Fault-tolerant Asynchronous Control for FM-II Model-based 2D Markov Jump Systems Under Actuator Failures and Mode Mismatches

Chengyi Han, Yu Zhang, Qing Chen, Taiping Jiang, and Jianping Zhou

***Abstract*—This paper deals with fault-tolerant asynchronous control for Fornasini-Marchesini second model-based two-dimensional Markov jump systems under actuator failures and mode mismatches. The actuator failures are modeled as norm-bounded uncertainties and the mode mismatches between the plant and the designed controller are characterized by a hidden Markov model. Employing the Lyapunov direct method, a criterion is established to ensure the asymptotic mean square stability and $\mathcal{H}_\infty$ noise suppression performance of the closed-loop system. Then, two design methods for the fault-tolerant asynchronous controller are proposed for the cases where the actuator fault matrix is known and unknown, respectively. The required controller gains can be determined through feasible solutions to the linear matrix inequalities. Finally, the effectiveness of these design methods is demonstrated by the Darboux equation.**

***Index Terms*—FM second model, Markov jump system, hidden Markov model, fault-tolerant control, asynchronous control.**

I. Introduction

Two-dimensional (2D) systems, which can also be called doubly-indexed systems, refer to a type of system that evolves based on independent variables along two different directions. Compared with one-dimensional systems, 2D systems can describe more complex dynamic modeling processes. Since the 1970s, 2D systems have found applications in various engineering fields, spanning from long-wall coal cutting to metal rolling, gas filtration processes, and digital filtering (see [1–3]). Among the 2D models, the Roesser model [4] and the Fornasini-Marchesini second (FM-II) model [5] stand out as two mainstream models. The former can be transformed into the latter via specific model transformations, suggesting that the FM-II model is more general.

In theoretical research within the control field, the study of Markov jump systems (MJSs) is crucial, as there may be structural or parameter mutations in actual systems, and MJSs can effectively simulate these scenarios by adhering to transition probabilities [6–10]. Over the past few decades, control and filtering of 2D MJSs based on the FM-II model have gained increasing attention. For instance, Dai et al. [11] studied the extended dissipative control via non-fragile state feedback. Under deficient uncertain transition probabilities, Wei et al. first dealt with the issue of filtering in [12] and then addressed the model approximation in [13]. Through the selection of components from an augmented vector subject to some algebraic constraints, Zhang et al. [14] developed a delay-dependent $\mathcal{H}_\infty$ filtering method to cope with such systems subject to interval delays.

It is noteworthy that the reliability of actuators is not considered in the existing references. For actuators that operate over the long term, it is difficult to guarantee that their working state remains normal. If actuator failures occur, they can lead to various unpredictable adverse effects on the system [15, 16]. In addition, most of the existing references assumed that the controlled system information could be completely captured when studying control/filtering problems. However, in networked engineering applications, the mode information on the controller or filter is likely to mismatch the actual system modes [17]. Omitting the mode mismatches may render the design methods of these references inapplicable. Thus, the natural question arises: can one design a control method for FM-II model-based 2D MJSs, effectively addressing actuator failures and mode mismatches? This question, however, has received limited attention in the literature, despite the ubiquity of actuator failures and mode mismatches in practical control systems.

Motivated by the above analysis, this paper deals with fault-tolerant asynchronous control for FM-II model-based 2D MJSs under actuator failures and mode mismatches. The actuator failures are modeled as norm-bounded uncertainties and the mode mismatches between the plant and the designed controller are characterized by a hidden Markov model. The aim is to design a fault-tolerant asynchronous controller (FTAC) to ensure the asymptotic mean square stability (AMSS) and $\mathcal{H}_\infty$ noise suppression performance of the closed-loop system. Employing the Lyapunov direct method, the required criteria are derived. Then, design methods for the FTAC are proposed for scenarios where the actuator fault matrix (AFM) is either known or unknown. The required controller gains can be determined through solving linear matrix inequalities, which can be efficiently achieved numerically. Finally, the effectiveness of these design methods is demonstrated using the Darboux equation.

II. Preliminaries

In this paper, the notations align with the definitions provided in [18, 19], except where specifically illustrated.

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Chengyi Han is a postgraduate student at the School of Computer Science and Technology, Anhui University of Technology, Ma'anshan 243032, China (e-mail: hanchengyi@ahut.edu.cn).

Yu Zhang is a postgraduate student at the School of Computer Science and Technology, Anhui University of Technology, Ma'anshan 243032, China (e-mail: yzhang@ahut.edu.cn).

Qing Chen is a postgraduate student at the School of Computer Science and Technology, Anhui University of Technology, Ma'anshan 243032, China (e-mail: chenqing@ahut.edu.cn).

Taiping Jiang is an associate professor at the School of Computer Science and Technology, Anhui University of Technology, Ma'anshan 243032, China (corresponding author, e-mail: tpjiang2008@163.com).

Jianping Zhou is a full professor at the School of Computer Science and Technology, Anhui University of Technology, Ma'anshan 243032, China (e-mail: jpzhou0@gmail.com).

A. 2D FM-II MJS

We consider the following discrete-time 2D MJS with a jump parameter, which is described by the FM-II model:

$$\begin{cases} x(i+1,j+1) = A_{1\gamma_{i,j+1}}x(i,j+1) + A_{2\gamma_{i+1,j}}x(i+1,j) \\ \qquad + B_{1\gamma_{i,j+1}}u_f(i,j+1) \\ \qquad + B_{2\gamma_{i+1,j}}u_f(i+1,j) \\ \qquad + E_{2\gamma_{i+1,j}}w(i+1,j) \\ \qquad + E_{1\gamma_{i,j+1}}w(i,j+1), \\ y(i,j) = C_{\gamma_{i,j}}x(i,j) + D_{\gamma_{i,j}}u_f(i,j) + F_{\gamma_{i,j}}w(i,j), \end{cases} \tag{1}$$

where $x(i,j) \in \mathbb{R}^{n_x}$, $u_f(i,j) \in \mathbb{R}^{n_u}$, $w(i,j) \in \mathbb{R}^{n_w}$, and $y(i,j) \in \mathbb{R}^{n_y}$ denote the system state, the fault input vector, the disturbance input and the system output, respectively. The real-valued matrices $A_{1\gamma_{i,j}}$, $A_{2\gamma_{i,j}}$, $B_{1\gamma_{i,j}}$, $B_{2\gamma_{i,j}}$, $C_{\gamma_{i,j}}$, $D_{\gamma_{i,j}}$, $E_{1\gamma_{i,j}}$, $E_{2\gamma_{i,j}}$, and $F_{\gamma_{i,j}}$, which depend on $\gamma_{i,j}$, are known and appropriately dimensioned. The random variable $\gamma_{i,j}$ obeys a Markov process with a transition probability matrix $\Lambda = \{\lambda_{pq}\}$. Unlike continuous-time systems, λ_{pq} is subject to

$$\begin{aligned} \lambda_{pq} =& \mathbf{Pr}\{\gamma_{i+1,j+1} = q | \gamma_{i,j+1} = p\} \\ =& \mathbf{Pr}\{\gamma_{i+1,j+1} = q | \gamma_{i+1,j} = p\}, \end{aligned}$$

with $\lambda_{i,j} \in [0,1]$ and $\sum_{q=1}^{k_1} \lambda_{pq} = 1$, where $p, q \in \mathcal{K}_1 = \{1, 2, \ldots, k_1\}$ [20, 21]. Furthermore, the boundary condition $(X_0,\ P_0)$ of 2D MJS (1) are defined as

$$\begin{aligned} X_0 &= \{x(0,j), x(i,0) \mid i,j = 0,1,2,\ldots\}, \\ P_0 &= \{\gamma_{0,j}, \gamma_{i,0} \mid i,j = 0,1,2,\ldots\}, \end{aligned}$$

and the zero boundary condition (ZBC) is given as $x(0,j) = x(i,0) = 0, i,j = 0,1,2,\ldots$.

B. FTAC

When designing the FTAC, we assume that precise information about $\gamma_{i,j}$ is not available. First, the asynchronous controller is detailed as follows:

$$u(i,j) = K_{\eta_{i,j}}x(i,j), \tag{2}$$

where $K_{\eta_{i,j}} \in \mathbb{R}^{n_u \times n_x}$ represents the controller gain that depend on the parameter $\eta_{i,j} \in \mathcal{K}_2 = \{1, 2, \ldots, k_2\}$; meanwhile, it satisfies the conditional probability matrix $\Pi = \{\pi_{ps}\}$ with

$$\begin{aligned} \pi_{ps} =& \mathbf{Pr}\{\eta_{i,j+1} = s \mid \gamma_{i,j+1} = p\} \\ =& \mathbf{Pr}\{\eta_{i+1,j} = s \mid \gamma_{i+1,j} = p\}, \end{aligned}$$

where $\pi_{ps} \in [0,1]$, $\sum_{s=1}^{k_2} \pi_{ps} = 1$, $\forall p \in \mathcal{K}_1, s \in \mathcal{K}_2$. In this way, $\{\Lambda, \Pi\}$ consists a hidden Markov model [22–25]. Then, we delve into the scenario of actuator failure in relation to this controller. The FTAC is as follows:

$$u_f(i,j) = \Theta_{\gamma_{i,j}}u(i,j), \tag{3}$$

where $\Theta_{\gamma_{i,j}}$ represents the fault matrix of the $\gamma_{i,j}$-th actuator [26–28], which has the following form:

$$\Theta_{\gamma_{i,j}} = diag\left\{\theta_{1\gamma_{i,j}}, \theta_{2\gamma_{i,j}}, \ldots, \theta_{n_u\gamma_{i,j}}\right\},$$

where $0 \le \underline{\theta}_{k\gamma_{i,j}} \le \theta_{k\gamma_{i,j}} \le \overline{\theta}_{k\gamma_{i,j}} \le 1\ (k = 1, 2, \ldots, n_u)$, $\underline{\theta}_{k\gamma_{i,j}}$ and $\overline{\theta}_{k\gamma_{i,j}}$ are known constants.

Remark 1. *The degree of actuator failure is determined by the upper and lower bounds $\overline{\theta}_{k\gamma_{i,j}}$ and $\underline{\theta}_{k\gamma_{i,j}}$ of the fault matrix parameters. When $\underline{\theta}_{k\gamma_{i,j}} = 1$, the actuator is completely normal. When $\overline{\theta}_{k\gamma_{i,j}} = 0$, the actuator is completely failed. When $\underline{\theta}_{k\gamma_{i,j}} \neq 1$ and $\overline{\theta}_{k\gamma_{i,j}} \neq 0$, the actuator is partially failed.*

Next, we define

$$\begin{aligned} \Theta_{0\gamma_{i,j}} &= diag\left\{\theta_{01}^{[\gamma_{i,j}]}, \theta_{02}^{[\gamma_{i,j}]}, \ldots, \theta_{0n_u}^{[\gamma_{i,j}]}\right\}, \\ \Theta_{1\gamma_{i,j}} &= diag\left\{\theta_{11}^{[\gamma_{i,j}]}, \theta_{12}^{[\gamma_{i,j}]}, \ldots, \theta_{1n_u}^{[\gamma_{i,j}]}\right\}, \end{aligned}$$

where $\theta_{0k}^{[\gamma_{i,j}]} = (\overline{\theta}_{k\gamma_{i,j}} + \underline{\theta}_{k\gamma_{i,j}})/2$, $\theta_{1k}^{[\gamma_{i,j}]} = (\overline{\theta}_{k\gamma_{i,j}} - \underline{\theta}_{k\gamma_{i,j}})/2$. Then, for $|\Delta_{\gamma_{i,j}}| = diag\{|\delta_{1\gamma_{i,j}}|, |\delta_{2\gamma_{i,j}}|, \ldots, |\delta_{n_u\gamma_{i,j}}|\}$, fault matrix can be rewritten as

$$\Theta_{\gamma_{i,j}} = \Theta_{0\gamma_{i,j}} + \Delta_{\gamma_{i,j}}, |\Delta_{\gamma_{i,j}}| \le \Theta_{1\gamma_{i,j}}. \tag{4}$$

Remark 2. *In (4), the AFM is decomposed into two parts: the known part and the unknown part. This decomposition can be conceptualized as the fault parameter values oscillating around the mean of the upper and lower bounds within these limits.*

Remark 3. *Obviously, whether the AFM corresponding to (4) is known depends on $\Delta_{\gamma_{i,j}}$. When $\Delta_{\gamma_{i,j}} = 0$ (i.e., $\underline{\theta}_{k\gamma_{i,j}} = \overline{\theta}_{k\gamma_{i,j}}$), the fault matrix of the $\gamma_{i,j}$-th actuator is known. When $\Delta_{\gamma_{i,j}} \neq 0$ (i.e., $\underline{\theta}_{k\gamma_{i,j}} \neq \overline{\theta}_{k\gamma_{i,j}}$), the fault matrix of the $\gamma_{i,j}$-th actuator is unknown.*

C. Problem formulation

Under $\gamma_{i,j+1} = p$ or $\gamma_{i+1,j} = p$, $A_{1\gamma_{i,j+1}}$, $A_{2\gamma_{i+1,j}}$, $B_{1\gamma_{i,j+1}}$, $B_{2\gamma_{i+1,j}}$, $E_{1\gamma_{i,j+1}}$, and $E_{2\gamma_{i+1,j}}$ can be abbreviated as A_{1p}, A_{2p}, B_{1p}, B_{2p}, E_{1p}, and E_{2p}. Then, the following closed-loop 2D MJS can be obtained through (1), (2) and (3):

$$\begin{cases} x(i+1,j+1) = \bar{A}_{1ps}x(i,j+1) + \bar{A}_{2ps}x(i+1,j) \\ \qquad + E_{1p}w(i,j+1) + E_{2p}w(i+1,j), \\ y(i,j) = \bar{C}_{ps}x(i,j) + F_p w(i,j), \end{cases} \tag{5}$$

where

$$\begin{aligned} \bar{A}_{1ps} &= A_{1p} + B_{1p}\Theta_p K_s, \\ \bar{A}_{2ps} &= A_{2p} + B_{2p}\Theta_p K_s, \\ \bar{C}_{ps} &= C_p + D_p\Theta_p K_s. \end{aligned}$$

Next, through the expansion of the classical definitions of 1D systems, we obtain the corresponding forms of these definitions in 2D systems. In addition, we include an assumption on the boundary condition and subsequently introduce three lemmas.

Assumption 1. *[29] Boundary X_0 satisfies*

$$\lim_{L\to\infty} \mathbb{E}\left\{\sum_{l=0}^{L}\left(\|x(0,l)\|^2 + \|x(l,0)\|^2\right)\right\} < \infty. \tag{6}$$

Definition 1. *[30] If the closed-loop 2D MJS (5) satisfies the following condition for any boundary condition $(X_0,\ P_0)$ when $w(i,j) \equiv 0$:*

$$\lim_{i+j\to\infty} \mathbb{E}\left\{\|x(i,j)\|^2\right\} = 0, \tag{7}$$

then the system has the AMSS.

Definition 2. *[31] The closed-loop 2D MJS (5) is said to have an $\mathcal{H}_\infty$ noise suppression performance μ, if*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mathbb{E}\left\{\left\|\begin{bmatrix}y(i,j+1)\\y(i+1,j)\end{bmatrix}\right\|^2\right\}\leq\mu^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}w(i,j+1)\\w(i+1,j)\end{bmatrix}\right\|^2 \tag{8}$$

holds for any $w(i,j)\in l_2\{[0,\infty),[0,\infty)\}$ under the ZBC.

Lemma 1. *[32] For a given matrix $Z=\begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix}$, where $Z_{11}\in\mathbb{R}^{r\times r}$, the following three conditions are equivalent:*
1) $Z<0$;
2) $Z_{11}<0, Z_{22}-Z_{21}Z_{11}^{-1}Z_{12}<0$;
3) $Z_{22}<0, Z_{11}-Z_{12}Z_{22}^{-1}Z_{21}<0$.

Lemma 2. *[33] For any two matrices X_1 and $X_2>0$ with appropriate dimensions, the following inequality holds:*

$$-X_1^TX_2^{-1}X_1\leq X_2-X_1^T-X_1.$$

Lemma 3. *[34] Let H, G, H_1, and H_2 be real matrices of suitable dimensions. Then*

$$H+H_1GH_2+H_2^TG^TH_1^T<0$$

holds for $G^TG<I$, if and only if there is a scalar $\sigma>0$ such that

$$H+\sigma^{-1}H_1H_1^T+\sigma H_2^TH_2<0.$$

The research purpose of this paper: We consider the design of the FTAC for scenarios involving actuator failure to ensure the AMSS and $\mathcal{H}_\infty$ noise suppression performance of the closed-loop 2D MJS (5). In the case of actuator failure, the conclusion is extended from the case where the fault matrix is known to the case where the fault matrix is unknown, i.e., when the AFM appears in the form of (4).

III. Stability and Performance Analysis

In this section, we provide sufficient conditions for the AMSS of the closed-loop 2D MJS (5) under $\mathcal{H}_\infty$ noise suppression performance.

Theorem 1. *Consider the closed-loop 2D MJS (5) that satisfies (6). Given scalars μ, ε_1, and ε_2 satisfying $\mu>0$, $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_1+\varepsilon_2=1$, for $\forall p\in\mathcal{K}_1$, $s\in\mathcal{K}_2$, if there exist matrices $R_p>0$, $Q_{ps}>0$, K_s, such that*

$$\sum_{s=1}^{k_2}\pi_{ps}\begin{bmatrix}Q_{ps} & 0\\ 0 & Q_{ps}\end{bmatrix}<\begin{bmatrix}\varepsilon_1R_p & 0\\ 0 & \varepsilon_2R_p\end{bmatrix}, \tag{9}$$

$$\begin{bmatrix}-\bar{R}_p^{-1} & 0 & 0 & \bar{A}_{1ps} & \bar{A}_{2ps} & E_{1p} & E_{2p}\\ * & -I & 0 & \bar{C}_{ps} & 0 & F_p & 0\\ * & * & -I & 0 & \bar{C}_{ps} & 0 & F_p\\ * & * & * & -Q_{ps} & 0 & 0 & 0\\ * & * & * & * & -Q_{ps} & 0 & 0\\ * & * & * & * & * & -\mu^2I & 0\\ * & * & * & * & * & * & -\mu^2I\end{bmatrix}<0, \tag{10}$$

hold, where $\bar{R}_p=\sum_{q=1}^{k_1}\lambda_{pq}R_q$, then the closed-loop 2D MJS (5) has AMSS and $\mathcal{H}_\infty$ noise suppression performance μ.

Proof: First, the AMSS of the system is proved. We introduce the Lyapunov function of the following form:

$$\begin{aligned}V_\kappa=&\varepsilon_1x^T(i,j+1)R_px(i,j+1)\\&+\varepsilon_2x^T(i+1,j)R_px(i+1,j),\\ V_{\kappa+1}=&x^T(i+1,j+1)R_qx(i+1,j+1),\end{aligned}$$

where κ is known as the global instant in the system, satisfying the condition $i+j=\kappa$. Define

$$\begin{aligned}\Delta V=&x^T(i+1,j+1)R_qx(i+1,j+1)\\&-\varepsilon_1x^T(i,j+1)R_px(i,j+1)\\&-\varepsilon_2x^T(i+1,j)R_px(i+1,j).\end{aligned} \tag{11}$$

According to the closed-loop 2D MJS (5) with $w(i,j)\equiv0$, we can obtain

$$\Delta V=\zeta^T(i,j)\Xi_{pqs}\zeta(i,j), \tag{12}$$

where

$$\Xi_{pqs}=\begin{bmatrix}\bar{A}_{1ps}^TR_q\bar{A}_{1ps}-\varepsilon_1R_p & \bar{A}_{1ps}^TR_q\bar{A}_{2ps}\\ * & \bar{A}_{2ps}^TR_q\bar{A}_{2ps}-\varepsilon_2R_p\end{bmatrix},$$
$$\zeta(i,j)=\begin{bmatrix}x(i,j+1)\\x(i+1,j)\end{bmatrix}.$$

Then, by performing the expectation operation on (12), we obtain

$$\mathbb{E}\{\Delta V\}=\mathbb{E}\left\{\zeta^T(i,j)\bar{\Xi}_{pqs}\zeta(i,j)\right\}, \tag{13}$$

where

$$\begin{aligned}\bar{\Xi}_{pqs}=&\sum_{x=1}^{k_2}\pi_{ps}\begin{bmatrix}\bar{A}_{1ps}^T\bar{R}_p\bar{A}_{1ps} & \bar{A}_{1ps}^T\bar{R}_p\bar{A}_{2ps}\\ * & \bar{A}_{2ps}^T\bar{R}_p\bar{A}_{2ps}\end{bmatrix}\\&-\begin{bmatrix}\varepsilon_1R_p & 0\\ 0 & \varepsilon_2R_p\end{bmatrix}.\end{aligned}$$

From (10), by utilizing Lemma 1 and scaling, we can deduce that

$$\begin{bmatrix}\bar{A}_{1ps}^T\bar{R}_p\bar{A}_{1ps} & \bar{A}_{1ps}^T\bar{R}_p\bar{A}_{2ps}\\ * & \bar{A}_{2ps}^T\bar{R}_p\bar{A}_{2ps}\end{bmatrix}<\begin{bmatrix}Q_{ps} & 0\\ 0 & Q_{ps}\end{bmatrix}. \tag{14}$$

Then, with the assistance of (14), we can derive the following from (13):

$$\begin{aligned}\mathbb{E}\{\Delta V\}<\mathbb{E}\Big\{\zeta^T(i,j)\Big(&\sum_{s=1}^{k_2}\begin{bmatrix}Q_{ps} & 0\\ 0 & Q_{ps}\end{bmatrix}\\&-\begin{bmatrix}\varepsilon_1R_p & 0\\ 0 & \varepsilon_2R_p\end{bmatrix}\Big)\zeta(i,j)\Big\}.\end{aligned} \tag{15}$$

For (9), $\exists\alpha>0$, α is the minimum eigenvalue of $\left(\begin{bmatrix}\varepsilon_1R_p & 0\\ 0 & \varepsilon_2R_p\end{bmatrix}-\sum_{s=1}^{k_2}\pi_{ps}\begin{bmatrix}Q_{ps} & 0\\ 0 & Q_{ps}\end{bmatrix}\right)$, and then the (15) is equivalent to

$$\mathbb{E}\{\Delta V\}\leqslant-\alpha\mathbb{E}\left\{\|\zeta(i,j)\|^2\right\}. \tag{16}$$

Summing up both sides of (16), we have

$$\mathbb{E}\left\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\|\zeta(i,j)\|^2\right\}\leq-\frac{1}{\alpha}\mathbb{E}\left\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\Delta V\right\}, \tag{17}$$

where d_1 and d_2 are any positive integers. It can be obtained from (11) that

$$\begin{aligned}&\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\Delta V\\=&\sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\left((\varepsilon_1+\varepsilon_2)x^T(i+1,j+1)R_qx(i+1,j+1)\right.\end{aligned}$$

$$
\begin{aligned}
&- \varepsilon_1 x^T(i,j+1)R_p x(i,j+1) \\
&-\varepsilon_2 x^T(i+1,j)R_p x(i+1,j)) \\
=&\varepsilon_1 \sum_{j=0}^{d_2} \left(x^T(d_1+1,j+1)R_{\gamma_{d_1+1,j+1}} x(d_1+1,j+1)\right. \\
&\left.-x^T(0,j+1)R_{\gamma_{0,j+1}} x(0,j+1)\right) \\
&+\varepsilon_2 \sum_{i=0}^{d_1} \left(x^T(i+1,d_2+1)R_{\gamma_{i+1,d_2+1}} x(i+1,d_2+1)\right. \\
&\left.-x^T(i+1,0)R_{\gamma_{i+1,0}} x(i+1,0)\right). \qquad (18)
\end{aligned}
$$

Combining (17) and (18), and scaling accordingly, we obtain

$$
\begin{aligned}
&\mathbb{E}\left\{\sum_{i=0}^{d_1}\sum_{j=0}^{d_2} \|\zeta(i,j)\|^2\right\} \\
\leq&\frac{1}{\alpha}\mathbb{E}\left\{\varepsilon_1 \sum_{j=0}^{d_2} x^T(0,j+1)R_{\gamma_{0,j+1}} x(0,j+1)\right. \\
&\left.+\varepsilon_2 \sum_{i=0}^{d_1} x^T(i+1,0)R_{\gamma_{i+1,0}} x(i+1,0)\right\}. \qquad (19)
\end{aligned}
$$

Let $\varepsilon = max\{\varepsilon_1, \varepsilon_2\}$, β be the maximum eigenvalue of $R_{\gamma_{0,j+1}}$ and $R_{\gamma_{i+1,0}}$, and let d_1 and d_2 both tend to infinity. Then, the above (19) is equivalent to

$$
\begin{aligned}
&\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \|\zeta(i,j)\|^2\right\} \\
\leq&\frac{\varepsilon\beta}{\alpha}\mathbb{E}\left\{\sum_{l=0}^{\infty}\left(\|x(0,l)\|^2 + \|x(l,0)\|^2\right)\right\}. \qquad (20)
\end{aligned}
$$

Next, by combining the above inequality with (6), it is easy to get that

$$
\mathbb{E}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \|\zeta(i,j)\|^2\right\} < \infty, \qquad (21)
$$

which obviously guarantees the establishment of (7). At this point, the AMSS is proven. Next, we consider the $\mathcal{H}_\infty$ noise suppression performance under the ZBC.

Recalling (11) for the closed-loop 2D MJS (5), we obtain the following equation:

$$
\begin{aligned}
\Delta V =&\xi^T(i,j)Y_1^T R_q Y_1 \xi(i,j) - \varepsilon_1 x^T(i,j+1)R_p x(i,j+1) \\
&- \varepsilon_2 x^T(i+1,j)R_p x(i+1,j), \qquad (22)
\end{aligned}
$$

where $\xi(i,j) = \begin{bmatrix} x(i,j+1) \\ x(i+1,j) \\ w(i,j+1) \\ w(i+1,j)\end{bmatrix}$, $Y_1 = \begin{bmatrix}\bar{A}_{1ps} & \bar{A}_{2ps} & E_{1p} & E_{2p}\end{bmatrix}$. Then, based on the form of (8), we introduce the following equation:

$$
\begin{aligned}
&\left\|\begin{bmatrix} y(i,j+1) \\ y(i+1,j)\end{bmatrix}\right\|^2 - \mu^2 \left\|\begin{bmatrix} w(i,j+1) \\ w(i+1,j)\end{bmatrix}\right\|^2 \\
=&y^T(i,j+1)y(i,j+1) + y^T(i+1,j)y(i+1,j) \\
&-\mu^2 w^T(i,j+1)w(i,j+1) - \mu^2 w^T(i+1,j)w(i+1,j). \qquad (23)
\end{aligned}
$$

Combining the closed-loop 2D MJS (5) state equation, we can obtain that (23) is equivalent to

$$
\begin{aligned}
&\left\|\begin{bmatrix} y(i,j+1) \\ y(i+1,j)\end{bmatrix}\right\|^2 - \mu^2 \left\|\begin{bmatrix} w(i,j+1) \\ w(i+1,j)\end{bmatrix}\right\|^2 \\
=&\xi^T(i,j)\left(Y_2^T Y_2 - \bar{I}\right)\xi(i,j), \qquad (24)
\end{aligned}
$$

where $Y_2 = \begin{bmatrix}\bar{C}_{ps} & 0 & F_p & 0 \\ 0 & \bar{C}_{ps} & 0 & F_p\end{bmatrix}$, $\bar{I} = diag\{0, 0, \mu I, \mu I\}$. Under the ZBC, we can derive the following condition from (18):

$$
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Delta V \geq 0. \qquad (25)
$$

Next, based on (25), we consider $\mathcal{H}_\infty$ noise suppression performance μ and set

$$
\begin{aligned}
J =& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\left\{\begin{bmatrix} y(i,j+1) \\ y(i+1,j)\end{bmatrix}^T \begin{bmatrix} y(i,j+1) \\ y(i+1,j)\end{bmatrix}\right. \\
&\left.-\mu^2 \begin{bmatrix} w(i,j+1) \\ w(i+1,j)\end{bmatrix}^T \begin{bmatrix} w(i,j+1) \\ w(i+1,j)\end{bmatrix}\right\} \qquad (26) \\
\leq& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\left\{\begin{bmatrix} y(i,j+1) \\ y(i+1,j)\end{bmatrix}^T \begin{bmatrix} y(i,j+1) \\ y(i+1,j)\end{bmatrix}\right. \\
&\left.-\mu^2 \begin{bmatrix} w(i,j+1) \\ w(i+1,j)\end{bmatrix}^T \begin{bmatrix} w(i,j+1) \\ w(i+1,j)\end{bmatrix} + \Delta V\right\}. \qquad (27)
\end{aligned}
$$

The above (27) combines (22) and (24), we have

$$
\begin{aligned}
J \leq& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\Big\{\xi^T(i,j)\sum_{s=1}^{k_2} \pi_{ps}(Y_1^T \bar{R}_p Y_1 \\
&+ Y_2^T Y_2 - \bar{I})\xi(i,j) - \varepsilon_1 x^T(i,j+1)R_p x(i,j+1) \\
&- \varepsilon_2 x^T(i+1,j)R_p x(i+1,j)\Big\}. \qquad (28)
\end{aligned}
$$

Using Lemma 1 on (10), we can get

$$
Y_1^T \bar{R}_p Y_1 + Y_2^T Y_2 - \bar{I} < \bar{Q}_{ps}, \qquad (29)
$$

where $\bar{Q}_{ps} = diag\{Q_{ps}, Q_{ps}, 0, 0\}$. According to (28) and (29), the following is satisfied:

$$
\begin{aligned}
J <& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\Big\{\xi^T(i,j)\sum_{s=1}^{k_2} \pi_{ps}\bar{Q}_{ps}\xi(i,j) \\
&- \varepsilon_1 x^T(i,j+1)R_p x(i,j+1) \\
&- \varepsilon_2 x^T(i+1,j)R_p x(i+1,j)\Big\} \\
=& \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\left\{\zeta^T(i,j)\left(\sum_{s=1}^{k_2} \pi_{ps}\begin{bmatrix} Q_{ps} & 0 \\ 0 & Q_{ps}\end{bmatrix}\right.\right. \\
&\left.\left.- \begin{bmatrix}\varepsilon_1 R_p & 0 \\ 0 & \varepsilon_2 R_p\end{bmatrix}\right)\zeta(i,j)\right\}. \qquad (30)
\end{aligned}
$$

From (9), it can be deduced that $J < 0$. Simultaneously, from the definition of J in (26), it follows that the $\mathcal{H}_\infty$ noise suppression performance condition in (8) is guaranteed. The proof is completed. ■

IV. FTAC Design

In this section, we first provide a design method for the FTAC based on known actuator fault matrix. Then, we extend this design method to the case where the AFM is unknown.

A. Known actuator fault matrix

To begin with, based on Theorem 1, we present the design method for the FTAC when the AFM is known.

Theorem 2. *Consider the closed-loop 2D MJS (5) that satisfies (6). If there exist a scalar $\bar{\mu} > 0$, a set of matrices $\tilde{R}_p > 0$, $\tilde{Q}_{ps} > 0$, $\tilde{K}_s$, $\tilde{M}_s$, such that*

$$\begin{bmatrix} -\varepsilon_1\tilde{R}_p & 0 & \bar{T}_p & 0 \\ * & -\varepsilon_2\tilde{R}_p & 0 & \bar{T}_p \\ * & * & -\hat{Q}_p & 0 \\ * & * & * & -\hat{Q}_p \end{bmatrix} < 0, \quad (31)$$

$$\begin{bmatrix} \Omega^1_{ps} & \Omega^2_{ps} & \Omega^3_{ps} \\ * & -I & 0 \\ * & * & -\hat{R} \end{bmatrix} < 0, \quad (32)$$

where

$$\hat{Q}_p = diag\left\{\tilde{Q}_{p1}, \tilde{Q}_{p1}, \ldots, \tilde{Q}_{pk_2}\right\},$$

$$\bar{T}_p = \left[\sqrt{\pi_{p1}}\tilde{R}_p \ \sqrt{\pi_{p2}}\tilde{R}_p \ \cdots \ \sqrt{\pi_{pk_2}}\tilde{R}_p\right],$$

$$\Omega^1_{ps} = diag\left\{\tilde{Q}_{ps} - M_s^T - M_s, \tilde{Q}_{ps} - M_s^T - M_s, -\bar{\mu}I, -\bar{\mu}I\right\},$$

$$\Omega^2_{ps} = \begin{bmatrix} C_pM_s + D_p\Theta_p\tilde{K}_s & 0 & F_p & 0 \\ 0 & C_pM_s + D_p\Theta_p\tilde{K}_s & 0 & F_p \end{bmatrix}^T,$$

$$\Omega^3_{ps} = \left[\sqrt{\lambda_{p1}}\tilde{Y}^T_{ps} \ \sqrt{\lambda_{p2}}\tilde{Y}^T_{ps} \ \cdots \ \sqrt{\lambda_{pk_1}}\tilde{Y}^T_{ps}\right],$$

$$\tilde{Y}_{ps} = \left[A_{1p}M_s + B_{1p}\Theta_p\tilde{K}_s \ A_{2p}M_s + B_{2p}\Theta_p\tilde{K}_s \ E_{1p} \ E_{2p}\right],$$

$$\hat{R} = diag\left\{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{k_1}\right\},$$

hold for $\forall p \in \mathcal{K}_1$, $s \in \mathcal{K}_2$, then the closed-loop 2D MJS (5) is asymptotically mean square stable with $\mathcal{H}_\infty$ noise suppression performance $\mu = \sqrt{\bar{\mu}}$, and the controller gain can be expressed in the following form:

$$K_s = \tilde{K}_s M_s^{-1}. \quad (33)$$

Proof: First, we demonstrate the equivalence between (9) and (31). Let us introduce the following notations: $\tilde{R}_p = R_p^{-1}$, $\tilde{Q}_{ps} = Q_{ps}^{-1}$.

By pre-multiplying and post-multiplying (31) with $diag\left\{R_p, R_p, I, I\right\}$ and its transpose matrix, it can be shown to be equivalent to

$$\begin{bmatrix} -\varepsilon_1 R_p & 0 & T_p & 0 \\ * & -\varepsilon_2 R_p & 0 & T_p \\ * & * & -\hat{Q}_p & 0 \\ * & * & * & -\hat{Q}_p \end{bmatrix} < 0, \quad (34)$$

where $T_p = \left[\sqrt{\pi_{p1}}I \ \sqrt{\pi_{p2}}I \ \cdots \ \sqrt{\pi_{pk_2}}I\right]$. Obviously, through Lemma 1, (34) is equivalent to (9).

Next, we prove that (10) can be guaranteed by (32). Here, we introduce the slack matrix M_s. The invertibility of M_s can be ensured by (32). According to Lemma 2, we have

$$-M_s^T\tilde{Q}_{ps}^{-1}M_s \le \tilde{Q}_{ps} - M_s^T - M_s. \quad (35)$$

From (32) and (35), it can be inferred that the following condition holds:

$$\begin{bmatrix} \bar{\Omega}^1_{ps} & \Omega^2_{ps} & \Omega^3_{ps} \\ * & -I & 0 \\ * & * & -\hat{R} \end{bmatrix} < 0, \quad (36)$$

where

$$\bar{\Omega}^1_{ps} = diag\left\{-M_s^T\tilde{Q}_{ps}^{-1}M_s, -M_s^T\tilde{Q}_{ps}^{-1}M_s, -\bar{\mu}I, -\bar{\mu}I\right\}.$$

Pre-multiplying and post-multiplying (36) by $diag\left\{(M_s^T)^{-1}, (M_s^T)^{-1}, I, \cdots, I\right\}$, we can obtain that

$$\begin{bmatrix} -\bar{Q}_{ps} - \bar{I} & Y_2^T & \bar{Y}_{ps} \\ * & -I & 0 \\ * & * & -\hat{R} \end{bmatrix} < 0, \quad (37)$$

where $\bar{Y}_{ps} = \left[\sqrt{\lambda_{p1}}Y_1^T \ \sqrt{\lambda_{p2}}Y_1^T \ \cdots \ \sqrt{\lambda_{pk_1}}Y_1^T\right]$. By utilizing Lemma 1 of (37), we obtain (29), which ensures that (10) is established. At this point, the proof ends. ∎

B. Unknown actuator fault matrix

Below, we extend the aforementioned design method for the FTAC to the case where the AFM is unknown.

Theorem 3. *Consider the closed-loop 2D MJS (5) that satisfies (6). Given scalars $\underline{\theta}_{k\gamma_{i,j}}$ and $\overline{\theta}_{k\gamma_{i,j}}$, if there exist scalars $\bar{\mu} > 0$, $\sigma > 0$, and a set of matrices $\tilde{R}_p > 0$, $\tilde{Q}_{ps} > 0$, $\tilde{K}_s$, $\tilde{M}_s$, for $\forall p \in \mathcal{K}_1$, $s \in \mathcal{K}_2$, such that (31) and the following condition hold:*

$$\begin{bmatrix} \Sigma & \Sigma_1 & \Sigma_2 \\ * & -\sigma I & 0 \\ * & * & -\sigma I \end{bmatrix} < 0, \quad (38)$$

where

$$\Sigma = \begin{bmatrix} \Omega^1_{ps} & \bar{\Omega}^2_{ps} & \bar{\Omega}^3_{ps} \\ * & -I & 0 \\ * & * & -\hat{R} \end{bmatrix},$$

$$\bar{\Omega}^2_{ps} = \begin{bmatrix} C_pM_s + D_p\Theta_{0p}\tilde{K}_s & 0 & F_p & 0 \\ 0 & C_pM_s + D_p\Theta_{0p}\tilde{K}_s & 0 & F_p \end{bmatrix}^T,$$

$$\bar{\Omega}^3_{ps} = \left[\sqrt{\lambda_{p1}}\hat{Y}^T_{ps} \ \sqrt{\lambda_{p2}}\hat{Y}^T_{ps} \ \cdots \ \sqrt{\lambda_{pk_1}}\hat{Y}^T_{ps}\right],$$

$$\hat{Y}_{ps} = \left[A_{1p}M_s + B_{1p}\Theta_{0p}\tilde{K}_s \ A_{2p}M_s + B_{2p}\Theta_{0p}\tilde{K}_s \ E_{1p} \ E_{2p}\right],$$

$$\Sigma_1 = \begin{bmatrix} W_a\hat{F} \\ 0 \end{bmatrix}, W_a = \begin{bmatrix} \tilde{K}_s^T & 0 \\ 0 & \tilde{K}_s^T \end{bmatrix}, \hat{F} = \begin{bmatrix} \Theta_{1p}^T & 0 \\ 0 & \Theta_{1p}^T \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} 0 & \sigma W_b & \sigma W_c \end{bmatrix}^T, W_b = \begin{bmatrix} D_p^T & 0 \\ 0 & D_p^T \end{bmatrix},$$

$$W_c = \left[\sqrt{\lambda_{p1}}W_d \ \sqrt{\lambda_{p2}}W_d \ \cdots \ \sqrt{\lambda_{pk_1}}W_d\right],$$

$$W_d = \begin{bmatrix} B_{1p} & B_{2p} \end{bmatrix}^T.$$

Then, the system controller designed when the AFM is unknown ensures that the closed-loop 2D MJS (5) is asymptotically mean square stable with $\mathcal{H}_\infty$ noise suppression performance μ, where $\mu = \sqrt{\bar{\mu}}$. Furthermore, the controller gain can be described by (33). The symbols $\tilde{R}_p$ and $\tilde{Q}_{ps}$ have the same meaning as Theorem 2.

Proof: Through Lemma 1, (38) can be equivalent to

$$\Sigma + \sigma^{-1}\begin{bmatrix} W_a \\ 0 \end{bmatrix}\hat{F}\hat{F}^T\begin{bmatrix} W_a \\ 0 \end{bmatrix}^T + \sigma\begin{bmatrix} 0 & W_b & W_c \end{bmatrix}^T\begin{bmatrix} 0 & W_b & W_c \end{bmatrix} < 0. \quad (39)$$

It can be obtained from (39)

$$\Sigma + \sigma^{-1}\begin{bmatrix} W_a \\ 0 \end{bmatrix}\tilde{F}\tilde{F}^T\begin{bmatrix} W_a \\ 0 \end{bmatrix}^T + \sigma\begin{bmatrix} 0 & W_b & W_c \end{bmatrix}^T\begin{bmatrix} 0 & W_b & W_c \end{bmatrix} < 0, \quad (40)$$

where

$$\tilde{F} = \begin{bmatrix} \Delta_p^T & 0 \\ 0 & \Delta_p^T \end{bmatrix}, |\Delta_p| \le \Theta_{1p}.$$

From (4) and (32), we can derive

$$\Sigma + \begin{bmatrix} W_a \\ 0 \end{bmatrix} \tilde{F} \begin{bmatrix} 0 & W_b & W_c \end{bmatrix} + \begin{bmatrix} 0 & W_b & W_c \end{bmatrix}^T \tilde{F}^T \begin{bmatrix} W_a \\ 0 \end{bmatrix}^T < 0. \quad (41)$$

Through Lemma 3, we can ensure the equivalence between (40) and (41), that is, (38) can ensure the establishment of (32). Thus, by (31) and (38), the AMSS and $\mathcal{H}_\infty$ noise suppression performance μ of the closed-loop 2D MJS (5) are guaranteed. Here, the proof ends. ■

V. Application Examples

In this section, we will utilize the Darboux equation [35] to validate the effectiveness of the controllers designed in Theorem 2 and 3, respectively. The dynamics of the Darboux equation can be expressed as

$$\frac{\partial s(x,t)}{\partial x \partial t} = a_{\gamma(x,t)} s(x,t) + b_{\gamma(x,t)} \frac{\partial s(x,t)}{\partial t} + c_{\gamma(x,t)} \frac{\partial s(x,t)}{\partial x} + d_{\gamma(x,t)} s(x,t) f(x,t). \quad (42)$$

Similar to the technique used in [36]. Here we can get the parameter matrices of the 2D MJS based on the FM-II model with two modes

$$A_{1p} = \begin{bmatrix} 1 + b_p \varDelta x & (b_p c_p + a_p) \varDelta x \\ 0 & 0 \end{bmatrix},$$
$$A_{2p} = \begin{bmatrix} 0 & 0 \\ \varDelta t & 1 + c_p \varDelta t \end{bmatrix},$$
$$B_{1p} = \begin{bmatrix} d_p \varDelta x \\ 0 \end{bmatrix}, B_{2p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then, by appropriately selecting other system matrices and setting $\varDelta x = 0.3$, $\varDelta t = 0.35$, $a_1 = 1.2$, $b_1 = -3$, $c_1 = -1$, $d_1 = 0.5$, $a_2 = 0.5$, $b_2 = -1$, $c_2 = -2$, $d_2 = 0.5$, we can obtain

Mode 1:

$$A_{11} = \begin{bmatrix} 0.1 & 1.26 \\ 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 0 \\ 0.35 & 0.65 \end{bmatrix}, B_{11} = \begin{bmatrix} 0.15 \\ 0 \end{bmatrix},$$
$$B_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_{11} = \begin{bmatrix} 0.2 \\ 0.04 \end{bmatrix}, E_{21} = \begin{bmatrix} 0.08 \\ 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.5 & 0.3 \end{bmatrix}, D_1 = \begin{bmatrix} 0.05 \end{bmatrix}, F_1 = \begin{bmatrix} -0.1 \end{bmatrix}.$$

Mode 2:

$$A_{12} = \begin{bmatrix} 0.7 & 0.75 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 \\ 0.35 & 0.3 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.15 \\ 0 \end{bmatrix},$$
$$B_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, D_2 = \begin{bmatrix} 0.05 \end{bmatrix}, F_2 = \begin{bmatrix} -0.2 \end{bmatrix}.$$

Under the asynchronous condition, the transition probability matrix Λ and the conditional probability matrix Π are described as follows:

$$\Lambda = \begin{bmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{bmatrix}, \Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{bmatrix}.$$

The conditional probability matrix Π under synchronization and mode independence are

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Pi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

respectively. Here we give the boundary value and disturbance as

$$x(0,j) = \begin{cases} [0.3 \quad 0.4], 0 \le j \le 10; \\ [0 \quad 0], j > 10. \end{cases}$$
$$x(i,0) = \begin{cases} [0.4 \quad 0.3], 0 \le i \le 10; \\ [0 \quad 0], i > 10. \end{cases}$$
$$w(i,j) = \begin{cases} 1.2, 0 \le i,j \le 10; \\ 0, elsewhere. \end{cases}$$

Under the asynchronous condition, we apply Theorem 2 with $\underline{\theta}_{k1} = \overline{\theta}_{k1} = 0.8$, $\underline{\theta}_{k2} = \overline{\theta}_{k2} = 0.9$ to solve for the control gains, resulting in

$$K_1 = [-2.7502 \quad -7.2514], K_2 = [-2.2424 \quad -8.1222],$$

with the optimal $\mathcal{H}_\infty$ noise suppression performance $\mu^* = 0.2211$. Under the conditions of synchronous, asynchronous, and mode-independent, the optimal performance μ^* of Theorem 2 with $\underline{\theta}_{k1} = \overline{\theta}_{k1} = 0.8$, $\underline{\theta}_{k2} = \overline{\theta}_{k2} = 0.9$ obtained respectively is shown in TABLE I. This table also supports our conjecture: in addressing the problem of actuator failure, the theoretical conditions for the synchronous case are relatively ideal, resulting in the smallest performance μ^*. The results for the asynchronous case account for the additional challenge of system mode mismatch. Although the performance μ^* is larger, the theoretical outcomes are more aligned with actual systems, making the trade-off of using asynchronous control relatively reasonable. The mode-independent case, which does not consider any mode conditions, obviously has the largest performance μ^*.

Similarly, in the asynchronous condition, solving Theorem 3 with $0.8I \le \Theta_1 \le I$, $0.7I \le \Theta_2 \le 0.9I$ (i.e., $\underline{\theta}_{k1} = 0.8$, $\overline{\theta}_{k1} = 1$, $\underline{\theta}_{k2} = 0.7$, $\overline{\theta}_{k2} = 0.9$) yields the corresponding control gains as

$$K_1 = [-2.9377 \quad -7.1765], K_2 = [-2.1360 \quad -7.9596],$$

TABLE I
THE OPTIMAL μ^* CORRESPONDING TO THE THREE SITUATIONS.

synchronous	asynchronous	mode-independent
0.2089	0.2211	0.2247

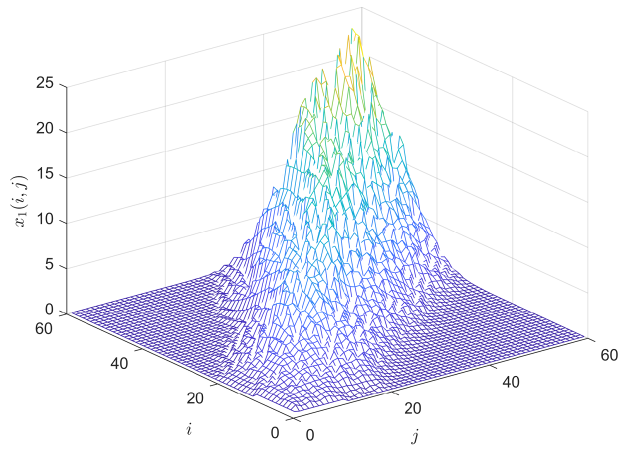

Fig. 1. Trajectory of x_1 of the unforced system.

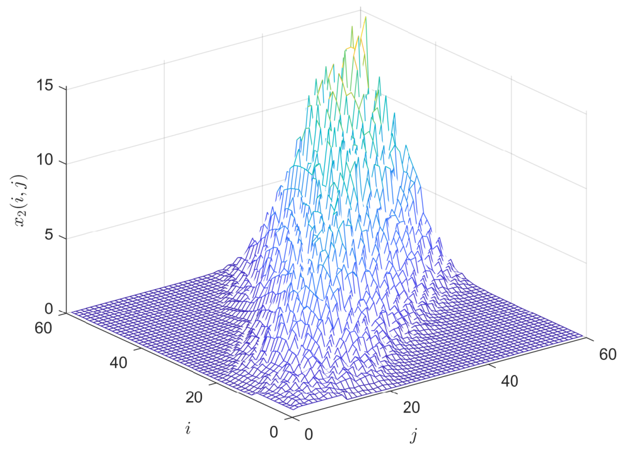

Fig. 2. Trajectory of x_2 of the unforced system.

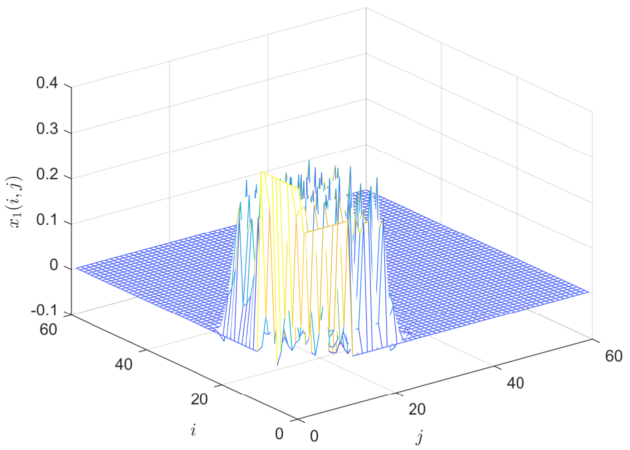

Fig. 3. Trajectory of x_1 under the control input.

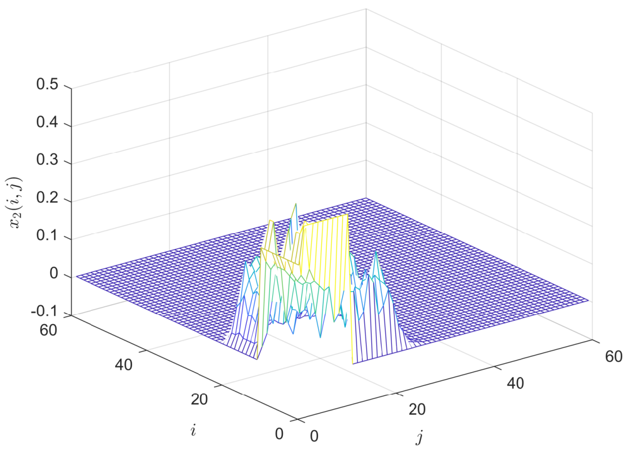

Fig. 4. Trajectory of x_2 under the control input.

which correspond to the optimal $\mathcal{H}_\infty$ performance $\mu^* = 0.2287$. Fig. 1 and Fig. 2 show the state trajectories of the two components of $x(i,j)$ of the system without control input, and the AFM is unknown. It is obvious that this system is unstable. Fig. 3 and Fig. 4 illustrate the trajectories of the two components of $x(i,j)$ of the closed-loop system. It can be observed that the trajectories of state $x(i,j)$ quickly converge to zero under the action of the controller. It is worth noting that, considering the system's equal importance in both directions, so we choose a diagonal traversal method. The specific traversal sequence rule is as follows:

$$\begin{cases} (i,j) < (h,v), & i+j = h+v \text{ and } i < h, \\ (i,j) < (h,v), & i+j < h+v. \end{cases}$$

TABLE II
THE OPTIMAL μ^* UNDER DIFFERENT FAULT DEGREES.

Actuator 1		Actuator 2		μ^*
$\underline{\theta}_{k1}$	$\overline{\theta}_{k1}$	$\underline{\theta}_{k2}$	$\overline{\theta}_{k2}$	
0.8	1	0.7	0.9	0.2287
0.7	0.9	0.6	0.8	0.2309
0.6	0.8	0.5	0.7	0.2345
0.5	0.7	0.4	0.6	0.2416
0.4	0.6	0.3	0.5	0.2607
0.3	0.5	0.2	0.4	0.3797
0.2	0.4	0.1	0.3	$\sim$

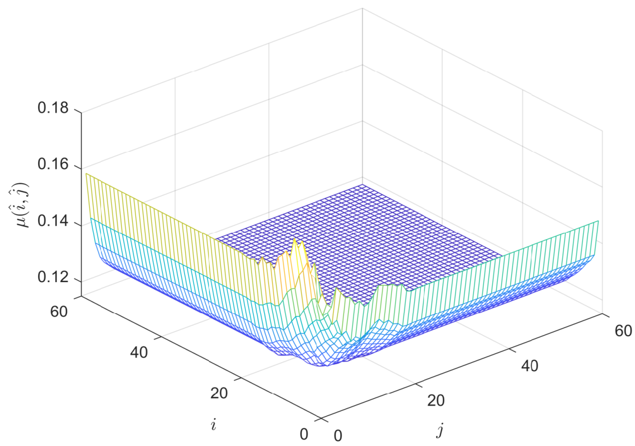

Fig. 5. Binary function $\mu(\hat{i},\hat{j})$ trajectory for Theorem 3 with $\underline{\theta}_{k1}=0.8$, $\overline{\theta}_{k1}=1$ and $\underline{\theta}_{k2}=0.7$, $\overline{\theta}_{k2}=0.9$.

TABLE II shows the optimal performance μ^* obtained by solving Theorem 3 when different actuators have different fault levels. We can observe from TABLE II that as the degree of actuator failure increases, the optimal performance μ^* also increases.

Based on (8), we introduce the following binary equation under the ZBC:

$$\mu(\hat{i},\hat{j}) = \sqrt{\frac{\sum_{i=0}^{\hat{i}} \sum_{j=0}^{\hat{j}} \left\| \begin{bmatrix} y(i,j+1) \\ y(i+1,j) \end{bmatrix} \right\|^2}{\sum_{i=0}^{\hat{i}} \sum_{j=0}^{\hat{j}} \left\| \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \end{bmatrix} \right\|^2}}.$$

The trajectory of the binary function $\mu(\hat{i},\hat{j})$ for Theorem 3 with $\underline{\theta}_{k1}=0.8$, $\overline{\theta}_{k1}=1$ and $\underline{\theta}_{k2}=0.7$, $\overline{\theta}_{k2}=0.9$ is shown in

Fig. 3. The trajectory of $\mu(\hat{i},\hat{j})$ in Fig. 5 converges to 0.1174, which is lower than the optimal performance $\mu^* = 0.2287$ obtained by solving Theorem 3. The above results all verify the effectiveness of the proposed design methods.

VI. Conclusion

This paper investigated the fault-tolerant asynchronous control problem for 2D MJS based on the FM-II mode under actuator failures and mode mismatches. The actuator failures were modeled as norm-bounded uncertainties, and the mode mismatches between the plant and the designed controller were characterized by a hidden Markov model. By employing the Lyapunov direct method, a sufficient condition for the AMSS and $\mathcal{H}_\infty$ noise suppression performance of the closed-loop 2D MJS was obtained. Then, by introducing a slack matrix and scaling to handle nonlinear terms, the FTAC was developed for cases where the AFM is known and unknown, respectively. Finally, the effectiveness of the design methods was validated through the Darboux equation.

References

[1] E. Rogers, K. Galkowski, and D. H. Owens, *Control Systems Theory and Applications for Linear Repetitive Processes*. Berlin, Germany: Springer, 2007.

[2] D. Bors and S. Walczak, "Application of 2D systems to investigation of a process of gas filtration," *Multidimensional Systems and Signal Processing*, vol. 23, no. 1-2, pp. 119–130, 2012.

[3] L. Mitiche and A. B. H. Adamou-Mitiche, "An efficient low order model for two-dimensional digital systems: Application to the 2D digital filters," *Journal of King Saud University-Computer and Information Sciences*, vol. 26, no. 3, pp. 308–318, 2014.

[4] R. Roesser, "A discrete state-space model for linear image processing," *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 1–10, 1975.

[5] E. Fornasini and G. Marchesini, "Doubly-indexed dynamical systems: State-space models and structural properties," *Mathematical Systems Theory*, vol. 12, no. 1, pp. 59–72, 1978.

[6] V. Karthick and V. Suvitha, "An analysis of three servers Markovian multiple vacation queueing system with servers breakdown," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 2, pp. 670–677, 2023.

[7] S. Dong, L. Liu, G. Feng, M. Liu, Z.-G. Wu, and R. Zheng, "Cooperative output regulation quadratic control for discrete-time heterogeneous multiagent Markov jump systems," *IEEE Transactions on Cybernetics*, vol. 52, no. 9, pp. 9882–9892, 2022.

[8] C. Xu, D. Tong, Q. Chen, W. Zhou, and P. Shi, "Exponential stability of Markovian jumping systems via adaptive sliding mode control," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 2, pp. 954–964, 2021.

[9] F. Zuhairoh, D. Rosadi, and A. R. Effendie, "Continuous-time hybrid Markov/semi-Markov model with sojourn time approach in the spread of infectious diseases," *IAENG International Journal of Computer Science*, vol. 50, no. 3, pp. 1108–1114, 2023.

[10] L. Yao, Z. Wang, X. Huang, Y. Li, Q. Ma, and H. Shen, "Stochastic sampled-data exponential synchronization of Markovian jump neural networks with time-varying delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 34, no. 2, pp. 909–920, 2023.

[11] M. Dai, Z. Huang, J. Xia, B. Meng, J. Wang, and H. Shen, "Non-fragile extended dissipativity-based state feedback control for 2-D Markov jump delayed systems," *Applied Mathematics and Computation*, vol. 362, p. 124571, 2019.

[12] Y. Wei, J. Qiu, H. R. Karimi, and M. Wang, "Filtering design for two-dimensional Markovian jump systems with state-delays and deficient mode information," *Information Sciences*, vol. 269, pp. 316–331, 2014.

[13] Y. Wei, J. Qiu, H. R. Karimi, and M. Wang, "Model approximation for two-dimensional Markovian jump systems with state-delays and imperfect mode information," *Multidimensional Systems and Signal Processing*, vol. 26, no. 3, pp. 575–597, 2015.

[14] R. Zhang, Y. Zhang, C. Hu, M.-H. Meng, and Q. He, "Delay-range-dependent $\mathcal{H}_\infty$ filtering for two-dimensional Markovian jump systems with interval delays," *IET Control Theory & Applications*, vol. 5, no. 18, pp. 2191–2199, 2011.

[15] R. Saravanakumar, A. Amini, R. Datta, and Y. Cao, "Reliable memory sampled-data consensus of multi-agent systems with nonlinear actuator faults," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 69, no. 4, pp. 2201–2205, 2022.

[16] Q. Zhu, G. Zhuang, J. Xia, and X. Xie, "Actuator fault and bernoulli random delay-based $\mathcal{H}_\infty$ reliable fuzzy control of singular hybrid vehicle suspension systems with impulsive perturbations," *Information Sciences*, vol. 647, p. 119493, 2023.

[17] C. Ge, C. Chang, Y. Liu, and C. Liu, "Sampled-data-based exponential synchronization of switched coupled neural networks with unbounded delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 117, p. 106931, 2023.

[18] G. Chen, C. Fan, J. Sun, and J. Xia, "Mean square exponential stability analysis for itô stochastic systems with aperiodic sampling and multiple time-delays," *IEEE Transactions on Automatic Control*, vol. 67, no. 5, pp. 2473–2480, 2022.

[19] K. Badie, M. Alfidi, M. Oubaidi, and Z. Chalh, "Parameter-dependent robust $\mathcal{H}_\infty$ filtering for uncertain two-dimensional discrete systems in the FM second model," *IMA Journal of Mathematical Control and Information*, vol. 37, no. 4, pp. 1114–1132, 2020.

[20] X. Qin, J. Dong, X. Zhang, T. Jiang, and J. Zhou, "$\mathcal{H}_\infty$ control of time-delayed Markov jump systems subject to mismatched modes and interval conditional probabilities," *Arabian Journal for Science and Engineering*, vol. 49, no. 5, pp. 7471–7486, 2024.

[21] F. Zuhairoh, D. Rosadi, and A. R. Effendie, "Multi-state discrete-time Markov chain svirs model on the spread of covid-19." *Engineering Letters*, vol. 30, no. 2, pp. 598–608, 2022.

[22] Z.-G. Wu, Y. Shen, P. Shi, Z. Shu, and H. Su, "$\mathcal{H}_\infty$ control for 2-D Markov jump systems in Roesser model," *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 427–432, 2019.

[23] J. Zhou, J. Dong, and S. Xu, "Asynchronous dissipative control of discrete-time fuzzy Markov jump systems with dynamic state and input quantization," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 11, pp. 3906–3920, 2023.

[24] S. Dong and M. Liu, "Adaptive fuzzy asynchronous control for nonhomogeneous Markov jump power systems under hybrid attacks," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 3, pp. 1009–1019, 2023.

[25] X. Ma, J. Dong, W. Tai, J. Zhou, and W. Paszke, "Asynchronous event-triggered $\mathcal{H}_\infty$ control for 2D Markov jump systems subject to networked random packet losses," *Communications in Nonlinear Science and Numerical Simulation*, vol. 126, p. 107453, 2023.

[26] X. Wang, Y. Wang, and W. Hou, "Non-fragile reliable passive control for switched systems using an event-triggered scheme,"

IAENG International Journal of Applied Mathematics, vol. 51, no. 2, pp. 431–438, 2021.

[27] J. Zhou, Y. Liu, J. Xia, Z. Wang, and S. Arik, "Resilient fault-tolerant anti-synchronization for stochastic delayed reaction–diffusion neural networks with semi-Markov jump parameters," *Neural Networks*, vol. 125, pp. 194–204, 2020.

[28] K. Badie, Z. Chalh, and M. Alfidi, "Output tracking control for 2-D discrete systems with actuator failures," *Optimal Control Applications and Methods*, vol. 44, no. 5, pp. 2517–2531, 2023.

[29] L. Wu, R. Yang, P. Shi, and X. Su, "Stability analysis and stabilization of 2-D switched systems under arbitrary and restricted switchings," *Automatica*, vol. 59, pp. 206–215, 2015.

[30] H. Trinh, N. T. Lan-Huong *et al.*, "Delay-dependent energy-to-peak stability of 2-D time-delay Roesser systems with multiplicative stochastic noises," *IEEE Transactions on Automatic Control*, vol. 64, no. 12, pp. 5066–5073, 2019.

[31] C. Yang, J. Liang, and X. Chen, "Distributed event-based $\mathcal{H}_\infty$ consensus filtering for 2-D TS fuzzy systems over sensor networks subject to DoS attacks," *Information Sciences*, vol. 641, p. 119079, 2023.

[32] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994.

[33] E.-K. Boukas, *Control of Singular Systems With Random Abrupt Changes*. Berlin, Germany: Springer, 2008.

[34] L. Xie, M. Fu, C. E. de Souza *et al.*, "$\mathcal{H}_\infty$ control and quadratic stabilization of systems with parameter uncertainty via output feedback," *IEEE Transactions on Automatic Control*, vol. 37, no. 8, pp. 1253–1256, 1992.

[35] W. Marszalek, "Two-dimensional state-space discrete models for hyperbolic partial differential equations," *Applied Mathematical Modelling*, vol. 8, no. 1, pp. 11–14, 1984.

[36] T. Kaczorek, *Two-dimensional Linear Systems*. Berlin, Germany: Springer, 1985.