Innovative Study on Complex Fuzzy Soft Graph and its Properties

R Suresh, V Veeramani and R Thamizharasi

Abstract - **This study aims to present complex fuzzy soft graphs (*cfsg*). We also obtain regular *cfsg*, strong *cfsg*, spanning *cfsg*, and complex fuzzy soft graph (*cfsg*) along with the definitions of the *cfsg* 's for path, bridge, strength, regular order and size. Furthermore, examples are provided for the union, intersection, complement, and Cartesian product of the *cfsg* procedures. This study also addresses the potential applications of the *cfsg* and energy of the *cfsg*.**

Index Terms— **Complex fuzzy soft set, complex fuzzy graph, complex fuzzy soft graph (*cfsg*), regular *cfsg*, spanning *cfsg*, and strong *cfsg*, energy of *cfsg*.**

I. INTRODUCTION

Due to the existence of uncertainty, graphs were extended into fuzzy graphs by Kaufmann [1] in 1973. Azriel Rosenfeld [2] developed a relation called fuzzy relation on fuzzy sets in 1975. Connectivity of fuzzy graphs is discussed in [3]. Later on, partial fuzzy sub graph, fuzzy spanning sub graph, complete fuzzy graph, fuzzy tree and fuzzy bridges were introduced and investigated as numerous fields, including environmental science, social science, geography, linguistics science and technology use fuzzy graphs.

Molodtsov [4] widespread this concept to other domains such as function consistency, game theory, optimization techniques, possibility and measurement theory. As a consequence, numerous investigators are growingly engaged on soft set research. Maji et al [5, 6] established the concept of soft sets with fuzzy elements in 2001, which is a mixture of fuzzy set and soft set. By combining soft sets and fuzzy sets, Zhicai [7] created applications for the decision-making model. The definitions of fuzzy graph theory, fuzzy relations and their applications—such as cognitive analysis and decision-making problems—can be found in [2] and [3].

Fuzzy soft sets and fuzzy soft graph (FSG) are clearly defined in [4,8] and [9] respectively. Also, some of its applications were discussed such as two-dimensional membership, uncertainty and the period. Incorporating uncertainty and periodicity together into applications is a complex issue. To resolve this issue, Ramot et al. [10] presented an interesting extension, called Complex Fuzzy Sets (CFS), in which the membership function is complex-valued, as contrasted with fuzzy complex numbers. Operations of the complex fuzzy sets, their properties and applications were discussed in [11, 12] and also membership of a CFS function is comprised into functions: an amplitude function and a phase function.

For multidimensional (Uncertainty, Periodic) data inside a single set of data, CFS manages uncertainties with degrees whose ranges have been extended from the real sub sets to the complex subgroup with the unit disc. Thirunavukkarasu et al. [13] pioneered the concept of the complex fuzzy graph (CFG) which capitalizes on these features. A complex fuzzy soft set (CFSS) was introduced in [14] and its applications were discussed in [15]. Soft sets are employed in the complex fuzzy sets both to describe periodical behavior and to represent in two-dimensional membership functions. A complex intuitionistic fuzzy graph and its properties were presented in [2] and the applications of complex dombi fuzzy graph were debated in [16]. Most of the authors referred the study [1] for extending the studies of complex fuzzy graphs like complex intuitionistic graphs, complex neutrosophic graphs, and the complex Pythagorean fuzzy graphs. Despite the fact that the concept of CFG is derived in [12], not enough details were given in this study. As a result, we sought to develop explicit notions of complex fuzzy graphs with an example. We also aimed to generate a complex fuzzy soft graph (*cfsg*) by improving the concept of complex fuzzy graph which was originally proposed in [12]. The complex intuitionistic fuzzy graphs and its soft graphs developed in [2,9]. Since the characteristics and the parameters of a complex intuitionistic fuzzy graphs were not discussed in this field, we aimed to extend the discussion over these concepts. Yoti hetty et al [17] introduced egularity in the semi-graph which results the concept of *cfsg*. The current investigation expands the mathematical notion of the complex fuzzy soft sets towards *cfsg* and also establishes the ideas of strong *cfsg*, complete *cfsg*, regular *cfsg*, and spanning *cfsg*. Further, we identify the possible applications of *cfsg*.

In this study, we discuss the existing concepts of *cfsg* in section I. In section II, an overview of the complex fuzzy sets and the complex fuzzy soft sets are presented with examples. Complex fuzzy soft graphs (*cfsg*), properties and the various types of *cfsg* are discussed in section III. In Section IV, operations such as union, intersection and complement of *cfsg* are proposed and the energy of the *cfsg* is discussed as well. We concluded with some potential directions for future research.

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II. COMPLEX FUZZY GRAPH & COMPLEX FUZZY SOFT SET

Definition 2.1 Ramot et al. [9] suggested a complex fuzzy set in which the existing membership function η, rather than being a one-dimensional membership function with the range [0,1], has been replaced by a complex-valued membership function or two-dimensional membership function with a range of the form

$$\eta_S(u) = p_S(u)e^{j\varpi_S(u)},\ j=\sqrt{-1},$$

where $p_s(u)$ is amplitude term, $\omega_s(u)$ is phase term of the complex membership function and both are real valued, resulting in the range as the complex plane's unit circle. The phase term holds significance in complex fuzzy sets, as demonstrated in [16].

Definition 2.2 Let $\dot{G}$ be a given graph of the form $\dot{G}$ = $(\dot{V},\dot{\sigma},\bar{E},\bar{\varphi})$ is said to be a CFG, where $\dot{V}$ is a set of vertices and $\bar{E} \subseteq \dot{V}X\dot{V}$ is a set of edges, such that

(i) The membership function of vertices defined by $\dot{\sigma}:\dot{V} \to [0,1]^2$ is a multifaceted function of the membership which is mapping from end points. i.e. $\dot{\sigma}$ is the process through which is the degrees of complex membership or two-dimensional membership functions which are assigned to the members of $\dot{V}$, $\bar{\varphi}:\bar{E} \to [0,1]^2$ is a function which maps elements of form $\bar{e} \in E:(\dot{u},\dot{v}) \to [0,1]^2$, where $\dot{u},\dot{v} \in \dot{V}$.

(ii) The complex membership function of the edge uv,
$$\varphi_S(uv) \le Min\{\sigma_S(u),\sigma_S(v)\}$$
$$= Min\{p_S(u),q_S(v)\}e^{jMin\{\varpi_S(u),v_S(v)\}},$$
Here complex membership function of the vertices $\dot{u},\dot{v}$ is defined by, $\sigma_S(\dot{u}) = p_S(\dot{u})e^{j\varpi_S(\dot{u})}$, $\sigma_S(\dot{v}) = q_S(\dot{v})e^{j v_S(\dot{v})}$.

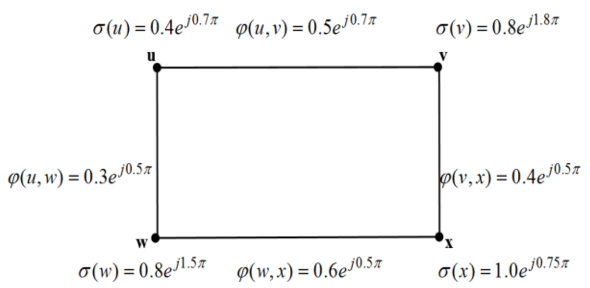


Fig. 1 Complex Fuzzy Graph

Definition 2.3 A graph representation H is of the form $H=(\dot{V},\dot{\sigma},\tilde{E},\tilde{\varphi})$ is said to be a sub graph of the CFG, such that

(i) $p_S(u) \le q_S(v)$ & $\varpi_S(u) \le v_S(v)$, here complex membership function of the vertices u,v in H defined by $\tilde{\sigma}_S(u) = p_S(u)e^{j\varpi_S(u)}$, $\tilde{\sigma}_S(v) = q_S(v)e^{j v_S(v)}$

(ii) $H=(\dot{V},\dot{\sigma},\tilde{E},\tilde{\varphi})$ itself satisfies the properties of CFG.

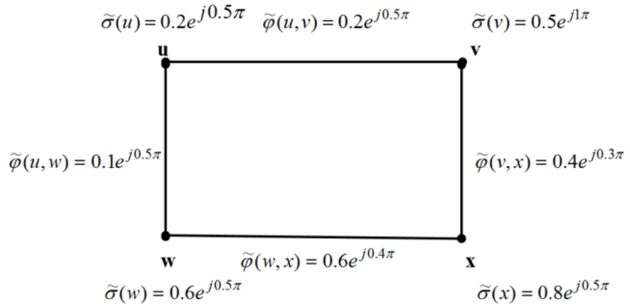


Fig. 2 CFG $H=(\dot{V},\dot{\sigma},\tilde{E},\tilde{\varphi})$

It clearly shows that $H=(\dot{V},\dot{\sigma},\tilde{E},\tilde{\varphi})$ is a complex fuzzy sub graph of $\dot{G}=(\dot{V},\dot{\sigma},\bar{E},\bar{\varphi})$ (Fig.2)

Definition 2.4 [14] Let Ω be an initial set, P_A represents a set of attributes that is not empty. Let $\psi(\Omega)$ is a power set in complex membership function Ω. A pair (ψ,Λ) is called a complex fuzzy soft set over Ω, where $\Lambda \subseteq P_A$ and ψ is a mapping given by $\psi:\Lambda \to \psi(\Omega)$. $\psi(e)$ can be written as $\psi(e) = \left\{\left(\dot{u},\chi_{\psi_\Lambda(\dot{u})}\right),\dot{u}\in\Omega, e \in P_A\right\}$, where $\chi_{\psi_\Lambda}(\dot{u})$ is the membership grade of an object $\dot{u}$.

Example 2.1: Let Ω= {C_1, C_2, C_3, C_4} represent the countries India, Russia, UK and USA respectively and it refers a Universe or Initial set. Consider P_Λ = {e_1(Unemployment rate), e_2(share market index), e_3(Inflation rate), e_4(Population growth)} be the parameters set for growth rate of countries and $\Lambda \subseteq P_\Lambda$, i.e. Λ={e_1, e_4}, then complex fuzzy soft Set (ψ, Λ), where $\psi:\ \Lambda \to \psi(\Omega)$, as represented in (2.1). Here complex fuzzy soft set (ψ, Λ) represents the membership grade for the parameters of unemployment rate and population growth with respect to countries. Phase terms play a significant role in representing the country's current situation of growth.

$$(\psi,\Lambda) = \left\{ \begin{array}{l} \psi(e_1) = \left\{ \begin{array}{l} \dfrac{\chi(e_1)=0.4e^{j0.7\pi}}{C_1}, \dfrac{\chi(e_1)=0.8e^{j1.8\pi}}{C_2}, \\ \dfrac{\chi(e_1)=0.8e^{j1.5\pi}}{C_3}, \dfrac{\chi(e_1)=1.0e^{j0.75\pi}}{c_4} \end{array} \right\}, \\ \psi(e_4) = \left\{ \begin{array}{l} \dfrac{\chi(e_4)=0.3e^{j0.4\pi}}{c_1}, \dfrac{\chi(e_4)=0.9e^{j0.8\pi}}{c_2}, \\ \dfrac{\chi(e_4)=0.7e^{j1.5\pi}}{c_3}, \dfrac{\chi(e_4)=0.75^{j0.75\pi}}{C_4} \end{array} \right\} \end{array} \right\} \quad (2.1)$$

Phase term takes values that lie between 0 and 2π. The USA is a developed country with a small population whereas India is a developing country with the largest population. So, we could not measure both the countries in some parameters like growth rate, unemployment rate, Inflation rate and etc. For assigning phase values in membership grade, we consider the particular nation's current situation of developing growth. 0 is considered for a low developing rate and 2π is considered for highly developing. So, we only add phase term for each membership function and its values lie in between 0 and 2π.

III. COMPLEX FUZZY SOFT GRAPH

Definition 3.1. Let $G=(V,E)$ be a simple graph. Let ρ_A be a non-empty set of parameters. Then, $(S_V(\tilde{a}),S_E(\tilde{a}))$ for all $\tilde{a}\in\rho_A$ is *cfsg* over G ∋,

(i) (S_V,ρ_A) is a complex fuzzy soft set over V(Vertices),

(S_E,ρ_A) is a complex fuzzy soft set over E (Edges),

(ii) $S_E(\tilde{a})(\dot{u}\dot{v})\le Min\{S_V(\tilde{a})(\dot{u}),S_V(\tilde{a})(\dot{v})\}=$

$Min\{p_S(\dot{u}),q_S(\dot{v})\}e^{jMin\{\varpi_S(\dot{u}),\nu_S(\dot{v})\}}$ and

$S_V(\tilde{a})(\dot{u})=$

$p_S(\dot{u})e^{j\varpi_S(\dot{u})}$, $S_V(\tilde{a})(\dot{v})=q_S(\dot{v})e^{j\nu_S(\dot{v})}$,

$\forall\dot{u},\dot{v}\in V$, For convenience, complex fuzzy soft graph $(S_V(\tilde{a}),S_E(\tilde{a}))$ is denoted by $C_{fs}(\tilde{G})$.

Example 3.1. Consider the vertices in the simple graph represented by the countries India, Russia, UK and USA as $\dot{V}=\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\}$ and $\overline{E}=\{\varsigma_1\varsigma_2,\varsigma_1\varsigma_3,\varsigma_2\varsigma_4,\varsigma_3\varsigma_4\}$ be the set of edges. i.e., relations between these countries. Let $P_A=\{e_1,e_2,e_3,e_4\}$ be represent the parameters of unemployment rate, share market index, inflation rate and population growth respectively and $S_V(e_1)=\{0.4e^{j0.7\pi},0.7e^{j1.7\pi},.7e^{j1.5\pi},1.0e^{j0.75\pi}\}$ is depicted as complex fuzzy soft membership function of Unemployment rate of the four countries $\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4$ respectively , ς

$$S_E(e_1)=\left\{\frac{0.3e^{j0.4\pi}}{\varsigma_1\varsigma_3},\frac{0.3e^{j0.4\pi}}{\varsigma_1\varsigma_2},\frac{0.4e^{j0.5\pi}}{\varsigma_3\varsigma_4},\frac{0.5e^{j0.6\pi}}{\varsigma_2\varsigma_4}\right\}$$ be

the relative membership function of unemployment rate with respect to the countries, then the complex fuzzy soft graph $C_{fs}(\tilde{G}(e_1))=(S_V(e_1),S_E(e_1))$ is represented as follows.

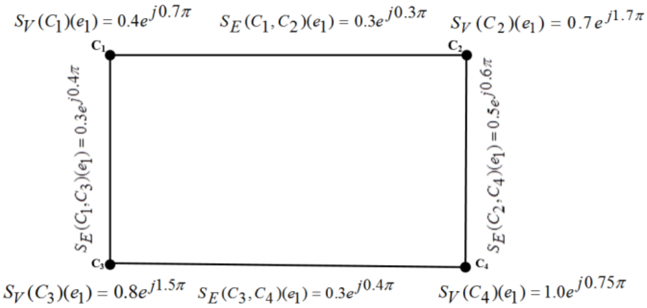


Fig. 3 *cfsg* $-C_{fs}(\tilde{G}(e_1))$

Definition 3.2. The *cfsg* $C_{fs}(\tilde{H})=(S_{\tilde{V}}(\dot{a}),S_{\tilde{E}}(\dot{a}))$ is said to be a sub-graph of $C_{fs}(\tilde{G})=(S_V(\dot{a}),S_E(\dot{a}))$, if

(i) $r_s(u)\le t_s(v)$, $\tau_s(u)\le\zeta_s(v)$,

where $S_{\tilde{V}}(a)(u)=r_s(u)e^{j\tau_s(u)}$, $S_{\tilde{V}}(a)(v)=t_s(v)e^{j\zeta_s(v)}$, for all $u,v\in V$ and $a\in P_A$.

(ii) $C_{fs}(\tilde{H})$ itself satisfies the properties of the complex fuzzy soft graph.

Definition 3.3. Let $C_{fs}(\tilde{G})=(S_V(a),S_E(a))$ is *cfsg*. The degree of the vertex $\dot{u}$ in $C_{fs}(\tilde{G})$ is defined as

$$D_g(\dot{u})=\sum_{\dot{y}\in V}|S_E(\dot{x},\dot{y})|,$$

and l is defined by, $l=\frac{\sum_{\dot{x}\in V}D_g(\dot{x})}{n}$, where $n=|V|$.

Example: 3.2.

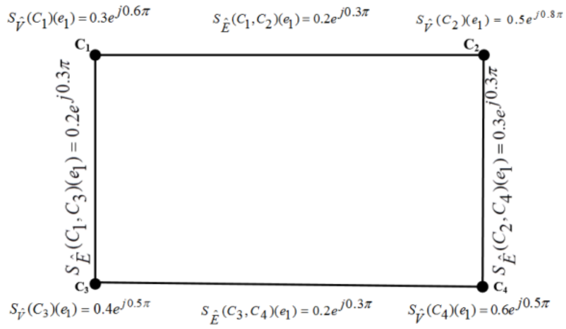


Fig. 4 $C_{fs}(\hat{H}(e_1))=\left(S_{\hat{V}}(e_1),S_{\hat{E}}(e_1)\right)$

Here $C_{fs}(\hat{H}(e_1))=\left(S_{\hat{V}}(e_1),S_{\hat{E}}(e_1)\right)$ is a sub graph of $C_{fs}(\tilde{G}(e_1))$.

Example3.3. From Example 2.3, $D_g(C_1)$=0.59, $D_g(C_2)$= 0.79, $D_g(C_3)$=0.69, $D_g(C_4)$=0.89, $D_g(x)$=2.96, for all $x\in V$, and $l=\frac{\sum_{x\in V}D_g(x)}{n}=0.74\le 1$, obtained from above mentioned formula.

Definition 3.4. If $|D_g(x)-l|\le\in$, $\forall\dot{x}\in V$, $0\le\in\le 1$, ($\in$ is small number), then $C_{fs}(\tilde{G})$ is regular.

From the Fig. 5, $|D_g(x)-l|\le\in\le 1$. Hence Fig. 5 is regular *cfsg*.

Example: 3.4.

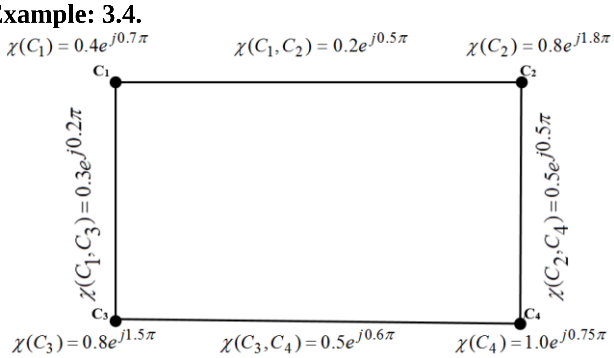


Fig. 5 Regular *cfsg*

Definition 3.5. A *cfsg* $C_{fs}(\tilde{G}(\hat{a}))$ is said to be strong *cfsg*, if $|S_E(\tilde{a})(\dot{u}\dot{v})|=Min\{|S_V(\tilde{a})(\dot{u})|,|S_V(\tilde{a})(\dot{v})|\}$, for all $\tilde{a}\in\rho_A$, $\dot{u}\dot{v}\in E$.

Example: 3.5

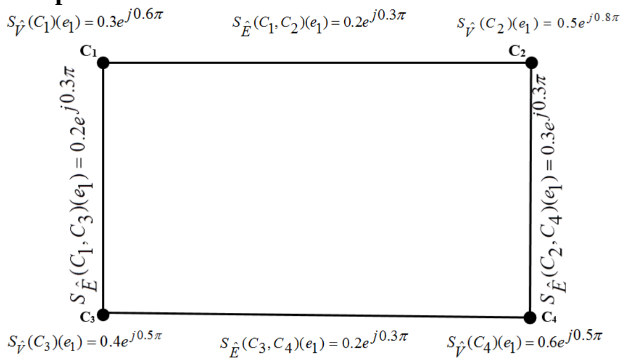


Fig. 6 $C_{fs}(\widetilde{G}(e_1)) -$ **Strong *cfsg***

From Fig.6,

$S_E(C_1,C_2)(e_1) = Min\{S_V(C_1)(e_1), S_V(C_2)(e_1)\}$

$= Min\{0.3e^{j0.7\pi}, 0.8e^{j0.6\pi}\}$

$= Min\{0.3,0.8\}e^{jMin\{0.7,0.6\}}$

$= 0.3e^{j0.6\pi}$

Similarly,

$S_E(C_3,C_4)(e_1) = 0.3e^{j0.4\pi}$, $S_E(C_1,C_3)(e_1) = 0.3e^{j0.4\pi}$,

$S_E(C_2,C_4)(e_1) = 0.5e^{j0.6\pi}$.

Hence Fig. 3.4 is Strong *cfsg*.

Definition 3.6. Let $(S_V(\tilde{a}), S_E(\tilde{a}))$ be the vertex and the edge set of a *cfsg* over Ĝ. Then the order of a *cfsg* is denoted by O(Ĝ) is defined as:

$$O(\hat{G}) = \left(\sum_{\dot{u}_i \in V} p_S(\dot{u}_i) e^{j \sum_{\dot{u}_i \in V} \varpi_S(\dot{u}_i)} \right)$$

The size of a *cfsg* over Ĝ is defined by,

$$S(\hat{G}) = \left(\sum_{\dot{u}_i \dot{v}_i \in E} \mu_E(\tilde{a})(\dot{u}\dot{v}) e^{j \sum_{\dot{u}_i \dot{v}_i \in E} \alpha_E(\tilde{a})(\dot{u}\dot{v})} \right)$$

Definition 3.7. Let $(S_V(\tilde{a}), S_E(\tilde{a}))$ be the vertex and the edge set of a *cfsg* over Ĝ. The degree of an edge $\dot{u}_i\dot{u}_j \in S_E(\tilde{a})$ is as defined

$d_{\hat{G}}(\dot{u}_i\dot{u}_j) = d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i\dot{u}_j)$,

Where,

$$d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i\dot{u}_j) = d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i) + d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_j) - 2\mu_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_j)$$

$$= \sum_{\substack{\dot{u}_i\dot{u}_j \in S_E(\tilde{a}) \\ k \neq j}} \mu_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_k) e^{j \sum_{\substack{\dot{u}_i\dot{u}_j \in S_E(\tilde{a}) \\ k \neq j}} \alpha_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_k)}$$

$$+ \sum_{\substack{\dot{u}_j\dot{u}_k \in S_E(\tilde{a}) \\ k \neq i}} \mu_{S_E(\tilde{a})}(\dot{u}_j\dot{u}_k) e^{j \sum_{\substack{\dot{u}_j\dot{u}_k \in S_E(\tilde{a}) \\ k \neq i}} \alpha_{S_E(\tilde{a})}(\dot{u}_j\dot{u}_k)}$$

Definition 3.8. A *cfsg* $C_{fs}(\widetilde{G}(\tilde{a}))$ is said to be complete, if $|S_E(\tilde{a})(\dot{u}\dot{v})| = Min\{|S_V(\tilde{a})(\dot{u})|, |S_V(\tilde{a})(\dot{v})|\}, \forall \tilde{a} \in \rho_A$ and $\dot{u}, \dot{v} \in V$. Fig. 7 illustrated for represent the complete *cfsg.*

Definition 3.9. A *cfsg* $C_{fs}(\widetilde{G}(\tilde{a}))$ is said to be empty, if $|S_E(\tilde{a})(\dot{u}\dot{v})| = 0, \forall \tilde{a} \in \rho_A$ and $\dot{u}, \dot{v} \in V$.

Definition 3.10. A *cfsg* $C_{fs}(\widetilde{G}(\tilde{a}))$ on a given set V and $\dot{v}_0, \dot{v}_n$ be two given vertices such that $n \in \mathrm{N}$, then a distinct sequence of vertices P: $\dot{v}_0, \dot{v}_1, \dot{v}_2, \ldots, \dot{v}_n$ in $C_{fs}(\widetilde{G}(\tilde{a}))$ is called a path of length n from $\dot{v}_0$ to $\dot{v}_n$.

Definition 3.11. A *cfsg* $C_{fs}(\widetilde{G}(\tilde{a}))$ on a given set V and $\dot{v}_i$, $\dot{v}_j$ be two given vertices such that $i > j$ and $i, j \in \mathrm{N}$, then Max $\{|S_E(\tilde{a})(\dot{u}\dot{v})|\}$ in $C_{fs}(\widetilde{G}(\tilde{a}))$ is called the strength between $\dot{v}_i$ and $\dot{v}_j$.

Definition 3.12. A *cfsg* $C_{fs}(\widetilde{G}(\tilde{a}))$ on a given set V, then an edge $\dot{u}\dot{v}$ in $C_{fs}(\widetilde{G}(\tilde{a}))$ is called the bridge, if the strengths of each path P from $\dot{u}$ to $\dot{v}$, not involving $\dot{u}\dot{v}$, were less than $|S_E(\tilde{a})(\dot{u}\dot{v})|$, $\forall \tilde{a} \in \rho_A$ and $\dot{u}, \dot{v} \in V$.

From Fig.7 , C_2C_4 is the bridge in the path- $C_1C_2C_3C_4$.

Definition 3.13. The complex fuzzy soft sub-graph $C_{fs}(\widetilde{H}) = (S_{\widetilde{V}}(\tilde{a}), S_{\widetilde{E}}(\tilde{a}))$ is said to be spanning sub graph of $C_{fs}(\widetilde{G}(\tilde{a}))$, if $S_{V_e}(x_i) = S_{\widetilde{V}_e}(x_i)$,for all $x_i \in V$, $e \in A$.

Fig. 8 illustrated for representing spanning sub graph of *cfsg*,

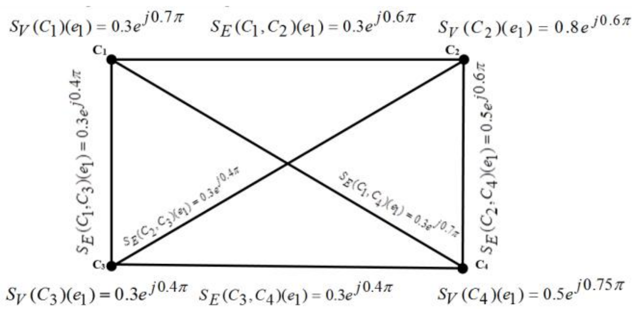


Fig. 7. Complete *cfsg*

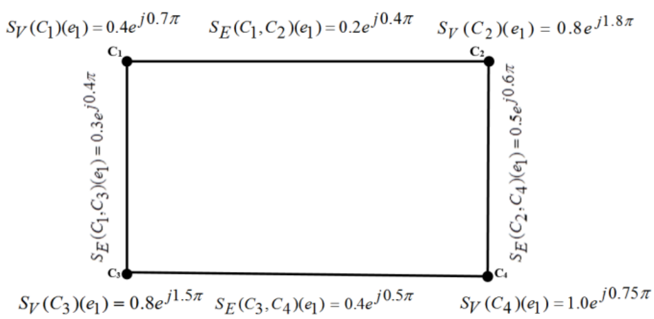


Fig. 8 spanning *cfsg* of Fig. 6

Theorem 3.1. Let $\widehat{G}=(V,E)$ be a simple graph. Let ρ_A be a non-empty set of parameters. Then, $(S_V(\tilde{a}),S_E(\tilde{a}))$ for all $\tilde{a}\in\rho_A$ is *cfsg* on a cycle graph $\hat{C}$. Then

$$\sum_{\dot{u}_i\in V} d_{\widehat{G}}(\dot{u}_i)=\sum_{\dot{u}_i\dot{u}_i\in E} d_{\widehat{G}}(\dot{u}_i,\dot{u}_{i+1})$$

Proof. Let $(S_V(\tilde{a}),S_E(\tilde{a}))$ be a *cfsg* and let $\hat{C}$ be a cycle $\dot{u}_1\dot{u}_2..\dot{u}_n\dot{u}_1$. Then $\sum_{i=1}^{n} d_{\widehat{G}}(\dot{u}_i,\dot{u}_{i+1})=\sum_{i=1}^{n} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i,\dot{u}_{i+1})$

Consider,

$$\sum_{i=1}^{n} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i,\dot{u}_{i+1})=d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_1\dot{u}_2)+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_2\dot{u}_3)+\ldots.+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_n\dot{u}_1)$$

$$=d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_1)+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_2)-2d_{S_E(\tilde{a})}(\dot{u}_1\dot{u}_2)e^{2j\alpha_{S_E}(\tilde{a})(\dot{u}_1\dot{u}_2)}$$

$$+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_2)+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_3)-2d_{S_E(\tilde{a})}(\dot{u}_2\dot{u}_3)e^{2j\alpha_{S_E}(\tilde{a})(\dot{u}_2\dot{u}_3)}$$

$$+\ldots$$

$$+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_n)+d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_1)-2d_{S_E(\tilde{a})}(\dot{u}_n\dot{u}_1)e^{2j\alpha_{S_E}(\tilde{a})(\dot{u}_n\dot{u}_1)}$$

$$=2\sum_{\dot{u}_i\in E} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i)-2\sum_{i=1}^{n}\mu_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_{i+1})e^{2i\sum_{i=1}^{n}\alpha_{S_E}(\tilde{a})(\dot{u}_i\dot{u}_{i+1})}$$

$$=\sum_{\dot{u}_i\in E} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i)+\sum_{\dot{u}_i\in E} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i)$$

$$-2\sum_{i=1}^{n}\mu_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_{i+1})e^{2i\sum_{i=1}^{n}\alpha_{S_E}(\tilde{a})(\dot{u}_i\dot{u}_{i+1})}$$

$$=\sum_{\dot{u}_i\in E} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i)+2\sum_{i=1}^{n}\mu_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_{i+1})e^{2i\sum_{i=1}^{n}\alpha_{S_E}(\tilde{a})(\dot{u}_i\dot{u}_{i+1})}$$

$$-2\sum_{i=1}^{n}\mu_{S_E(\tilde{a})}(\dot{u}_i\dot{u}_{i+1})e^{2i\sum_{i=1}^{n}\alpha_{S_E}(\tilde{a})(\dot{u}_i\dot{u}_{i+1})}$$

$$\sum_{i=1}^{n} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i,\dot{u}_{i+1})=\sum_{\dot{u}_i\in E} d_{\mu e^{j\alpha}(\tilde{a})}(\dot{u}_i)$$

Definition 3.14. Let $C_{fs}(\tilde{G})=(S_V(\tilde{a})(\tilde{u}),S_E(\tilde{a})(\tilde{u}))$, for all $\tilde{a}\in\rho_A,\tilde{u}\in V$ is a complex fuzzy soft graph. The adjacency matrix *A* of a *cfsg*, in order $n\times n$ can be written as two adjacent matrices, one is membership values that are amplitude function of $S_E(\tilde{a})(\tilde{u})$ and the other is adjacency matrix containing the membership values of phase function of $S_E(\tilde{a})(\tilde{u})$.

i.e., A(*cfsg*)= $A\left(\mu_{S_E}(\tilde{a})(\dot{x})\right),A\left(\frac{1}{2\pi}\alpha_{S_E}(\tilde{a})(\dot{x})\right)$,

Definition 3.15. The eigen values of an adjacency matrix of A(*cfsg*) is defined as ($\lambda_{A_i},\lambda_{P_j}$), where $\lambda_{A_i},\lambda_{P_j}$ is the set of eigen values of $A(\mu_{S_E}(\tilde{a})(\dot{x}))$, $A(\frac{1}{2\pi}\alpha_{S_E}(\tilde{a})(\dot{x}))$ respectively.

Definition 3.16. The energy of an complex fuzzy soft graph $C_{fs}(\tilde{G})$ is defined as $\left(\sum_{i=1}^{n}|\lambda_{A_i}|,\sum_{j=1}^{n}|\lambda_{P_j}|\right)$, where $\sum_{i=1}^{n}|\lambda_{A_i}|$ is defined as an energy of the Amplitude matrix and $\sum_{j=1}^{n}|\lambda_{P_j}|$ is an energy of the Phase term matrix.

Example: 3.6

For a complex fuzzy soft graph $C_{fs}(\tilde{G})$ in Fig.9, the adjacency matrices of $C_{fs}(\tilde{G})$ are

$$A\left(\mu_{S_E}(\tilde{a})(\dot{x})\right)=\begin{pmatrix}0&0.3&0.3&0\\0.3&0&0&0.5\\0.3&0&0&0.3\\0&0.5&0.3&0\end{pmatrix}\text{and}$$

$$A\left(\frac{1}{2\pi}\alpha_{S_E}(\tilde{a})(\dot{x})\right)==\begin{pmatrix}0&0.15&0.2&0\\0.15&0&0&0.3\\0.2&0&0&0.2\\0&0.3&0.2&0\end{pmatrix}$$

Eigen values of $A\left(\mu_{S_E}(\tilde{a})(\dot{x})\right)$={-0.716228, 0.716228, -0.0837722,- 0.0837722}

Eigen values of $A\left(\frac{1}{2\pi}\alpha_{S_E}(\tilde{a})(\dot{x})\right)$= {-0.43325, 0.43325, -0.0692441, 0.0692441}; Energy of $A\left(\mu_{S_E}(\tilde{a})(\dot{x})\right)$= 1.6

Energy of $A\left(\frac{1}{2\pi}\alpha_{S_E}(\tilde{a})(\dot{x})\right)$= 1.005.

IV. Operations and Applications of Complex Fuzzy Soft Graph

Definition 4.1. Let $C_{fs}(\tilde{G})=(S_V(\tilde{a})(\dot{u}),S_E(\tilde{a})(\dot{u}))$, for all $\tilde{a}\in\rho_A,\dot{u}\in V$ is a *cfsg*, then the union of two *cfsg* $G^1_{\rho_{A_1},\Gamma_1}=\left((\rho_{A_1},S^1_{V_e}),(\rho_{A_1},S^1_{E_e})\right)$, $G^2_{\rho_{A_2},\Gamma_2}=\left((\rho_{A_2},S^2_{V_e}),(\rho_{A_2},S^2_{E_e})\right)$ is defined by $G^2_{\varsigma,\Gamma_3}=\left((\varsigma,S^3_{V_e}),(\varsigma,S^3_{E_e})\right)$(say), where $\Gamma_1,\Gamma_2\subset V$, $\rho_{A_1},\rho_{A_2}\in\rho_A$, $\varsigma=\rho_{A_1}\cup\rho_{A_2}$, $\Gamma_3=\Gamma_1\cup\Gamma_2$ and

$S^3_{V_e}(x_i)=S^1_{V_e}(x_i)$ for all $V_e\in\rho_{A_1}\setminus\rho_{A_2}$ and $x_i\in\Gamma_1\setminus\Gamma_2$,

$=0$ for all $V_e\in\rho_{A_1}\setminus\rho_{A_2}$ and $x_i\in\Gamma_2\setminus\Gamma_1$,

$=$ $S^1_{V_e}(x_i)$ for all $V_e\in\rho_{A_1}\setminus\rho_{A_2}$ and $x_i\in\Gamma_1\cap\Gamma_2$,

$=$ $S^2_{V_e}(x_i)$ for all $V_e\in\rho_{A_2}\setminus\rho_{A_1}$ and $x_i\in\Gamma_2\setminus\Gamma_1$,

$=0$ for all $V_e\in\rho_{A_2}\setminus\rho_{A_1}$ and $x_i\in\Gamma_1\setminus\Gamma_2$,

$=S^2_{V_e}(x_i)$ for all $V_e\in\rho_{A_2}\setminus\rho_{A_1}$ and $x_i\in\Gamma_1\cap\Gamma_2$, $=$ $Max\left\{S^1_{V_e}(x_i),S^2_{V_e}(x_i)\right\}$ for all $V_e\in\rho_{A_1}\cap\rho_{A_2}$ and $x_i\in\Gamma_1\cap\Gamma_2$,

$=S^1_{V_e}(x_i),\forall V_e\in\rho_{A_1}\cap\rho_{A_2}$ & $x_i\in\Gamma_1\setminus\Gamma_2$,

$=S^2_{V_e}(x_i),\forall V_e\in\rho_{A_1}\cap\rho_{A_2}$ & $x_i\in\Gamma_2\setminus\Gamma_1$, and

$S^3_{E_e}(x_i,x_j) = S^1_{E_e}(x_i,x_j)$ if $E_e \in \rho_{A_1} \setminus \rho_{A_2}$ and $(x_i,x_j) \in (\Gamma_1\times\Gamma_1)\setminus(\Gamma_2\times\Gamma_2)$,

$= 0$ if $E_e \in \rho_{A_1} \setminus \rho_{A_2}$ and $(x_i,x_j) \in (\Gamma_2\times\Gamma_2)\setminus(\Gamma_1\times\Gamma_1)$,

$= S^1_{E_e}(x_i,x_j)$ for all $E_e \in \rho_{A_1} \setminus \rho_{A_2}$ and $(x_i,x_j) \in (\Gamma_1\times\Gamma_1)\cap(\Gamma_2\times\Gamma_2)$,

$= S^2_{E_e}(x_i,x_j)$ for all $E_e \in \rho_{A2} \setminus \rho_{A_1}$ and $(x_i,x_j) \in (\Gamma_2\times\Gamma_2)\setminus(\Gamma_1\times\Gamma_1)$,

$= 0$ if $E_e \in \rho_{A_2} \setminus \rho_{A_1}$ and $(x_i,x_j) \in (\Gamma_1\times\Gamma_1)\setminus(\Gamma_2\times\Gamma_2)$

$= S^2_{E_e}(x_i,x_j)$ for all $E_e \in \rho_{A2} \setminus \rho_{A_1}$ and $(x_i,x_j) \in (\Gamma_1\times\Gamma_1)\setminus(\Gamma_2\times\Gamma_2)$,

$= Max\left\{S^1_{E_e}(x_i), S^2_{E_e}(x_j)\right\}$ for all $E_e \in \rho_{A_1} \cap \rho_{A_2}$ and $(x_i,x_j) \in (\Gamma_1\times\Gamma_1)\cap(\Gamma_2\times\Gamma_2)$,

$= S^1_{E_e}(x_i)$ for all $E_e \in \rho_{A_1} \cap \rho_{A_2}$ and $(x_i,x_j) \in (\Gamma_1\times\Gamma_1)\setminus(\Gamma_2\times\Gamma_2)$,

$= S^2_{E_e}(x_i)$ for all $E_e \in \rho_{A_1} \cap \rho_{A_2}$ and $(x_i,x_j) \in (\Gamma_2\times\Gamma_2)\setminus(\Gamma_1\times\Gamma_1)$.

Definition 4.2 Let $C_{fs}(\tilde{G}) = (S_V(\tilde{a})(\dot{u}), S_E(\tilde{a})(\dot{u}))$, for all $\tilde{a} \in \rho_A, \dot{u} \in V$ and let $\Gamma_1,\Gamma_2 \subset V$, $\rho_{A_1},\rho_{A_2} \in \rho_A$. The intersection of two complex fuzzy soft graphs $G^1_{\rho_{A_1},\Gamma_1} = \left((\rho_{A_1}, S^1_{Ve}),(\rho_{A_1}, S^1_{Ee})\right)$, $G^2_{\rho_{A_2},\Gamma_2} = \left((\rho_{A_2}, S^2_{V_e}),(\rho_{A_2}, S^2_{E_e})\right)$ is defined by $G^2_{\varsigma,\Gamma_3} = \left((\varsigma, S^3_{V_e}),(\varsigma, S^3_{E_e})\right)$(say), where $\varsigma = \rho_{A_1} \cap \rho_{A_2}, \Gamma_3 = \Gamma_1 \cap \Gamma_2$ and

$S^3_{V_e}(x_i) = Min\left\{S^1_{V_e}(x_i), S^2_{V_e}(x_j)\right\} \forall\, x_i \in \Gamma_3, e \in \varsigma$ and

$S^3_{Ee}(x_i,x_j)$

$= Min\left\{S^1_{E_e}(x_i,x_j), S^2_{E_e}(x_i,x_j)\right\}, \forall x_i,x_j \in \Gamma_3, e \in \varsigma.$

Remark 4.1: Union of two strong complex fuzzy soft graph need not be a strong complex fuzzy soft graph.

Remark 4.2: Intersection of two strong complex fuzzy soft graph is a complex fuzzy soft graph.

Definition 4.3 Let $C_{fs}(\tilde{G}) = (S_V(a)(\tilde{u}), S_E(a)(\tilde{u}))$, for all $\tilde{a} \in \rho_A, \tilde{u} \in V$ is a complex fuzzy soft graph.
Then the complement of $C_{fs}(\tilde{G})$ is defined by $\overline{C}_{fs}(\tilde{G}) = (S_{\overline{V}}(\tilde{a})(\tilde{u}), S_{\overline{E}}(\tilde{a})(\tilde{u}))$, such that

(i) $|V| = |\overline{V}|$

(ii) $\mu_{S_V}(\tilde{a})(\dot{x}) e^{j\alpha_{S_V}(\tilde{a})(\dot{x})} = \mu_{S_{\overline{V}}}(\tilde{a})(\dot{x}) e^{j\alpha_{S_{\overline{V}}}(\tilde{a})(\dot{x})}$ $\forall \dot{x} \in V$,

(iii) $\mu_{S_{\overline{E}}}(a)(xy) e^{j\alpha_{S_E}(a)(xy)}$

$$= \begin{cases} 0; if \quad \mu_{S_E}(a)(\dot{x}\dot{y}) e^{j\alpha_{S_E}(\tilde{a})(\dot{x}\dot{y})} \neq 0 \\ Min\{\mu_{S_V}(\tilde{a})(\dot{x}), \mu_{S_V}(\tilde{a})(\dot{y})\} e^{jMin\{\alpha_{S_V}(\tilde{a})(\dot{x}), \alpha_{S_V}(\tilde{a})(\dot{y})\}} \\ ;\text{if } \mu_{S_E}(\tilde{a})(\dot{x}\dot{y}) e^{j\alpha_{S_E}(\tilde{a})(\dot{x}\dot{y})} = 0, \forall \dot{x},\dot{y} \in V \end{cases}$$

Example: 4.1.

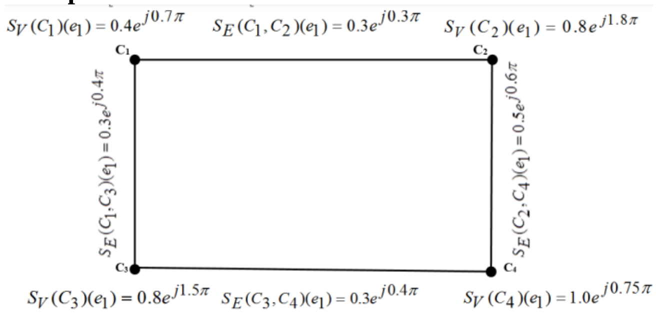


Fig. 9 *cfsg-1*

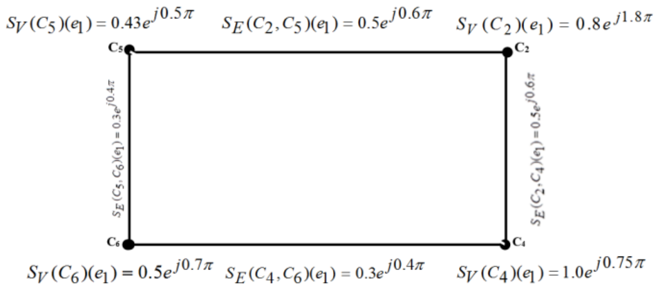
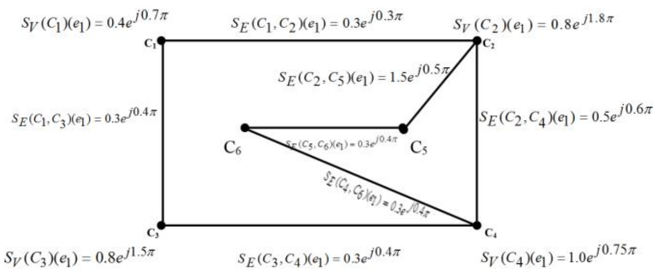


Fig. 10 *cfsg-2*



Union of *cfsg*
Fig. 11 (*cfsg-1* $\bigcup$ *cfsg-2*)

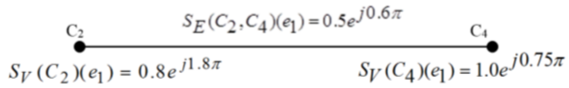


Fig. 12 Intersection of *cfsg*
(*cfsg-1* $\bigcap$ *cfsg-2*)

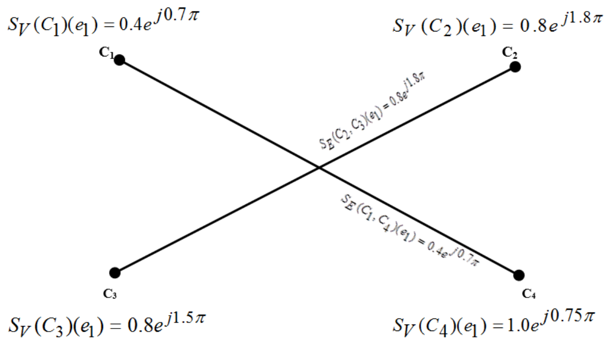


Fig. 13 Complement of *cfsg-1*

Cartesian product of *cfsg*

The cartesian product of complex fuzzy soft graphs $C_{fs}(\tilde{G}_1), C_{fs}(\tilde{G}_2)$ is defined as $C_{fs}(\tilde{G}_1) \times C_{fs}(\tilde{G}_2) = C_{fs}(\tilde{G}) = (S_V(a)(\tilde{u}), S_E(a)(\tilde{u}))$, for all $\tilde{a} \in \rho_A, \tilde{u} \in V$ is a complex fuzzy soft graph.

$(i) \left(S_{V_1}(a) \times S_{V_2}(a) \right)(\tilde{u}_1, \tilde{u}_2) = \min\{r_1(\tilde{u}_1), r_2(\tilde{u}_2)\} \times e^{\min(\tau_1(\tilde{u}_1), \tau_2(\tilde{u}_2))}$

, *for* $\tilde{u}_1, \tilde{u}_2 \in V$

$(ii) \left(S_{E_1}(a) \times S_{E_2}(a) \right)((u, \tilde{u}_2), (u, \tilde{v}_2)) = \min\{r_1(u), R_2(u)\} \times e^{\min(\tau_1(u), \varpi_2(u))}$,

where $R_2(u) \le \min\{r_2(\tilde{u}_2), r_2(\tilde{v}_2)\}$;

$\varpi_2(u) \le \min\{\tau_2(\tilde{u}_2), \tau_2(\tilde{v}_2)\}$ for all $u \in S_{V_1}$

and $(\tilde{u}_2, \tilde{v}_2) \in S_{E_2}$

$(iii) \left(S_{E_1}(a) \times S_{E_2}(a) \right)((\tilde{u}_1, u), (\tilde{v}_1, u)) = \min\{R_1(u), r_2(u)\} \times e^{\min(\varpi_1(u), \tau_2(u))}$, where

$R_1(u) \le \min\{r_1(\tilde{u}_1), r_2(\tilde{v}_1)\}; \varpi_2(u) \le \min\{\tau_1(\tilde{u}_1), \tau_1(\tilde{v}_1)\}$

for all $u \in S_{V_2}$

and $(\tilde{u}_1, \tilde{v}_1) \in S_{E_1}$.

Applications of *cfsg* in Decision Making Problems

As mentioned in the example 2.1, Let $\Omega = \{C_1, C_2, C_3, C_4\}$ be represent the countries India, Russia, UK and USA respectively. Let e_1 and e_4 be the unemployment rate and population growth of the given countries. By using the *cfsg*, we compare the unemployment rate and population growth of between the countries.

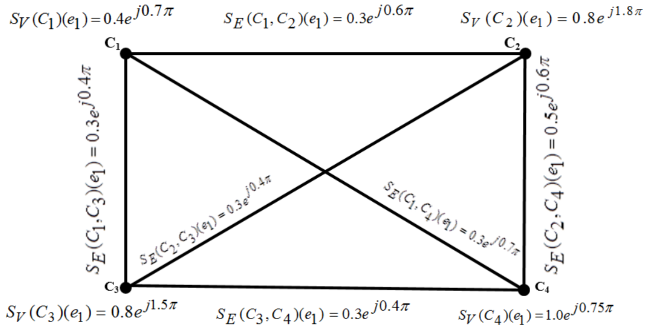


Fig. 14 *cfsg* – Representation of Unemployment Rate

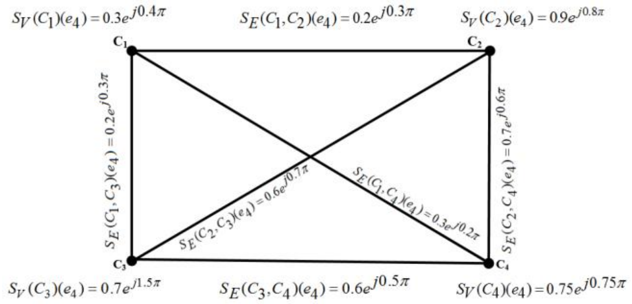


Fig. 15 *cfsg* – Representation of Population growth

The complex fuzzy soft set theory has numerous applications in dealing with uncertainties, the periodic occurrences, and the parameterized sets from everyday concerns. For example, we can identify the growth rates of the countries with respect to the parameters using the figure 14 and 15. This membership grade is helpful to find the growth of each country. In Quantum theory, the graph's energy is used by connecting the graph edges to the electron energy of certain kinds of molecule which motivates the further study. Energy of complex fuzzy soft graph is more efficient since it yields accurate results while dealing with periodicity, uncertainties in the parameterized sets. In the near future, we plan to derive the Eurlerian and Hamiltonian cycles which are identified from this complex fuzzy soft graph. Also, we plan to develop a Hamming index of the product of the two complex fuzzy soft graphs as mentioned in [18].

V. Conclusion

As part of this study, we explored few graph-theoretic concepts and integrated a hypothesis on energy of complex fuzzy soft graph. In specific, we developed a new concept called complex fuzzy soft graph, as well as its path, strength and the bridge. The energy of the adjacency matrix of the complex fuzzy soft graph is examined together with the illustration of various operations such as union, intersection, complement and the Cartesian product of a complex fuzzy soft graph. As a result, the complex fuzzy soft graph appears to be promising and opening up a wide range of opportunities for any further study.

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