamma}_{25} = \hat{\Sigma}_{26}
\end{aligned}$$

and other blocks unspecified are the same as Theorem 1, then under the controller (38) with gain $\mathcal{K} = (Y_1^T)^{-1}\hat{\mathcal{K}}$, disturbance observer gains $L_m = P_2^{-1}\hat{L}_m$, $L_s = P_3^{-1}\hat{L}_s$, and trigger matrix Λ, the SS (39) is exponentially stable and has the prescribed $\mathcal{H}_\infty$ disturbance-rejection performance.

IV. Numerical simulation

The effectiveness of the proposed results is verified below by using two Chua's circuits with unknown disturbances. As in [29], the master system and the slave system are described by the following forms:

$$\begin{cases}\dot{\delta}_{m1}(t) = a[\delta_{m2}(t) - h(\delta_{m1}(t))] + d_{m1}(t)\\ \dot{\delta}_{m2}(t) = \delta_{m1}(t) - \delta_{m2}(t) + \delta_{m3}(t) + d_{m2}(t)\\ \dot{\delta}_{m3}(t) = -b\delta_{m2}(t) + d_{m2}(t)\\ \sigma_m(t) = \delta_{m1}(t)\end{cases} \quad (44)$$

$$\begin{cases}\dot{\delta}_{s1}(t) = a[\delta_{s2}(t) - h(\delta_{s1}(t))] + d_{s1}(t) - 4\omega_1(t) + u_1(t)\\ \dot{\delta}_{s2}(t) = \delta_{s1}(t) - \delta_{s2}(t) + \delta_{s3}(t) + d_{s2}(t) + 3\omega_2(t) + u_2(t)\\ \dot{\delta}_{s3}(t) = -b\delta_{s2}(t) + d_{s2}(t) + 5\omega_3(t) + u_3(t)\\ \sigma_s(t) = \delta_{s1}(t)\end{cases} \quad (45)$$

where

$$\begin{aligned}h(\delta_{i1}(t)) =& m_1\delta_{i1}(t) + \frac{1}{2}(m_0 - m_1)\\ &\times(|\delta_{i1}(t) + c| - |\delta_{i1}(t) - c|),\ (i = m, s).\end{aligned}$$

When $a = 9$, $b = 14.28$, $m_0 = -\frac{1}{7}$, $m_1 = \frac{2}{7}$, $c = 1$, then the Chua's circuits can be transformed into the following Lur'e form by

$$A = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix},\ \mathcal{H} = \begin{bmatrix} -a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix}$$

$$C = D = \begin{bmatrix} 1, 0, 0 \end{bmatrix},\ B_0 = \begin{bmatrix} -4, 3, 5 \end{bmatrix}^T,\ \omega(t) = \frac{1}{1+t^2}$$

$$d_m(t) = \begin{bmatrix} d_{m1}(t) \\ d_{m2}(t) \end{bmatrix},\ d_s(t) = \begin{bmatrix} d_{s1}(t) \\ d_{s2}(t) \end{bmatrix},\ u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

and $f(\epsilon) = \frac{1}{2}(|\epsilon + c| - |\epsilon - c|)$ belonging to sector $[0, 1]$.

Set $\delta_m(0) = \text{col}\{1, 0.5, -0.5\}$, $\delta_s(0) = \text{col}\{-1, 0.1, 0.1\}$. In the absence of disturbances and control input, the aforementioned Chua's circuits exhibit double-scroll attractors, as shown in Fig. 1.

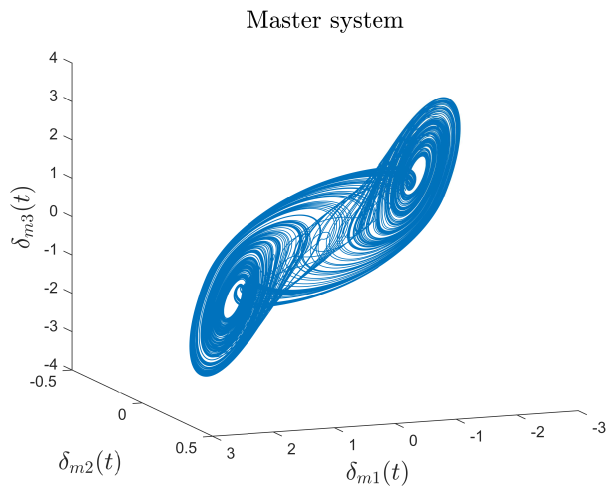

Fig. 1. The double-scroll attractor.

In what follows, we focus on verifying the exponential stability of SS (15) under the controller (14), along with the disturbance observers (7) and (8). The associated parameter matrices are provided as follows:

$$W_m = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix},\ W_s = \begin{bmatrix} 0 & 0.4 \\ -0.4 & 0 \end{bmatrix}$$

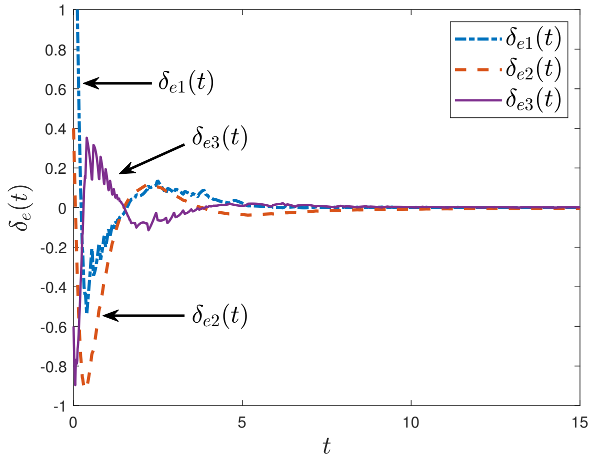

Fig. 2. Master-slave systems synchronization error.

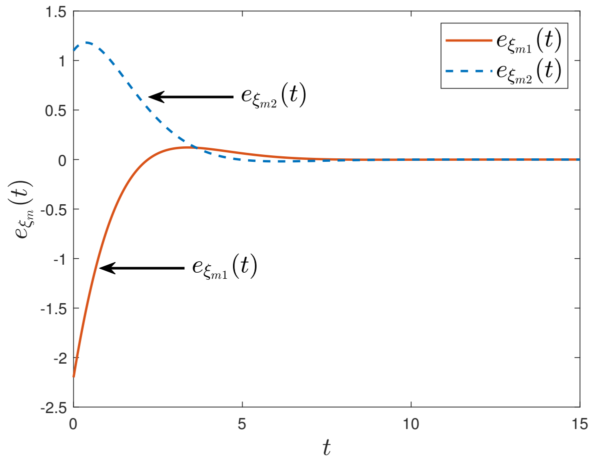

Fig. 3. Exogenous disturbance estimation errors for $d_m(t)$.

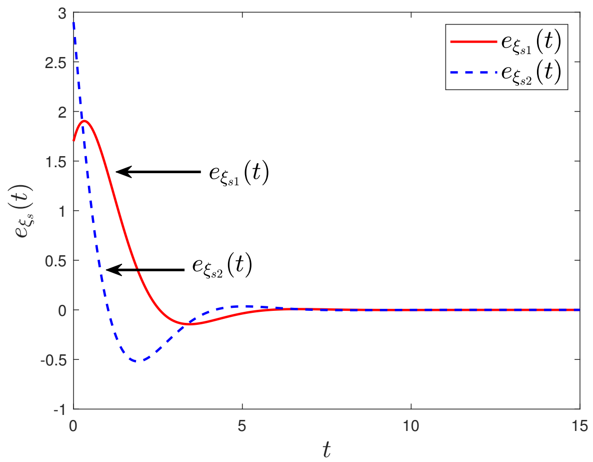

Fig. 4. Exogenous disturbance estimation errors for $d_s(t)$.

$$\mathcal{Z}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathcal{Z}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D_m = D_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \ V_m = V_s = I$$

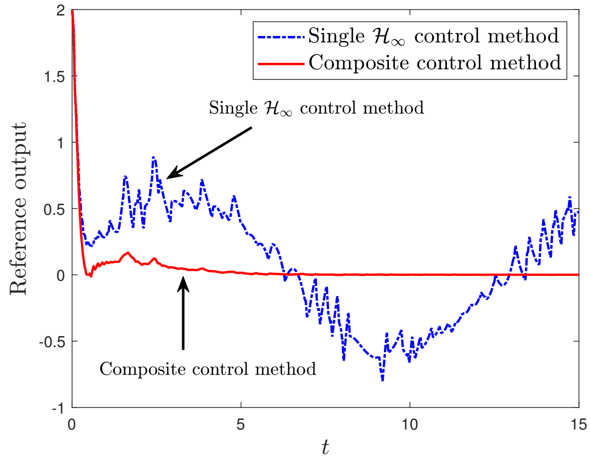

Fig. 5. Reference output under different control methods.

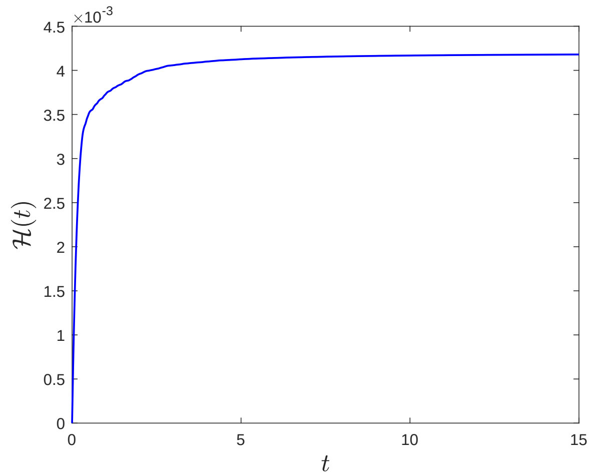

Fig. 6. The trajectory of $\mathcal{H}(t)$.

Take $\alpha = 0.1$, $\eta = 0.1$, $\theta = 0.2$, $h = 0.05$, and $\gamma = 2$. Then, by solving the inequalities (34)-(37) in Theorem 2, the desired gains and trigger matrix can be calculated as

$$\mathcal{K}_1 = \begin{bmatrix} -4.6801 & -2.0132 & 0.7999 \\ -1.0780 & -1.8924 & 1.0885 \\ 0.7314 & 2.5876 & -3.1280 \end{bmatrix}$$

$$\mathcal{K}_2 = \begin{bmatrix} -7.3932 & -5.4091 & 3.2602 \\ -0.9572 & -2.6040 & 0.5909 \\ 3.4341 & 7.0774 & -7.8341 \end{bmatrix}$$

$$L_m = \begin{bmatrix} -0.6806 & 0.0011 & -0.0000 \\ -0.0953 & -0.7685 & 0.0000 \end{bmatrix}$$

$$L_s = \begin{bmatrix} -0.8647 & 3.1786 & -2.6015 \\ -0.5980 & -0.8226 & -0.0045 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 6.2986 & 2.1315 & -0.3487 \\ 2.1315 & 7.1011 & -2.2913 \\ -0.3487 & -2.2913 & 5.6271 \end{bmatrix}.$$

As illustrated in Fig. 2, it is apparent that the synchronization error progressively converges to zero under the influence of the composite controller, thereby indicating that synchronization between the master and slave systems has been successfully achieved.

Subsequently, we analyze the estimation errors of external disturbances $d_m(t)$ and $d_s(t)$, respectively. As shown in Figs.

TABLE I
h_{max} FOR VARIOUS η.

Methods	h_{max} ($\gamma = 2$)			
	$\eta = 0$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$
Theorem 2	0.1019	0.0984	0.0951	0.0920
Corollary 1	0.0890	0.0852	0.0816	0.0783

TABLE II
γ_{min} FOR VARIOUS η.

Methods	γ_{min} ($h = 0.05$)			
	$\eta = 0$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.3$
Theorem 2	0.6823	0.6925	0.7033	0.7148
Corollary 1	0.9031	0.9298	0.9586	0.9899

3 and 4, estimation errors $e_{\xi_m}(t)$ and $e_{\xi_s}(t)$ gradually tend to zero, indicating that the estimated value of the disturbances are infinitely close to the real disturbances.

The reference outputs under single $\mathcal{H}_\infty$ control method and composite control method are compared, as shown in Fig. 5. It can be seen from the figure that the effect of using composite control method in suppressing disturbances is significantly better than that of a single $\mathcal{H}_\infty$ control method. Moreover, as in [30], we define

$$\mathcal{H}(t) = \sqrt{\frac{\int_0^t z^T(s)z(s)\,ds}{\int_0^t \omega^T(s)\omega(s)\,ds}}$$

to characterize the $\mathcal{H}_\infty$ disturbance-rejection performance. The curve of $\mathcal{H}(t)$ under zero initial condition is shown in Fig. 6, it can be found that $\mathcal{H}(\infty) = 0.0042 < \gamma_{\min} = 0.6925$. In summary, the numerical simulation demonstrates the effectiveness of the proposed event-triggered disturbance observer-based design.

Finally, to illustrate the superiority of the controller (14) with switched gain over the controller (38) with a fixed gain, we compare the maximum allowable sampling interval h_{max} and optimal $\mathcal{H}_\infty$ disturbance-rejection performance γ_{min} based on Theorem 2 and Corollary 1. The results, presented in Tables I and II, clearly indicate that, under the same parameter conditions, the controller with switched gain not only permits a larger sampling interval but also demonstrates superior $\mathcal{H}_\infty$ disturbance-rejection performance.

V. Conclusion

This paper investigated the issue of anti-disturbance synchronization for LSs using a hierarchical composite control mechanism (14). This mechanism integrated two control approaches: DOBC and switched-gain ETC, ensuring the $\mathcal{H}_\infty$ exponential stability of the SS (15) in the presence of multiple disturbances. A condition on the $\mathcal{H}_\infty$ exponential stability was established in Theorem 1 utilizing a piecewise-defined and time-dependent Lyapunov function and several inequalities. Based on the condition, a co-design was proposed for the gains of the event-triggered controller and disturbance observer in Theorem 2. For comparison, the hierarchical composite control with a fixed gain was also considered, and the corresponding design approach was presented in Corollary 1. Finally, the effectiveness of the proposed composite anti-disturbance synchronization control mechanism was validated through an example involving master-slave Chua's circuits.

References

[1] L. O. Chua, "Chua's circuit: An overview ten years later," *Journal of Circuits, Systems, and Computers*, vol. 4, no. 02, pp. 117–159, 1994.

[2] J. J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities." *Proceedings of the National Academy of Sciences*, vol. 79, no. 8, pp. 2554–2558, 1982.

[3] A. Dridi, R. L. Filali, and M. Benrejeb, "Discrete-time chaotic multimodels synchronization and apllication to secure communication," in *2017 International Conference on Advanced Systems and Electric Technologies (IC_ASET)*, Hammamet, Tunisia, January 2017, pp. 430–435.

[4] H. Liu, H. Wan, K. T. Chi, and J. Lü, "An encryption scheme based on synchronization of two-layered complex dynamical networks," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 11, pp. 2010–2021, 2016.

[5] K. H. Gularte, J. C. Gómez, M. E. V. Melgar, and J. A. Vargas, "Chaos synchronization and its application in parallel cryptography," in *2021 IEEE 5th Colombian Conference on Automatic Control (CCAC)*, Ibague, Colombia, October 2021, pp. 198–203.

[6] D. Wei, L. Chen, J.-S. Wang, and X. Yao, "Design of automatic control system for constant tension and linear speed of rewinder machine." *Engineering Letters*, vol. 29, no. 1, pp. 213–222, 2021.

[7] R. Kavikumar, R. Sakthivel, O.-M. Kwon, and P. Selvaraj, "Robust tracking control design for fractional-order interval type-2 fuzzy systems," *Nonlinear Dynamics*, vol. 107, no. 4, pp. 3611–3628, 2022.

[8] J. Zhou, J. Dong, S. Xu, and C. K. Ahn, "Input-to-state stabilization for Markov Jump systems with dynamic quantization and multimode injection attacks," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 54, no. 4, pp. 2517–2529, 2024.

[9] P. Pal, V. Mukherjee, H. Alemayehu, G. G. Jin, and G. Feyisa, "Generalized adaptive backstepping sliding mode control for synchronizing chaotic systems with uncertainties and disturbances," *Mathematics and Computers in Simulation*, vol. 190, pp. 793–807, 2021.

[10] N. Noroozi, M. Roopaei, and P. Karimaghaee, "Adaptive control and synchronization in a class of partially unknown chaotic systems," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 19, no. 2, p. 023121, 2009.

[11] M. Hamdy, M. Magdy, and S. Helmy, "Control and synchronization for two Chua systems based on intuitionistic fuzzy control scheme: A comparative study," *Transactions of the Institute of Measurement and Control*, vol. 43, no. 7, pp. 1650–1667, 2021.

[12] L. D'Alfonso, G. Fedele, and P. Pugliese, "Robust output tracking, disturbance attenuation and synchronization for a class of Lur'e systems: A high-gain, fractional-order control approach," *Nonlinear Dynamics*, vol. 112, no. 2, pp. 1011–1022, 2024.

[13] Z. Tang, J. H. Park, Y. Wang, and W. X. Zheng, "Synchronization on Lur'e cluster networks with proportional delay: Impulsive effects method," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 7, pp. 4555–4565, 2021.

[14] H. Huang and G. Feng, "Robust $\mathcal{H}_\infty$ synchronization of chaotic Lur'e systems," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 18, no. 3, p. 033113, 2008.

[15] L. He, W. Wu, J. Zhou, and G. Yao, "Input-to-state stable synchronization for delayed Lurie systems via sampled-data control." *Discrete*

& Continuous Dynamical Systems-Series B, vol. 28, no. 2, pp. 1553–1570, 2023.

[16] W.-H. Chen, “Disturbance observer based control for nonlinear systems,” *IEEE/ASME Transactions on Mechatronics*, vol. 9, no. 4, pp. 706–710, 2004.

[17] B. Kaviarasan, O. M. Kwon, M. J. Park, and R. Sakthivel, “Composite synchronization control for delayed coupling complex dynamical networks via a disturbance observer-based method,” *Nonlinear Dynamics*, vol. 99, pp. 1601–1619, 2020.

[18] G. Chen, Z. Jiang, H. Tan, C. Zhang, J. Wei, W. Chen, and H. Long, “Sliding-mode design with disturbance observer optimized by improved HBA algorithm for frequency regulation of multi-area power system,” *IAENG International Journal of Applied Mathematics*, vol. 52, no. 4, pp. 994–1013, 2022.

[19] J. Zhao and M. Chen, “Variable-horizon predictive tracking control for air-breathing hypersonic vehicle based on composite disturbance observer,” *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 237, no. 12, pp. 2817–2832, 2023.

[20] S. A. Emami, A. Banazadeh, P. Hajipourzadeh, P. Castaldi, and S. A. Fazelzadeh, “Disturbance observer-based adaptive neural guidance and control of an aircraft using composite learning,” *Control Engineering Practice*, vol. 134, p. 105463, 2023.

[21] B. Zhao, X. Yao, and W. X. Zheng, “Fixed-time composite anti-disturbance control for flexible-link manipulators based on disturbance observer,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 71, no. 7, pp. 3390–3400, 2024.

[22] Z. Yan, X. Huang, Y. Fan, J. Xia, and H. Shen, “Threshold-function-dependent quasi-synchronization of delayed memristive neural networks via hybrid event-triggered control,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 11, pp. 6712–6722, 2021.

[23] W. Wu, L. He, J. Zhou, Z. Xuan, and S. Arik, “Disturbance-term-based switching event-triggered synchronization control of chaotic Lurie systems subject to a joint performance guarantee,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 115, p. 106774, 2022.

[24] J. Zhou, D. Xu, W. Tai, and C. K. Ahn, “Switched event-triggered $\mathcal{H}_\infty$ security control for networked systems vulnerable to aperiodic DoS attacks,” *IEEE Transactions on Network Science and Engineering*, vol. 10, no. 4, pp. 2109–2123, 2023.

[25] D. Liu, X. Ouyang, N. Zhao, and Y. Luo, “Adaptive event-triggered control with prescribed performance for nonlinear system with full-state constraints.” *IAENG International Journal of Applied Mathematics*, vol. 54, no. 3, pp. 390–397, 2024.

[26] J. A. Suykens, P. F. Curran, and L. O. Chua, “Robust synthesis for master-slave synchronization of Lur’e systems,” *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 46, no. 7, pp. 841–850, 1999.

[27] J.-G. Lu and D. J. Hill, “Global asymptotical synchronization of chaotic Lur’e systems using sampled data: A linear matrix inequality approach,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 55, no. 6, pp. 586–590, 2008.

[28] C.-K. Zhang, Y. He, and M. Wu, “Improved global asymptotical synchronization of chaotic Lur’e systems with sampled-data control,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 56, no. 4, pp. 320–324, 2009.

[29] M. Zhao, R. Liu, and Y. Gao, “Dissipative lag synchronization of chaotic Lur’e systems with unknown disturbances,” *IMA Journal of Mathematical Control and Information*, vol. 34, no. 1, pp. 123–138, 2017.

[30] Z. Wan, X. Ma, Y. Zhang, T. Jiang, and J. Zhou, “Event-triggered $\mathcal{H}_\infty$ controller design for Lurie systems with switching exponential time-varying gains,” *Computational and Applied Mathematics*, vol. 42, no. 6, p. 281, 2023.