Adjacent Vertex Reducible Edge Labeling Algorithm for Several Compound Graphs

Liangjing Sun, Jingwen Li, Cong Huang, Xin Gao

***Abstract*—Let $G(p,q)$ be a simple graph, where p is the number of vertices and q is the number of edges if there exists a one-to-one mapping f: $E(G)\rightarrow\{1,2,\ldots,|E|\}$, so that for any two vertices $uv\in E(G)$, if $d(u)=d(v)$, then $S(u)=S(v)$, where $S(u)=\sum_{uw\in E(G)}f(uw)$ and $d(u)$ represents the degree of vertex u, f is called the Adjacent Vertex Reducible Edge Labeling (AVREL) of G. Building on the current graph labeling algorithm, a heuristic search algorithm is designed, and this algorithm is used to label random graphs with less than 12 vertices and obtain the result set of adjacent vertex reducible edge labeling. Based on the analysis of the result set and combining it with the known theorem, the adjacent vertex reducible edge labeling law of other compound graphs is obtained, and the related proof is given.**

***Index Terms*—Adjacent Vertex Reducible Edge Labeling, special graphs, compound graphs, algorithm**

I. Introduction

In the mid-1960s, the problem of graph labeling first emerged, becoming one of the most focused research topics in the field of graph theory. In 1967, Rosa et al.[1] put forward a beautiful conjecture that "every tree is beautiful.". In 1997, Burris et al.[2] put forward the idea of vertex-distincting edge coloring along with associated conjectures. In 2002, MacDougall et al.[3] proposed the vertex-magic total labeling, which has since attracted increasing attention and research from scholars. In 2007, the literature[4] made new progress in the study of graceful labeling, super-magic total labeling, and harmonic labeling, and successfully proved the related conjecture. In 2009, Zhang Zhongfu et al.[5] expanded on the idea of distinguishing coloring by introducing the concept of reducible coloring. Many scholars have since studied this concept, leading to a series of research finding[6][7]. In 2023, the literature[8] expanded on the theory of adjacent vertex-reducible edge labeling, deriving theorems and conjectures for path graphs, cycle graphs, star graphs, fan graphs, wheel graphs, tree graphs, and their compound graphs. The correctness of these theorems and conjectures was verified using mathematical proofs and computational algorithms.

In frequency allocation, to reduce interference between base stations and users, it is necessary to ensure that different frequencies are assigned to different base stations. This issue can be expressed as a graph theory problem by abstracting the network topology into an undirected graph, where base stations are represented as vertices, and channels between them as edges. The issue of channel frequency allocation is then transformed into the problem of labeling the edges associated with each vertex in the graph, with the condition that each edge receives a different labeling. Building on the research of various scholars, this paper presents an algorithm for adjacent vertex reducible edge labeling, based on concepts such as vertex sum reducible edge coloring[9][10], adjacent vertex distinguishable edge coloring[11], and vertex magic total labeling[12]. This algorithm aims to address the adjacent vertex reducible edge labeling problem for special and compound graphs, starting with reducible coloring and incorporating vertex magic total labeling. The algorithm's labeling results are analyzed, corresponding theorems are summarized, and proofs are offered.

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II. Preliminary Knowledge

Definition 1: Let $G(V,E)$ be a simple graph. If there exists one-to-one mapping f: $E(G)\rightarrow\{1,2,\ldots,|E|\}$, so that for any two vertices $uv\in E(G)$, if $d(u)=d(v)$, then $S(u)=S(v)$, where $S(u)=\sum_{uw\in E(G)}f(uw)$ and $d(u)$ represents the degree of vertex u, then f is called the Adjacent Vertex Reducible Edge Labeling (AVREL) of G. The graph G is termed AVREL graph; otherwise, it is referred to as non-AVREL graph.

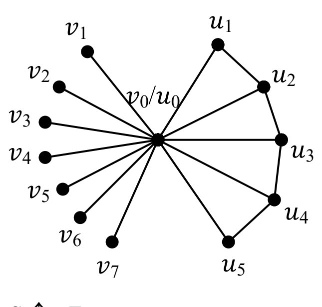


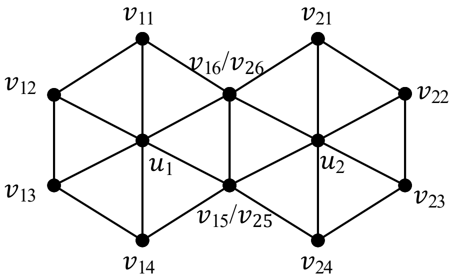
Fig. 1. Example of $S_7 \uparrow_{aa} F_5$.

Definition 2: Let G_1 and G_2 be simple connected graphs belonging to the path graph (P_n), cycle graph (C_n), star graph

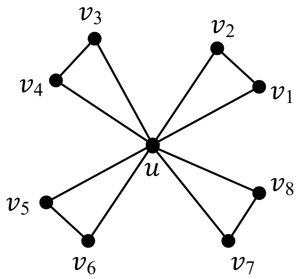
(S_n), fan graph (F_n), wheel graph (W_n) and complete graph (K_n) where symbol a represents center nodes of star, fan and wheel graphs, degree-1 vertices of path graph and any vertices. Symbol b represents non-center nodes of star and wheel graphs, degree-2 vertices of fan graphs, and degree-2 vertices of path graphs. Symbol d represents nodes at a distance of 2 from the previous node without passing through the center node. The compound graph $G_1 \uparrow_{aa} G_2$ refers to connecting the a node of G_1 to the a node of G_2, as shown in Fig. 1.

Definition 3: Let G_n be a simple graph. The compound graph $G_n \downarrow G_n (n \geq 3)$ refers to the G_n formed by sharing edges with itself, as shown in Fig. 2(a).

Definition 4: The friendship graph T_n is the compound graph formed by n copies of the cycle graph C_3 with common vertices, as shown in Fig. 2(b).



(a) $W_6 \downarrow W_6$



(b) T_4

Fig. 2. Example of $W_6 \downarrow W_6$ and T_n.

Definition 5: Let G_1 and G_2 be two simple graphs with the number of vertices n_1 and n_2 and the number of edges m_1 and m_2, respectively. We define the corona graph of G_1 and G_2 as a graph that connects each vertex of G_1 to each vertex of a copy of G_2, represented as $G_1 \circ G_2$. The vertex number of $G_1 \circ G_2$ is $n_1(1+n_2)$, and the edge number is $m_1 + n_1 n_2 + n_1 m_2$. For example, $G_1 = C_4$, $G_2 = P_2$, and the corona graph of vertex $C_4 \circ P_2$ of graph G_1 and G_2 are shown in Fig. 3.

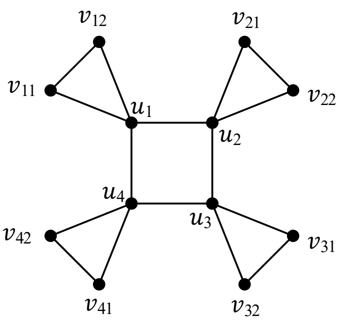


Fig. 3. Example of $C_4 \circ P_2$.

Definition 6: Let G_1 and G_2 be two simple graphs with the number of vertices n_1 and n_2, and the number of edges m_1 and m_2. The generalized corona graph of G_1 and G_2 is defined as follows: for each b node of G_1 (where b represents non-center nodes of star, wheel, friendship graphs, degree-2 vertices of fan graphs, and degree-2 vertices of path graphs), connect it to copy of the a node of G_2 (where a represents center nodes of star, fan, wheel, friendship graphs, degree-1 vertices of path graphs, and any vertices of cycle graphs). This results in a compound graph denoted as $G_1^b \circ G_2^a$. When several nodes of G_1 are connected with a nodes of G_2, the generalized corona graph is abbreviated as $G_1^{abc} \circ G_2^a$. For example, $G_1 = W_8$ and $G_2 = S_3$, the generalized corona graph as $W_8^b \circ S_3^a$ of graphs G_1 and G_2, as shown in Fig. 4.

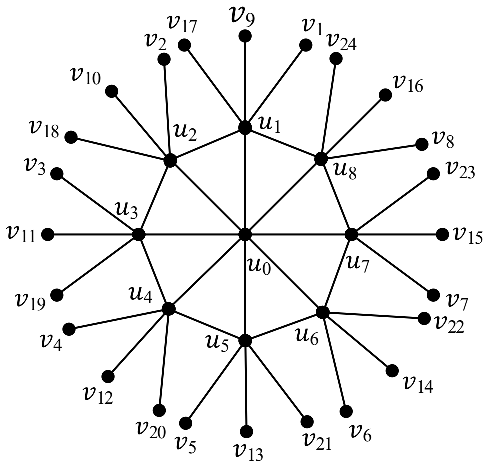


Fig. 4. Example of $W_8^b \circ S_3^a$.

III. AVREL ALGORITHM

A. Preparation phase

Based on the definition of AVREL, a graph classification function is defined using the graph's degree sequence. The graph is divided into two classes: the graph with the same degree sequence of neighboring vertex and the other with a different degree sequence of neighboring vertex; the adjacent vertex degree sequence has the same labeling, and the labeling number is continuous. A balance function is then defined according to the labeling conditions of AVREL, and this balance function is used to determine whether the labeling meets the necessary conditions.

The basic principles of the AVREL algorithm involve utilizing permutations to generate a solution space and recursively searching the solution space to obtain labeling that satisfyingly restrictive condition.

(1) Generate the solution space based on the preparatory work and recursively search through it.

(2) Use a balancing function to filter out graph sets that satisfy AVREL conditions and output the labeling results as an adjacency matrix.

(3) We classify the graph sets as AVREL if they satisfy the condition, and as non-AVREL if they don't.

We are setting the labeling balance constraint condition based on the principles of the AVREL algorithm:

(1) $d(u)=d(v)$, for $uv \in E(G)$, where $S(u)=\sum_{uw \in E(G)} f(uw)=S(v)=\sum_{vw \in E(G)} f(vw)=k$, and k is a constant.

(2) A one-to-one mapping f: $E(G)\rightarrow\{1,2,\ldots,|E|\}$ exists, and the labeling numbers are consecutive.

B. Pseudocode for AVREL Algorithm

```
Input    The adjacency matrix of the graph G(p,q)
Output   AVREL graph or non-AVREL graph
begin
    Read the AdjustMatrix, the adjacency matrix of
    graph G, and initialize the LabelMatrix for labeling.
    Input the number of vertices, the number of edges,
    the degree sequence, the classification function, and
    the solution space
    while(φ(p,q))!=null)
        search φ(p,q)
        if G. isBalance ← true
            LabelMatrix ← AdjustMatrix
            break
        end if
    end while
        if G. isBalance ← false
            Output G is not AVREL
        end if
        else
            Output LabelMatrix
        end else
end
```

C. Analysis of Experimental Results

Table I lists the number of AVREL graphs in the single circle and double circle graphs, ranging from 4 to 12 vertices. From Table I, it can be seen that the larger the vertices are, the proportion of single circle graphs that satisfy AVREL graphs increases gradually, and when the vertices are 8, the proportion reaches the maximum, and tends to be smooth after that; double circle graphs are exactly the opposite, and when the number of vertices is 8, the proportion of double circle graphs that satisfy AVREL graphs decreases and tends to be smooth after that. After that, it tends to stabilise.

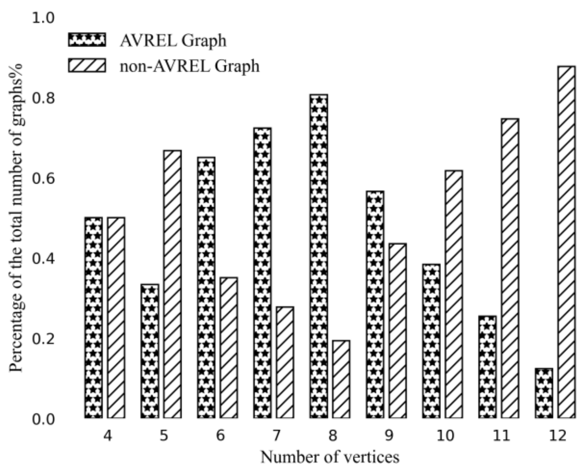


Fig. 5. Percentage of AVREL and non-AVREL graphs in all random graphs within 4-12 vertices.

In Fig. 5, we conducted experiments on all random graphs, ranging from 4 to 12 vertices. All random graphs within the range of 4 to 12 vertices exhibit a proportion of AVREL and non-AVREL graphs. We can observe that the proportion of the AVREL graph gradually increases, reaches its maximum when there are 8 vertices, and then gradually decreases.

TABLE I
STATISTICS FOR AVREL GRAPHS WITH 4 TO 12 VERTICES IN SINGLE AND DOUBLE CIRCLE GRAPHS

(p,q)	Total number of graphs/ piece	AVREL graph number/ piece	(p,q)	Total number of graphs/ piece	AVREL graph number/ piece
(4,4)	2	0	(8,9)	236	104
(4,5)	1	1	(9,9)	240	85
(5,5)	5	1	(9,10)	797	330
(5,6)	5	3	(10,10)	657	226
(6,6)	13	4	(10,11)	1412	568
(6,7)	19	10	(11,11)	1806	599
(7,7)	33	11	(11,12)	2675	1215
(7,8)	67	32	(12,12)	5026	1659
(8,8)	89	31	(12,13)	6121	2510

Fig. 6 and Fig. 7 shows some of the labeling results for AVREL.

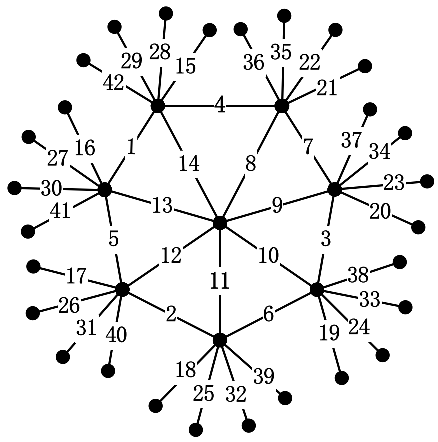


Fig. 6. Labeling results of $G(36,54)$

IV. THEOREMS AND PROOFS

Theorem 1: The generalized corona graph $W_n^b \circ S_m^a\left(n\equiv 1(\bmod\ 2), m\equiv 0(\bmod\ 2), n\geq 3\right)$ is AVREL graph.

Proof: Let the vertex set be $V\left(W_n^b \circ S_m^a\right)=\{v_1,v_2,\cdots,v_{mn}\}\cup\{u_1,u_2,...,u_n\}$ and the edge set be $E\left(W_n^b \circ S_m^a\right)=\{u_nu_1\}\cup\{u_iu_{i+1}\mid 1\leq i\leq n-1\}\cup\left\{u_0u_i,u_iv_i,u_iv_{n+i},u_iv_{(m-2)n+i},u_iv_{(m-1)n+i}\mid 1\leq i\leq n\right\}$ of the generalized corona graph $W_n^b \circ S_m^a$.

When $n\equiv 1(\bmod\ 2), m\equiv 0(\bmod\ 2)$ and $n\geq 3$, the adjacent vertex reducible edge labeling of the generalized corona graph $W_n^b \circ S_m^a$ is:

Let $n=2k+1, m=2l, k,l=1,2,\cdots$

$$f\left(u_0u_i\right)=4k-i+3, i=1,2,\cdots,2k+1$$

$$f\left(u_iu_{i+1}\right)=\begin{cases}\dfrac{i+1}{2} & ,i=1,3,\cdots,2k-1\\[2ex] k+1+\dfrac{i}{2}, & i=2,4,\cdots,2k\end{cases}$$

$$f\left(u_nu_1\right)=k+1$$

$$f\left(u_iv_{n+i}\right)=8k-i+5, i=1,2,\cdots,2k+1$$

$$f\left(u_iv_{(m-2)n+i}\right)=4kl+2l+i\left(l=1,2,\cdots,m/2, i=1,2,\cdots,2k+1\right)$$

$$f\left(u_iv_{(m-1)n+i}\right)=8kl+4l-4k-i-1\left(l=2,3,\cdots,m/2, i=1,2,\cdots,2k+1\right)$$

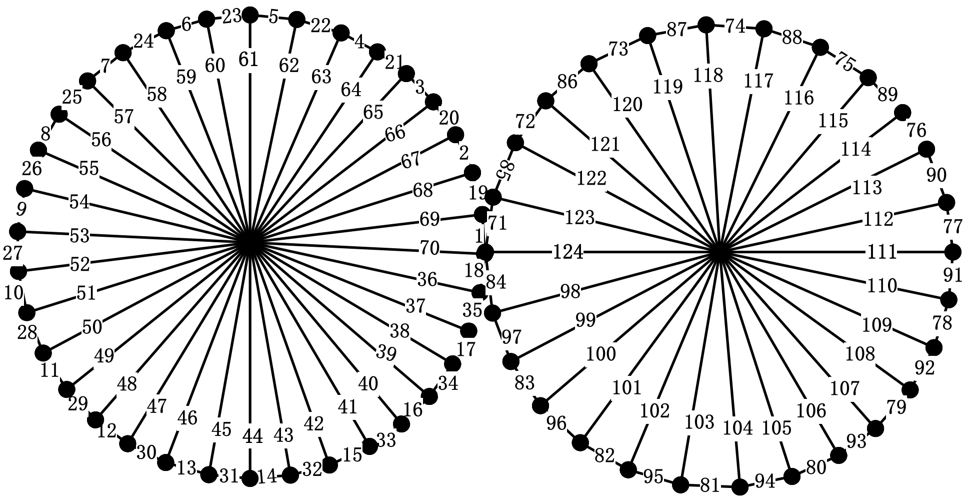


Fig. 7. Labeling results of $G(63,124)$

At this point, the graph contains degree-1 vertices, degree-n vertices, and degree-(m+3) vertices. Since the degree-1 and degree-n vertices are not adjacent, they do not need to be considered. It is only necessary to ensure that all adjacent degree-(m+3) vertices have the same sum of labeling. $\{u_1, u_2, \ldots, u_n\}$ are adjacent degree-(m+3) vertices. When $2 \le i \le 2k$, the sum of their labels:

$$\begin{aligned}
&Sum(u_i) = \sum\nolimits_{uu \in E(u_i)} f(uu) \\
&= \left\{ f(u_{i-1}u_i) + f(u_i u_{i+1}) + f(u_0 u_i) + \cdots + f\left(u_i v_{(m-2)n+i}\right) + f\left(u_i v_{(m-1)n+i}\right) \right\} \\
&\| \left\{ f(u_1 u_2) + f(u_n u_1) + f(u_0 u_1) + \cdots + f\left(u_1 v_{(m-2)n+1}\right) + f\left(u_1 v_{(m-1)n+1}\right) \right\} \\
&\| \left\{ f(u_{n-1}u_n) + f(u_n u_1) + f(u_0 u_n) + \cdots + f\left(u_n v_{(m-1)n}\right) + f(u_n v_{mn}) \right\} \\
&= \left\{ \frac{i}{2} + \frac{2k+i+2}{2} + (4k-i+3) + \cdots + (4kl+2l+i) + (8kl+4l-4k-i-1) \right\} \\
&\| \left\{ 1 + (k+1) + (4k+2) + \cdots + (4kl+2l+1) + (8kl+4l-4k-2) \right\} \\
&\| \left\{ (2k+1) + (k+1) + (2k+2) + \cdots + (4kl+2k+2l+1) + (8kl-6k+4l-2) \right\} \\
&= \left\{ 17k+11 + (l-1)(20k+11) + (l-1)(l-2)(6k+3) \right\} \\
&= \left\{ 20kl - 3k + 11l + (l-1)(l-2)(6k+3) \right\}
\end{aligned}$$

The symbol "‖" in the full text is represented as a logical or.

An example of the generalized corona graph $W_n^b \circ S_m^a$ is shown in Fig. 8.

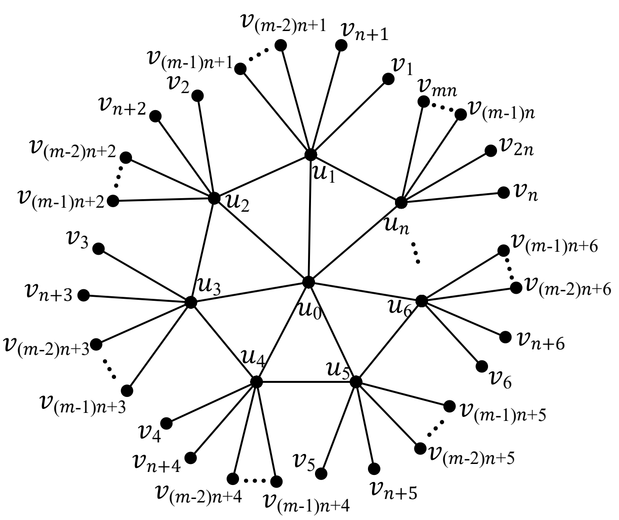


Fig. 8. Generalized corona graph $W_n^b \circ S_m^a$.

According to the AVREL definition, the function $f\left(E\left(W_n^b \circ S_m^a\right)\right) \to \{1, 2, \cdots, 2n+nm\}$ is a one-to-one mapping, and the sum of edge labels for adjacent vertices of the same degree is constant. The proof of Theorem 1 is complete.

Theorem 2: The generalized corona graph $W_n^b \circ P_2^a$ is AVREL graph.

Proof: Let the set of vertex be $V\left(W_n^b \circ P_2^a\right) = \{u_1, u_2, ..., u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ and the set of edge be $E\left(W_n^b \circ P_2^a\right) = \{u_0 u_i \mid 1 \le i \le n-1\} \cup \{u_n u_1\} \cup \{u_i v_i \mid 1 \le i \le n\}$ of the generalized corona graph $W_n^b \circ P_2^a$.

The adjacent vertex reducible edge labeling of the generalized corona graph $W_n^b \circ P_2^a$ is:

Let $n = 2k+1, m = 2k, k = 1, 2, \cdots$

$$\begin{aligned}
&f(u_0 u_i) = i, i = 1, 2, \cdots, 2k \\
&f(u_i u_{i+1}) = 4k - i + 2, i = 1, 2, \cdots, 2k+1 \\
&f(u_n u_1) = 4k + 2 \\
&f(u_i v_i) = 4k + i + 3, i = 1, 2, \cdots, 2k+1 \\
&f(u_n v_n) = 4k + 3
\end{aligned}$$

Currently, the figure contains degree-1 vertices, degree-4 vertices, and degree-n vertices. The degree-1 vertices are not adjacent, and the degree-n vertices form a separate point, so they are not considered. It is only necessary to ensure that all adjacent degree-4 vertices have the same sum of labels. $\{u_1, u_2, \ldots, u_n\}$ are adjacent degree-4 vertices. When $2 \le i \le 2k$, the sum of their labels:

$$\begin{aligned}
Sum(u_i) &= \sum\nolimits_{uu \in E(u_i)} f(uu) \\
&= \left\{ f(u_0 u_i) + f(u_{i-1}u_i) + f(u_i u_{i+1}) + f(u_i v_i) \right\} \\
&\quad \| \left\{ f(u_0 u_1) + f(u_n u_1) + f(u_1 u_2) + f(u_1 v_1) \right\} \\
&\quad \| \left\{ f(u_0 u_n) + f(u_{n-1}u_n) + f(u_n u_1) + f(u_n v_n) \right\} \\
&= \left\{ i + (4k-i+3) + (4k-i+2) + (4k+i+3) \right\} \\
&\quad \| \left\{ 1 + 2(2k+1) + 2(2k+1) - 1 + 2(2k+1) + 2 \right\} \\
&\quad \| \left\{ (2k+1) + (2k+2) + (4k+2) + (4k+3) \right\} \\
&= \{12k + 8\}
\end{aligned}$$

A graphical example of the generalized corona graph $W_n^b \circ P_2^a$ is shown below in Fig. 9.

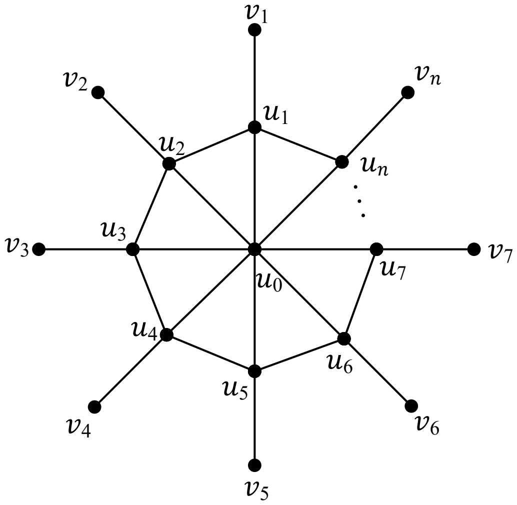


Fig. 9. Generalized corona graph $W_n^b \circ P_2^a$.

As defined by AVREL, it is possible to determine the function of the one-to-one mapping of $f\left(E\left(W_n^b \circ P_2^a\right)\right) \to \{1,2,\cdots,3n\}$, and the sum of edge labels for adjacent vertices of the same degree is constant. The proof of Theorem 2 is complete.

Theorem 3: For generalized corona graph $C_n^a \circ S_m^a$ $(n \ge 3, m \ge 2)$ except $n \equiv 0(mod\ 2), m \equiv 1(mod\ 2)$, all are AVREL graph.

Proof: Let the vertex set of the generalized corona graph $C_n^a \circ S_m^a$ be $V\left(C_n^a \circ S_m^a\right) = \{u_1,u_2,\cdots,u_n\} \cup \{v_1,v_2,\cdots,v_{mn}\}$, and the edge set be $E\left(C_n^a \circ S_m^a\right) = \{u_iu_{i+1} \mid 1 \le i \le n-1\} \cup \{u_nu_1\} \cup \left\{u_iv_i, u_iv_{(m-2)n+i}, u_iv_{(m-1)n+i} \mid 1 \le i \le n\right\}$.

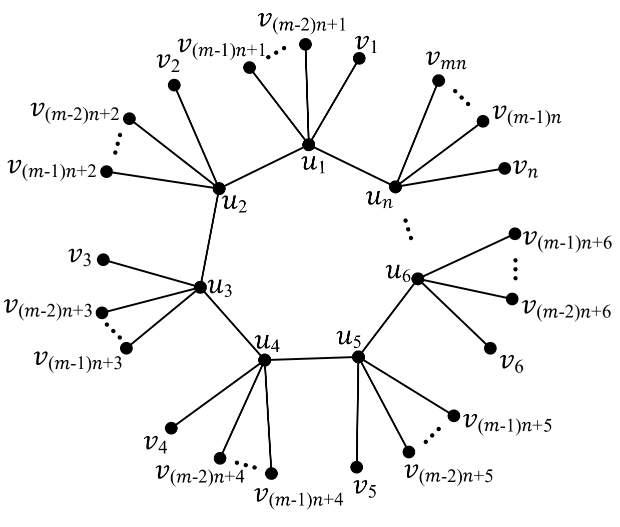


Fig. 10. Generalized corona graph $C_n^a \circ S_m^a\left(n,m \equiv 1(\text{mod } 2)\right)$.

Case 1: When $n,m \equiv 1(\text{mod } 2)$, the adjacent vertex reducible edge labeling of the generalized corona graph $C_n^a \circ S_m^a$ is:

Let $n = 2k+1, m = 2l+1, k,l = 1,2,\cdots$

$$f\left(u_iu_{i+1}\right) = \begin{cases} \dfrac{i+1}{2} & \left(1 \le i \le 2k+1, i \equiv 1(\text{mod } 2)\right) \\ k + \dfrac{i}{2} + 1 & \left(1 < i < 2k+1, i \equiv 0(\text{mod } 2)\right) \end{cases}$$

$f\left(u_nu_1\right) = k+1$

$f\left(u_iv_i\right) = 4k - i + 3 \quad 1 \le i \le 2k+1$

$f\left(u_iv_{(m-2)n+i}\right) = 4kl + 2(2k+l) - i + 3 \quad \left(1 \le i \le 2k+1, l = 1,2,\cdots,(m-1)/2\right)$

$f\left(u_iv_{(m-1)n+i}\right) = 4kl + 2l + i \quad \left(1 \le i \le 2k+1, l = 1,2,\cdots,(m-1)/2\right)$

When $n,m \equiv 1(\text{mod } 2)$, an example of the generalized corona graph $C_n^a \circ S_m^a$ is shown in Fig. 10.

In the graph, there are only degree-1 and degree-(m+2) vertices. One of the degree-1 vertices is not adjacent, so there is no need to consider it, only to ensure that the adjacent degree-(m+2) vertices are the same. Meanwhile, $\{u_1, u_2, \ldots, u_n\}$ are adjacent degree-(m+2) vertices in the graph. When $1 \le i \le n$, the sum of their labels is as follows:

$$Sum\left(u_i\right) = \sum_{uu \in E(u_i)} f(uu)$$

$$= \left\{ f\left(u_{i-1}u_i\right) + f\left(u_iu_{i+1}\right) + f\left(u_iv_i\right) + \cdots + f\left(u_iv_{(m-2)n+i}\right) + f\left(u_iv_{(m-1)n+i}\right) \right\}$$

$$\| \left\{ f\left(u_nu_1\right) + f\left(u_1u_2\right) + f\left(u_1v_1\right) + \cdots + f\left(u_1v_{(m-2)n+1}\right) + f\left(u_1v_{(m-1)n+1}\right) \right\}$$

$$\| \left\{ f\left(u_{n-1}u_n\right) + f\left(u_nu_1\right) + f\left(u_nv_n\right) + \cdots + f\left(u_nv_{(m-1)n}\right) + f\left(u_nv_{mn}\right) \right\}$$

$$= \left\{ \frac{i}{2} + \left(k + \frac{i}{2} + 1\right) + (4k - i + 3) + \cdots + (2l+2)(2k+1) + 2l(2k+1) + 1 \right\}$$

$$\| \left\{ (k+1) + 1 + (4k+2) + \cdots + (2l+2)(2k+1) + 2l(2k+1) + 1 \right\}$$

$$\| \left\{ (2k+1) + (k+1) + (2k+2) + \cdots + (2l+2)(2k+1) + 2l(2k+1) + 1 \right\}$$

$$= \left\{ 5k + 4 + l^2(4k+2) + l(8k+5) \right\}$$

From the proof above, it can be determined that for the generalized corona graph $C_n^a \circ S_m^a(n \ge 3, m \ge 2)$ when $n,m \equiv 1(\text{mod } 2)$, for adjacent vertices with the same degree, the total of their edge labels is constant. According to the AVREL definition, case 1 is proven to be an AVREL graph.

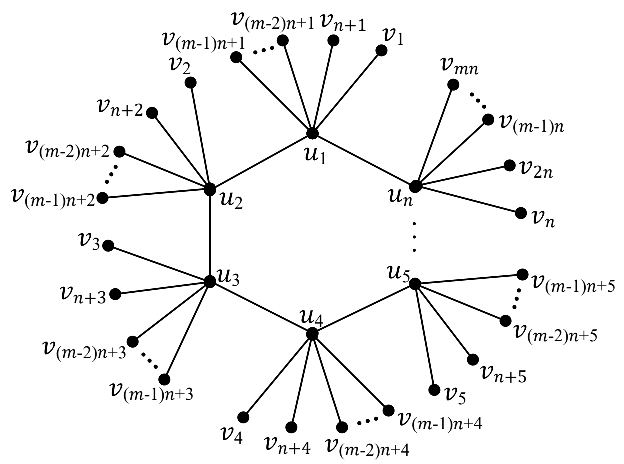


Fig. 11. Generalized corona graph $C_n^a \circ S_m^a\left(m \equiv 0(\text{mod } 2)\right)$.

Case 2: When $m \equiv 0(\text{mod } 2)$, the adjacent vertex reducible edge labeling of the generalized corona graph $C_n^a \circ S_m^a$ is:

Let $m = 2k, k = 1,2,\cdots$

$f\left(u_iu_{i+1}\right) = i \quad 1 \le i \le n-1$

$f\left(u_nu_1\right) = n$

$f\left(u_1v_1\right) = n+1$

$$f\left(u_i v_i\right)=2n-i+2 \quad 2\le i\le n$$
$$f\left(u_i v_{n+i}\right)=3n-i+1 \quad 1\le i\le n$$
$$f\left(u_i v_{(m-2)n+i}\right)=(2k+1)n-i+1 \quad (1\le i\le n, k=2,3,\cdots,m/2)$$
$$f\left(u_i v_{(m-1)n+i}\right)=(2k-1)n+i \quad (1\le i\le n, k=2,3,\cdots,m/2)$$

When $m\equiv 0(\mathrm{mod}\ 2)$, an example of the generalized corona graph $C_n^a \circ S_m^a$ is shown in Fig. 11.

In the graph, there are degree-1 and degree-(m+2) vertices. One of the degree-1 vertices is not adjacent, so there is no need to consider it, only to ensure that the adjacent degree-(m+2) vertices are the same. Meanwhile, $\{u_1, u_2, \dots, u_n\}$ are adjacent degree-(m+2) vertices in the graph. When $1\le i\le n$, the sum of their labels:

$$Sum\left(u_i\right)=\sum\nolimits_{uu\in E(u_i)} f\left(uu\right)$$
$$=\left\{f\left(u_{i-1}u_i\right)+f\left(u_i u_{i+1}\right)+f\left(u_i v_i\right)+\cdots+f\left(u_i v_{(m-2)n+i}\right)+f\left(u_i v_{(m-1)n+i}\right)\right\}$$
$$\|\left\{f\left(u_n u_1\right)+f\left(u_1 u_2\right)+f\left(u_1 v_1\right)+\cdots+f\left(u_1 v_{(m-2)n+1}\right)+f\left(u_1 v_{(m-1)n+1}\right)\right\}$$
$$\|\left\{f\left(u_{n-1}u_n\right)+f\left(u_n u_1\right)+f\left(u_n v_n\right)+\cdots+f\left(u_n v_{(m-1)n}\right)+f\left(u_n v_{mn}\right)\right\}$$
$$=\left\{i-1+i+2n-i+2+3n-i+1+\cdots+4kn+1\right\}$$
$$\|\left\{n+1+n+1+3n+\cdots+4kn+1\right\}$$
$$\|\left\{(n-1)+n+(n+2)+(2n+1)+\cdots+4kn+1\right\}$$
$$=\left\{5n+2+2nk^2+2nk-4n+k-1\right\}$$
$$=\left\{2nk^2+2nk+n+k+1\right\}$$

It can be determined by the above proof that the total of the edge labeling of the adjacent vertices with the same degree is constant for the generalized corona $C_n^a \circ S_m^a (n\ge 3, m\ge 2)$ when $m\equiv 0(\mathrm{mod}\ 2)$. According to the AVREL definition, case 2 is a proof of the AVREL graph.

In summary, the functions of one-to-one mapping for $f\left(E\left(C_n^a \circ S_m^a\right)\right)\to\{1,2,\cdots,n+nm\}$ can be determined from case 1 and case 2, and the total of the edge labeling of adjacent vertices with the same degree is a constant. According to the AVREL definition, Theorem 3 proved.

Theorem 4: The generalized corona graph $F_n^{abc}\circ S_m^a (m\equiv 0(\mathrm{mod}\ 2))$ is AVREL graph.

Proof: Let the vertex set be $V\left(F_n^{abc}\circ S_m^a\right)=\{u_0,\dots,u_n\}\cup \{v_1,v_2,\cdots,v_{2n+m}\}$ and the edge set be $E\left(F_n^{abc}\circ S_m^a\right)=\{u_0u_i \mid 1\le i\le n\} \cup\{u_i v_{mi+j} \mid 1\le j\le m, 0\le i\le n\}\cup\{u_i u_{i+1} \mid 1\le i\le n-1\}$ of the generalized corona graph $F_n^{abc}\circ S_m^a$.

When $m\equiv 0(mod\ 2)$ the adjacent vertex reducible edge labeling of the generalized corona graph $F_n^{abc}\circ S_m^a$ is:

Let $m=2k, k=1,2,\cdots$

$$f\left(u_i v_{im+j}\right)=\begin{cases}(j+1)n+2k+i-1, & j\equiv 1(\mathrm{mod}\ 2)\\ (j+2)n+2k-i, & j\equiv 0(\mathrm{mod}\ 2)\end{cases} 1\le i\le n$$
$$f\left(u_0 u_i\right)=2i-1 \quad i=1,2,\cdots,n$$
$$f\left(u_0 v_i\right)=2n+i-1 \quad i=1,2,\cdots,m$$

Case 1: When $n\equiv 0(mod\ 2)$

$$f\left(u_i u_{i+1}\right)=\begin{cases}n-i+1 & i\equiv 1(mod\ 2)\\ 2n-i & i\equiv 0(mod\ 2)\end{cases} i=1,2,\cdots,n-1$$

Case 2: When $n\equiv 1(mod\ 2)$

$$f\left(u_i u_{i+1}\right)=\begin{cases}2n-i-1 & i\equiv 1(mod\ 2)\\ n-i+1 & i\equiv 0(mod\ 2)\end{cases} i=1,2,\cdots,n-1$$

For the generalized corona graph $F_n^{abc}\circ S_m^a$, its example graph is shown in Fig. 12.

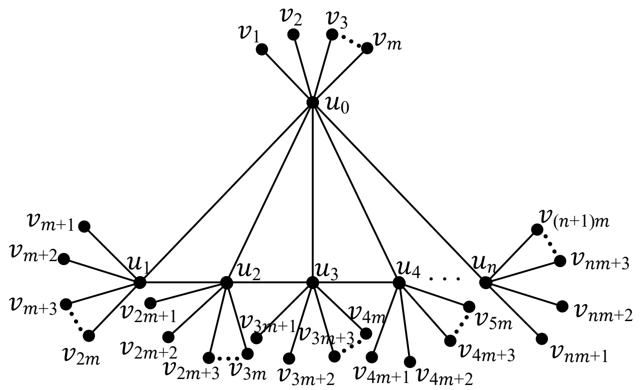


Fig. 12. Generalized corona graph $F_n^{abc}\circ S_m^a$.

There are degree-1, degree-(m+2), degree-(m+3), and degree-(m+n) vertices that are not adjacent, so there is no need to consider them, only to ensure that the adjacent degree-(m+3) vertices are the same. Meanwhile, $\{u_2, u_3, \dots, u_{n-1}\}$ are adjacent degree-(m+3) vertices in the graph. When $2\le i\le n-1$, the sum of their labels:

Case 1: When $n\equiv 0(mod\ 2)$

$$Sum\left(u_i\right)=\sum\nolimits_{uu\in E(u_i)} f\left(uu\right)$$
$$=f\left(u_{i-1}u_i\right)+f\left(u_i u_{i+1}\right)+f\left(u_0 u_i\right)+\cdots+f\left(u_i v_{mi+j}\right)$$
$$=n-i+2+(2n-i)+(2i-1)+\cdots+(2kn+2k)+(i-1)+(2k+2)n+2k-i$$
$$=(4k+5)n+4k$$

Case 2: When $n\equiv 1(mod\ 2)$

$$Sum\left(u_i\right)=\sum\nolimits_{uu\in E(u_i)} f\left(uu\right)$$
$$=f\left(u_{i-1}u_i\right)+f\left(u_i u_{i+1}\right)+f\left(u_0 u_i\right)+\cdots+f\left(u_i v_{mi+j}\right)$$
$$=2n-i+(n-i+1)+(2i-1)+\cdots+(2kn+2k)+(i-1)+(2k+2)n+2k-i$$
$$=(4k+5)n+4k-1$$

According to the AVREL definition, the function can be determined to establish a one-to-one mapping for $f\left(E\left(F_n\circ S_m^{abc}\right)\right)\to\left\{1,2,\cdots,(n+1)m+2n-1\right\}$, and the edge label sum for adjacent vertices of identical degrees is constant. Theorem 4 roved.

Theorem 5: The compound graph $W_{n_1}\uparrow_{bb} W_{n_2}\uparrow_{bd}\cdots\uparrow_{bd} W_{n_t} (n_1\ne n_2\ne\cdots\ne n_t, n_t\ne 6)$ is AVREL graph.

Proof: Let $V\left(W_{n_1}\uparrow_{bb} W_{n_2}\uparrow_{bd}\cdots\uparrow_{bd} W_{n_t}\right)=\{u_1,u_2,\dots,u_t\}\cup \left\{v_{11},v_{12},\cdots,v_{m_t}\right\}$ be the vertex set of the compound graph $W_{n_1}\uparrow_{bb} W_{n_2}\uparrow_{bd}\cdots\uparrow_{bd} W_{n_t}$, and $E\left(W_{n_1}\uparrow_{bb} W_{n_2}\uparrow_{bd}\cdots\uparrow_{bd} W_{n_t}\right)= \left\{v_{m_t}v_{t1}\right\}\cup\{u_t v_{ti} \mid 1\le i\le n_t\}\cup\left\{v_{ti}v_{t(i+1)} \mid 1\le i\le n_t-1\right\}$ be the edge set. The central vertices are denoted by $\{u_1, u_2, \dots, u_t\}$, and the initial labels of the non-central vertex in the first graph corresponds to the vertex connected to it in the second graph. Similarly, the initial label of the non-central vertex in the second graph corresponds to the vertex connected to it in the first graph, and so on.

An example graph of the compound graph $W_{n_1} \uparrow_{bb} W_{n_2} \uparrow_{bd} \cdots \uparrow_{bd} W_{n_t}$ is shown in Fig. 13.

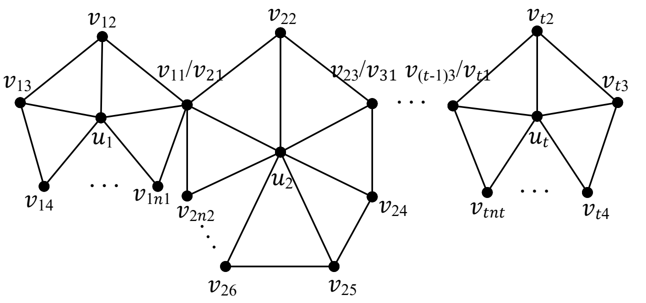


Fig. 13. Compound graph $W_{n_1} \uparrow_{bb} W_{n_2} \uparrow_{bd} \cdots \uparrow_{bd} W_{n_t} \left(n_1 \ne n_2 \ne \cdots \ne n_t\right)$.

When $n_1 \ne n_2 \ne \cdots \ne n_t, n_t \ne 6$, the adjacent vertex reducible edge labeling of the compound graph $W_{n_1} \uparrow_{bb} W_{n_2} \uparrow_{bd} \cdots \uparrow_{bd} W_{n_t}$ is:

Let $t = k, k = 1, 2, \cdots$

$$f\left(v_{ti} v_{t(i+1)}\right) = \frac{i+1}{2} + 2\sum_{k=1}^{t-1} n_k, i \equiv 1(mod\ 2)$$

$$f\left(v_{ti} v_{t(i+1)}\right) = \begin{cases} \dfrac{n_k + i + 1}{2} + 2\sum\limits_{k=1}^{t-1} n_k, n_k \equiv 1(mod\ 2), i \equiv 0(mod\ 2) \\ \dfrac{n_k + i}{2} + 2\sum\limits_{k=1}^{t-1} n_k \quad , n_k \equiv 0(mod\ 2), i \equiv 0(mod\ 2) \end{cases}$$

$$f\left(v_{tn_t} v_{t1}\right) = \begin{cases} \dfrac{n_k + 1}{2} + 2\sum\limits_{k=1}^{t-1} n_k, n_k \equiv 1(mod\ 2) \\ n_k + 2\sum\limits_{k=1}^{t-1} n_k \quad , n_k \equiv 0(mod\ 2) \end{cases}$$

$$f\left(u_t v_{ti}\right) = 2n_k - i + 1 + 2\sum_{k=1}^{t-1} n_k, i = 1, 2, \cdots, n_t 且 n_t \ne 6$$

The graph has degree-3, degree-6, and degree-$\left(n_1 \cup \cdots \cup n_t\right)$ vertices. Specifically, degree-$\left(n_1 \cup \cdots \cup n_t\right)$ and degree-6 vertices are not adjacent, so they are not considered. One need only match the labels of adjacent degree-3 vertices on each cycle graph are the same. The set $\left\{v_{t2}, v_{t3}, \cdots, v_{tn_t}\right\}$ represents adjacent degree-3 vertices. When $2 \le i \le n_k$, the sum of their labels is:

Case 1: When $n_k \equiv 1(mod\ 2)$

$$Sum\left(v_{ti}\right) = \sum\nolimits_{uu \in E\left(v_{ti}\right)} f\left(uu\right)$$
$$= \left\{ f\left(v_{t(i-1)} v_{ti}\right) + f\left(v_{ti} v_{t(i+1)}\right) + f\left(u_t v_{ti}\right) \right\}$$
$$\| \left\{ f\left(v_{t(n_t-1)} v_{tn_t}\right) + f\left(v_{tn_t} v_{t1}\right) + f\left(u_t v_{tn_t}\right) \right\}$$
$$= \left\{ 1 + 2\sum_{k=1}^{t-1} n_k + \left(\frac{n_k + 3}{2} + 2\sum_{k=1}^{t-1} n_k\right) + \left(2n_k - 1 + 2\sum_{k=1}^{t-1} n_k\right) \right\}$$
$$\| \left\{ \frac{n_k + n_k - 1 + 1}{2} + 2\sum_{k=1}^{t-1} n_k + \left(\frac{n_k + 1}{2} + 2\sum_{k=1}^{t-1} n_k\right) + \left(n_k + 1 + 2\sum_{k=1}^{t-1} n_k\right) \right\}$$
$$= \left\{ 6\sum_{k=1}^{t-1} n_k + \frac{5n_k + 3}{2} \right\}$$

Case 2: When $n_k \equiv 0(mod\ 2)$

$$Sum\left(v_{ti}\right) = \sum\nolimits_{uu \in E\left(v_{ti}\right)} f\left(uu\right)$$
$$= \left\{ f\left(v_{t(i-1)} v_{ti}\right) + f\left(v_{ti} v_{t(i+1)}\right) + f\left(u_t v_{ti}\right) \right\}$$
$$\| \left\{ f\left(v_{t(n_t-1)} v_{tn_t}\right) + f\left(v_{tn_t} v_{t1}\right) + f\left(u_t v_{tn_t}\right) \right\}$$
$$= \left\{ \left(1 + 2\sum_{k=1}^{t-1} n_k\right) + \left(\frac{n_k + 2}{2} + 2\sum_{k=1}^{t-1} n_k\right) + \left(2n_k - 1 + 2\sum_{k=1}^{t-1} n_k\right) \right\}$$
$$\| \left\{ \frac{n_k - 1 + 1}{2} + 2\sum_{k=1}^{t-1} n_k + \left(n_k + 2\sum_{k=1}^{t-1} n_k\right) + \left(2n_k - n_k + 1 + 2\sum_{k=1}^{t-1} n_k\right) \right\}$$
$$= \left\{ 6\sum_{k=1}^{t-1} n_k + \frac{5n_k + 2}{2} \right\}$$

According to the AVREL definition, the one-to-one mapping function $f\left(E\left(W_{n_1} \uparrow_{bb} W_{n_2} \uparrow_{bd} \cdots \uparrow_{bd} W_{n_t}\right)\right) \to \{1, 2, \cdots,$ $2\left(n_1 + n_2 + \cdots + n_t\right)\}$ can be determined, and the total of edge labels for adjacent vertices with identical degrees remains constant. This concludes the proof of Theorem 5.

Theorem 6: The compound graph $W_n \downarrow W_n \left(n \ge 3, n \ne 5\right)$ is AVREL graph.

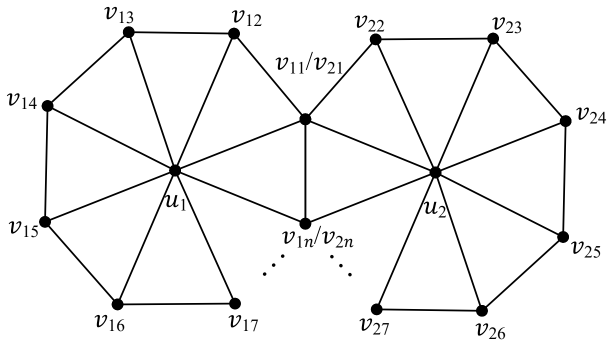


Fig. 14. Compound graph $W_n \downarrow W_n \left(n \equiv 0(mod\ 2)\right)$.

Proof: Let the vertex set be $V\left(W_n \downarrow W_n\right) = \{u_1, u_2\} \cup \{v_{11}, v_{12}, \cdots, v_{2n}\}$ and the set of edge be $E\left(W_n \downarrow W_n\right) = \{u_t v_{ti} \mid 1 \le i \le n\} \cup \left\{v_{ti} v_{t(i+1)} \mid 1 \le i \le n-1\right\} \cup \{v_{tn} v_{t1}\}$ of the compound graph $W_n \downarrow W_n$. In which, the central nodes are $\{u_1, u_2\}$, and the edge $v_{1n}v_{11}$ of the first wheel graph is connected to the edge $v_{2n}v_{21}$ of the second wheel graph.

Case 1: When $n \equiv 0(mod\ 2)$, the adjacent vertex reducible edge labeling of the compound graph $W_n \downarrow W_n$ is:

$$f\left(v_{ti} v_{t(i+1)}\right) = \begin{cases} \dfrac{i+1}{2} \quad , t = 1, i = 1, 3, \cdots, n-1 \\ \dfrac{3n + i - 1}{2}, t = 2, i = 1, 3, \cdots, n-1 \end{cases}$$

$$f\left(v_{ti} v_{t(i+1)}\right) = \begin{cases} \dfrac{i+n}{2}, t = 1, i = 2, 4, \cdots, n \\ \dfrac{i}{2} + n, t = 2, i = 2, 4, \cdots, n \end{cases}$$

$$f\left(v_{tn} v_{t1}\right) = n$$
$$f\left(u_2 v_{21}\right) = 3n$$
$$f\left(u_t v_{ti}\right) = \begin{cases} 3n - i \quad , t = 1, i = 1, 2, \cdots, n \\ 4n - i + 1, t = 2, i = 1, 2, \cdots, n \end{cases}$$

When $n \equiv 0(mod\ 2)$, the example graph of compound graph $W_n \downarrow W_n$ is shown in Fig. 14.

The graph contains degree-3, degree-5, and degree-n vertices. Since the degree-n vertices are not adjacent, they do not need to be considered. It is only necessary to ensure that the sum of the labels for the two adjacent degree-5 vertices and others for each adjacent degree-3 vertex in the wheel graph is the same. In the graph, v_{11}/v_{21} and v_{1n}/v_{2n} are adjacent degree-5 vertices, and $\{v_{t2}, v_{t3}, \ldots, v_{t(n-1)}\}$ is an adjacent degree-3 vertex. When $2 \le i \le t(n-1)$, their sum of labels is:

$$\begin{aligned} Sum(v_{ti}) &= \sum\nolimits_{uu \in E(v_{ti})} f(uu) \\ &= f\left(v_{t(i-1)}v_{ti}\right) + f\left(v_{ti}v_{t(i+1)}\right) + f(u_t v_{ti}) \\ &= \begin{cases} 1 + \dfrac{n+2}{2} + 3n - 2 & ,t=1 \\ \dfrac{3n+1-1}{2} + 1 + n + 4n - 1, & t=2 \end{cases} \\ &= \begin{cases} \dfrac{7n}{2} & ,t=1 \\ \dfrac{13n}{2}, & t=2 \end{cases} \end{aligned}$$

Then $Sum(v_{2i}) = Sum(v_{1i}) + 3n$

Vertices of the same degree that share an edge:

$$\begin{aligned} Sum(v_{ti}) &= \sum\nolimits_{uu \in E(v_{ti})} f(uu) \\ &= \left\{ \sum_{t=1}^{2} f\left(v_{ti}v_{t(i+1)}\right) + \sum_{t=1}^{2} f(u_t v_{t1}) + f(v_{tn}v_{t1}) \right\} \\ &\quad \| \left\{ \sum_{t=1}^{2} f\left(v_{t(n-1)}v_{tn}\right) + \sum_{t=1}^{2} f(u_t v_{tn}) + f(v_{tn}v_{t1}) \right\} \\ &= \left\{ 1 + \frac{3n}{2} + 3n - 1 + 3n + n \right\} \\ &\quad \| \left\{ \frac{n-1+1}{2} + \frac{3n+n-1-1}{2} + 2n + (3n+1) + n \right\} \\ &= \left\{ \frac{17n}{2} \right\} \end{aligned}$$

Case 2: When $n \equiv 1(mod\ 2)$, the adjacent vertex reducible edge labeling of the compound graph $W_n \downarrow W_n$ is:

$$f\left(v_{ti}v_{t(i+1)}\right) = \begin{cases} \dfrac{i+1}{2} & ,t=1, i=1,3,\cdots,n-2 \\ 2n - \dfrac{i+1}{2}, & t=2, i=1,3,\cdots,n-2 \end{cases}$$

$$f\left(v_{ti}v_{t(i+1)}\right) = \begin{cases} \dfrac{i+n+1}{2} & ,t=1, i=2,6,\cdots,n-1 \text{且} i \ne 4 \\ 2n - \dfrac{n-i+1}{2}, & t=2, i=2,6,\cdots,n-1 \text{且} i \ne 4 \end{cases}$$

$$f\left(v_{t(n-1)}v_{tn}\right) = n$$

$$f(v_{tn}v_{t1}) = \begin{cases} \dfrac{n+1}{2} & ,t=1 \\ \dfrac{3n-1}{2}, & t=2 \end{cases}$$

$$f(u_t v_{ti}) = \begin{cases} 3n - i - 1, & t=1, i=1,2,\cdots,n \\ 3n + i & ,t=2, i=1,2,\cdots,n \end{cases}$$

$$f(u_1 v_{1n}) = \begin{cases} 3n-1, & t=1 \\ 3n & ,t=2 \end{cases}$$

When $n \equiv 1(mod\ 2)$, the example graph of compound graph $W_n \downarrow W_n$ is shown in Fig. 15.

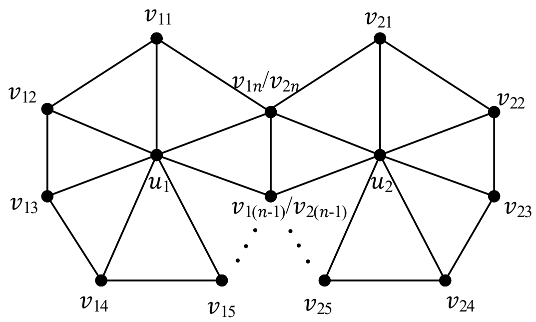


Fig. 15. Compound graph $W_n \downarrow W_n\left(n \equiv 1(mod\ 2)\right)$.

The graph has degree-3, degree-5, and degree-n vertices. Among them, the degree-n vertices are non-adjacent, so they need not be considered. On each wheel graph, it is only necessary to ensure that adjacent pairs of degree-5 vertices share the same labels as adjacent degree-3 vertices. $v_{1(n-1)}/v_{2(n-1)}$ and v_{1n}/v_{2n} are adjacent degree-5 vertices, and $\{v_{t1}, v_{t2}, \ldots, v_{t(n-2)}\}$ are adjacent degree-3 vertices. When $1 \le i \le t(n-2)$, the sum of their labels is:

$$\begin{aligned} & Sum(v_{ti}) = \sum\nolimits_{uu \in E(v_{ti})} f(uu) \\ & = \left\{ f\left(v_{t(i-1)}v_{ti}\right) + f\left(v_{ti}v_{t(i+1)}\right) + f(u_t v_{ti}) \right\} \| \left\{ f(v_{tn}v_{t1}) + f(v_{t1}v_{t2}) + f(u_t v_{t1}) \right\} \\ & = \begin{cases} \dfrac{n+1}{2} + 1 + 3n - 2 & ,t=1 \\ \dfrac{3n-1}{2} + 2n - 1 + 3n + 1, & t=2 \end{cases} \\ & = \begin{cases} \dfrac{7n-1}{2} & ,t=1 \\ \dfrac{13n-1}{2}, & t=2 \end{cases} \end{aligned}$$

Then $Sum(v_{2i}) = Sum(v_{1i}) + 3n$

Vertices of the same degree that share an edge:

$$\begin{aligned} Sum(v_{ti}) &= \sum\nolimits_{uu \in E(v_{ti})} f(uu) \\ &= \left\{ \sum_{t=1}^{2} f(v_{tn}v_{t1}) + \sum_{t=1}^{2} f(u_t v_{tn}) + f\left(v_{t(n-1)}v_{tn}\right) \right\} \\ &\quad \| \left\{ \sum_{t=1}^{2} f\left(v_{t(n-2)}v_{t(n-1)}\right) + \sum_{t=1}^{2} f\left(u_t v_{t(n-1)}\right) + f\left(v_{t(n-1)}v_{tn}\right) \right\} \\ &= \left\{ \frac{n+1}{2} + \frac{3n-1}{2} + (2n-1) + 4n + n \right\} \\ &\quad \| \left\{ \frac{n-1}{2} + \left(2n - \frac{n-1}{2}\right) + 3n - (n-1) - 1 + 3n + n - 1 + n \right\} \\ &= \{9n-1\} \end{aligned}$$

According to the AVREL definition, the one-to-one mapping function $f\left(E\left(W_n \downarrow W_n\right)\right) \to \{1,2,\cdots,4n\}$ can be determined, and the sum of the edge labels for adjacent vertices of the same degree is constant. The proof of Theorem 6 is complete.

Theorem 7: The generalized corona graph $T_n^b \circ S_m^a \left(m \equiv 1 (mod\ 2)\right)$ is AVREL graph.

Proof: Let the vertex set be $V\left(T_n^b \circ S_m^a\right) = \{u\} \cup \{v_1, v_2, \cdots, v_{2n}\} \cup \left\{v_{2n+1}, v_{2n+2}, \cdots, v_{(2m+2)n}\right\}$ and the set of edge be $E\left(T_n^b \circ S_m^a\right) = \{v_i v_{i+1} \mid 1 \le i \le 2n-1\} \cup \left\{v_i v_{2mn+i}, v_i v_{2(m-1)n+i} \mid 1 \le i \le n\right\} \cup \{uv_i \mid 1 \le i \le 2n\}$ of the generalized corona graph $T_n^b \circ S_m^a$.

An example graph of generalized corona graph $T_n^b \circ S_m^a$ is shown in Fig. 16.

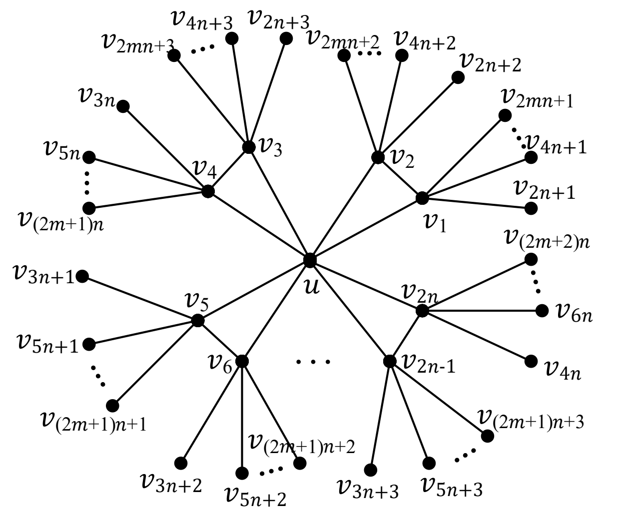


Fig. 16. Generalized corona graph $T_n^b \circ S_m^a$.

When $m \equiv 1(\text{mod } 2)$, the adjacent vertex reducible edge labeling of the generalized corona graph $T_n^b \circ S_m^a$ is:

Let $m = 2k+1, k = 0,1,2,\cdots$

$$f(uv_i) = i, i = 1,2,\cdots,2n$$

$$f(v_i v_{i+1}) = 2n + \frac{i+1}{2}, i = 1,2,\cdots,2n-1$$

When $k = 0,1,\cdots,(m-1)/2$:

$$f(v_i v_{2mn+i}) = \begin{cases} (4k+3)n + i - 1, i = 2,4,\cdots,n \\ (4k+3)n + i + 1, i = 1,3,\cdots,n \end{cases}$$

When $k = 1,2\cdots,(m-1)/2$:

$$f\left(v_i v_{2(m-1)n+i}\right) = (4k+1)n + i, i = 1,2,\cdots,n$$

The graph has degree-1, degree-(m+2), and degree-2n vertices. Among them, the degree-1 and degree-2n vertices are non-adjacent, so they do not need to be considered. It is only necessary to ensure that the labels of the two adjacent degree-(m+2) vertices are the same. $\{v_1, v_2, \ldots, v_{2n}\}$ are adjacent degree-(m+2) vertices, and when $1 \le i \le 2n$, the sum of their labels is:

Case 1: When $i \equiv 1(mod\ 2)$

$$Sum(v_i) = \sum_{uu \in E(v_i)} f(uu)$$
$$= f(uv_i) + f(v_i v_{i+1}) + f(v_i v_{2n+i}) + \cdots + f(v_i v_{2mn+i}) + f\left(v_i v_{2(m-1)n+i}\right)$$
$$= i + 2n + \frac{i+1}{2} + (3n+i+1) + \cdots + \left[(2m+1)n + i + 1\right] + \left[(2m-1)n + i\right]$$
$$= \frac{1}{2}\left[20mn + m - 26n + (2m+3)i + 2\right]$$

Case 2: When $i \equiv 0(mod\ 2)$

$$Sum(v_i) = \sum_{uu \in E(v_i)} f(uu)$$
$$= f(uv_i) + f(v_{i-1}v_i) + f(v_i v_{2n+i}) + \cdots + f(v_i v_{2mn+i}) + f\left(v_i v_{2(m-1)n+i}\right)$$
$$= i + 2n + \frac{i}{2} + (3n+i-1) + \cdots + (2m+1)n + i - 1 + (2m-1)n + i$$
$$= \frac{1}{2}\left[20mn - m - 26n + (2m+3)i - 1\right]$$

It is possible to determine a function that establishes a one-to-one mapping for $f\left(E\left(T_n^b \circ S_m^a\right)\right) \to \{1,2,\cdots,2nm+3n\}$ according to the AVREL definition, and the total of edge labels for adjacent vertices with identical degrees remains constant. The proof of Theorem 7 is complete.

Theorem 8: The corona graph $W_n \circ \left(C_3 \uparrow_{aa} P_2\right)$ $\left(n \equiv 1(\text{mod } 2)\right)$ is AVREL graph.

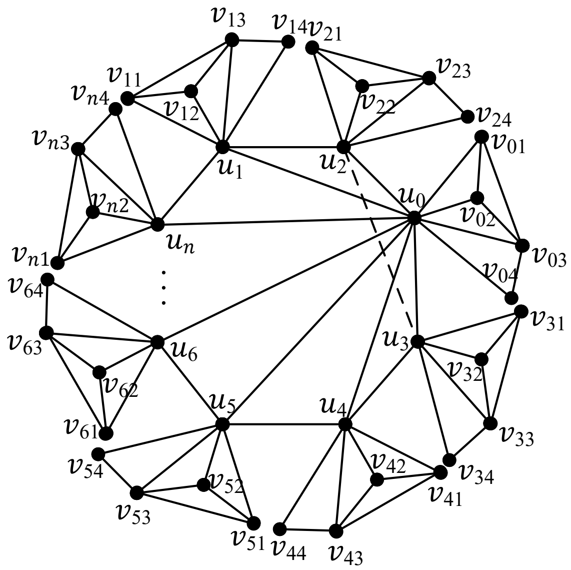


Fig. 17. Corona graph $W_n \circ \left(C_3 \uparrow_{aa} P_2\right)$.

Proof: Let the vertex set of corona graph $W_n \circ \left(C_3 \uparrow_{aa} P_2\right)$ be $V\left(W_n \circ \left(C_3 \uparrow_{aa} P_2\right)\right) = \{u_1, u_2, ..., u_n\} \cup \{v_{11}, v_{12}, \cdots, v_{n4}\}$ and the edge set be $E\left(W_n \circ \left(C_3 \uparrow_{aa} P_2\right)\right) = \left\{v_{ij} v_{i(j+2)} \mid 1 \le i \le n, j = 1\right\} \cup \left\{v_{ij} v_{i(j+1)} \mid 1 \le i \le n, 1 \le j \le 3\right\} \cup \{u_n u_1\} \cup \left\{u_i v_{ij} \mid 1 \le i \le n, 1 \le j \le 4\right\} \cup \{u_i u_{i+1} \mid 1 \le i \le n-1\}$.

When $n \equiv 1(\text{mod } 2)$, the adjacent vertex reducible edge labeling of the corona graph $W_n \circ \left(C_3 \uparrow_{aa} P_2\right)$ are labeled:

Let $n = 2k+1, k = 1,2,\cdots$

$$f(u_i v_{ij}) = \begin{cases} 4k + i + 2 \quad (0 \le i \le 2k+1, j = 1) \\ i \quad (0 \le i \le 2k+1, j = 2) \\ 4k - i + 3 \quad (0 \le i \le 2k+1, j = 3) \\ 8k - i + 5 \quad (0 \le i \le 2k+1, j = 4) \end{cases}$$

$$f\left(v_{ij} v_{i(j+1)}\right) = \begin{cases} 12k - i + 7 \quad (0 \le i \le 2k+1, j = 1) \\ 12k + i + 6 \quad (0 \le i \le 2k+1, j = 2) \\ 16k - i + 9 \quad (0 \le i \le 2k+1, j = 3) \end{cases}$$

$$f\left(v_{ij}v_{i(j+2)}\right)=8k+i+4 \quad \left(0\le i\le 2k+1, j=1\right)$$
$$f\left(u_0u_i\right)=18k+i+9 \quad 1\le i\le 2k+1$$
$$f\left(u_iu_{i+1}\right)=\begin{cases}18k-\dfrac{i-1}{2}+9 & \left(1\le i\le 2k+1, i\equiv 1\left(\bmod\ 2\right)\right)\\ 17k-\dfrac{i}{2}+9 & \left(1< i< 2k+1, i\equiv 0\left(\bmod\ 2\right)\right)\end{cases}$$
$$f\left(u_nu_1\right)=17k+9$$

For the corona graph $W_n\circ\left(C_3\uparrow_{aa}P_2\right)$, it is example graph in Fig. 17.

The graph has degree-2, degree-3, degree-4, degree-7, and degree-n vertices. Moreover, there are degree-3 vertices and degree-7 vertices in the adjacent same degree vertices, so degree-2, degree-4 and degree-n vertices need not be considered, just make sure that the adjacent degree-3 vertices are the same as the adjacent degree-7 vertices on corona graph. In the graph, $\{v_{11}, v_{12}, \ldots, v_{n1}, v_{n2}\}$ and $\{u_1, u_2, \ldots, u_n\}$ are adjacent degree-3 vertices and degree-7 vertices, the sum of their labels is:

$$\begin{aligned}S_3&=\left\{\sum\nolimits_{uu\in E(u_i)}f\left(uu\right)\right\}\\&=\left\{f\left(u_iv_{i1}\right)+f\left(v_{i1}v_{i2}\right)+f\left(v_{i1}v_{i3}\right)\right\}\,\|\,\left\{f\left(u_iv_{i2}\right)+f\left(v_{i1}v_{i2}\right)+f\left(v_{i2}v_{i3}\right)\right\}\\&=\left\{\left(4k+3\right)+\left(12k+6\right)+\left(8k+5\right)\right\}\,\|\,\left\{1+\left(12k+6\right)+\left(12k+7\right)\right\}\\&=\left\{24k+14\right\}\,\|\,\left\{24k+14\right\}\\&=\left\{24k+14\right\}\end{aligned}$$

$$\begin{aligned}S_7&=\left\{\sum\nolimits_{uu\in E(u_i)}f\left(uu\right)\right\}\\&=\left\{f\left(u_0u_1\right)+f\left(u_nu_1\right)+f\left(u_1u_2\right)+f\left(u_1v_{11}\right)+f\left(u_1v_{12}\right)+f\left(u_1v_{13}\right)+f\left(u_1v_{14}\right)\right\}\,\|\cdots\\&\|\left\{f\left(u_0u_n\right)+f\left(u_{n-1}u_n\right)+f\left(u_nu_1\right)+f\left(u_nv_{n1}\right)+f\left(u_nv_{n2}\right)+f\left(u_nv_{n3}\right)+f\left(u_nv_{n4}\right)\right\}\\&=\left\{\left(18k+10\right)+\left(17k+9\right)+\left(18k+9\right)+\left(4k+3\right)+1+\left(4k+2\right)+\left(8k+4\right)\right\}\,\|\cdots\|\\&\left\{\left(20k+10\right)+\left(16k+9\right)+\left(17k+9\right)+\left(6k+3\right)+\left(2k+1\right)+\left(2k+2\right)+\left(6k+4\right)\right\}\\&=\left\{69k+38\right\}\,\|\cdots\|\,\left\{69k+38\right\}\\&=\left\{69k+38\right\}\end{aligned}$$

According to the AVREL definition, it is possible to determine that the one-to-one mapping function $f\left(E\left(W_n\circ\left(C_3\uparrow_{aa}P_2\right)\right)\right)\to\{1,2,\cdots,8n\}$ and the sum of the edge labels for adjacent vertices of the same degree are constant. This concludes the proof of Theorem 8.

V. Conclusion

This paper designs a novel adjacent vertex reducible edge labeling algorithm to address the signal interference problem in frequency allocation, based on existing approaches like vertex sum reducible edge coloring, adjacent vertex distinguishable edge coloring, and vertex magic total labeling. The algorithm iteratively finds the optimal solution, labeling path graphs, circle graphs, star graphs, fan graphs, wheel graphs, friendship graphs, and their compound graphs within a finite number of vertices. When a graph $G(V, E)$ satisfies certain conditions, such graphs have AVREL labeling patterns, according to analysis. We derive the labeling patterns, summarize the theorems, and provide relevant proofs.

References

[1] A. ROSA, "On certain valuations of the vertices of a graph," Theory of Graphs, pp. 349-355,1967.

[2] A C BURRIS, and R H SCHELP, "Vertex-distinguishing proper edge-colorings," Journal of Graph Theory, vol. 26, no. 2, pp. 73-82, 1997.

[3] J A MACDOUGALL, M MILLER, and WD WALLIS, "Vertex-magic total labeling of graphs" Utilitas Mathematics, vol. 61, pp. 3-21, 2002.

[4] Yue Xi, "The Research on Labelling Problems in Graph," Journal of Dalian University of Technology, 2007.

[5] Jing-Wen Li, Zhong-Fu Zhang, and En-Qiang Zhu, "Adjacent Vertex Reducible Edge-Total Coloring of Graphs," 2009 2nd International Conference on Biomedical Engineering and Informatics, IEEE, pp. 1-3, 2009.

[6] Li Wang, Jing-Wen, and Li-Jing Zhang, "Adjacent Vertex Reducible Total Labeling of Graphs" IAENG International Journal of Computer Science, vol. 50, no. 2, pp. 715-726, 2023.

[7] Li Wang, Jing-Wen Li, Chen Song, and Li-Jing Zhang, "Adjacent Vertex Reducible Total Labeling of Corona Graph" Engineering Letters, vol. 31, no. 2, pp. 618-626, 2023.

[8] Shu-Cheng Zhang, "Research on Sum Reducible Coloring and Reducible Labeling Algorithms of Random Graphs" Lanzhou Jiaotong University, 2022.

[9] Jing-Wen Li, Yu-Mei Kang, Shu-Cheng Zhang, and Ruo Luo, "The Vertex Sum Reducible Edge Coloring for Graphs" Journal of Wuhan University (Natural Science Edition), vol. 68, no. 05, pp. 487-495, 2022.

[10] Ruo Luo, Jing-Wen Li, and Shu-Cheng Zhang, "Adjacent points sum reducible edge coloring of some joint graphs" Journal of Central China Normal University (Natural Sciences), vol. 57, no. 02, pp. 201-207, 2023.

[11] Xiu-Qing Jia, "On the $D(2)$-Vertex Distinguishing Coloring of Some Graphs" Lanzhou Jiaotong University, 2023.

[12] Xiao-Hui Xi, Jing-Wen Li, and Shuai Sun, "Vertex-Magic Total Labelings of Some Graphs" Journal of Southwest China Normal University (Natural Science Edition), vol. 45, no. 08, pp. 18-24, 2020.

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