# On the Crossing Number of the Cartesian Product of a Triangular Snake Graph with Path, Cycle, Star and 3-Vertex Graphs

Mhaid Mhdi Alhajjar, Amaresh Chandra Panda, and Siva Prasad Behera

***Abstract*—The crossing number is a key indicator of the graph's complexity and difficulty in visualization. In this paper, we study the crossing number of the Cartesian product of triangular snake graph $TS_m$ with path $P_n$ (on $n+1$ vertices), cycle $C_n$, star $K_{1,n}$ and 3-vertex graphs. We prove that $cr(TS_5 \times P_n) = 2(n-2)\ \forall\ n \geq 1$ and establish a conjecture for the general case. Moreover, we obtain the crossing number of the graph $TS_m \times C_n$ by proving that $cr(TS_m \times C_n) = n\lfloor\frac{m}{2}\rfloor$ for $m, n \geq 3$. Furthermore, we consider the graph $TS_5 \times K_{1,n}$ and establish $cr(TS_5 \times K_{1,n}) = n(n-1)\ \forall\ n \geq 3$, also a conjecture is provided for the general case. Finally, we extend the study to 3-vertex graphs by proving that $cr(TS_m \times P_2) = \lfloor\frac{m}{2}\rfloor$ for $m \geq 5$.**

***Index Terms*—Crossing Number, Triangular Snake Graph, Subdivision, Edge-Disjoint Subgraphs, Contraction the Edge.**

## I. Introduction

THE smallest number of crossings that occur among all of $G$'s drawings in the plane is known as the $crossing\ number$, or $cr(G)$. The following four prerequisites must be met in order to study this concept:

1) No edge crosses over to itself.
2) The edges that are adjacent do not cross.
3) Two edges can only be crossed at one point at most.
4) Three or more edges do not cross at the same point.

In Figure 1, the aforementioned conditions are clarified.

A drawing is considered *good* if it satisfies those conditions. If $cr_D(G) = cr(G)$, ($cr_D(G)$ is the number of crossings in $D$), then $D$ is known as $optimal$ drawing of $G$. Let $cr_D(u, v)$ be the number of crossings founded among the edges incident on $u$ or $v$ in $D$.

Let $L$ be a graph constructed from $G$ by inserting vertices of degree 2 into $G$'s edges, then $L$ is referred to as a $subdivision$ of $G$ [1]. Clearly, $cr(G) = cr(L)$ and $G$ is planar if and only if $L$ is planar. Every subdivision of $K_5$ and $K_{3,3}$ is therefore nonplanar (based on the nonplanarity of $K_5$ and $K_{3,3}$).

The $Cartesian\ product$ of two graphs $H$ and $L$, denoted by $H \times L$, has vertices set $V(H) \times V(L)$ and edges set $E(H \times L) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2\ and\ v_1v_2 \in E(L)\ or\ v_1 = v_2\ and\ u_1u_2 \in E(H)\}$. Thus for each edge $u_1u_2$ of $H$ and each edge $v_1v_2$ of $L$, there are four edges in $H \times L$ (Figure 2).

Manuscript received April 8, 2024; revised September 12, 2024.

Mhaid Mhdi Alhajjar is an associate researcher in the Department of Mathematics, C.V. Raman Global University, Bhubaneswar, 752054, India. Corresponding author; e-mail: (mhaid.alhajjar1991@gmail.com).

Amaresh Chandra Panda is an associate professor in the Department of Mathematics at C.V. Raman Global University, Bhubaneswar, 752054, India. E-mail: (amaresh.panda@cgu-odisha.ac.in).

Siva Prasad Behera is an assistant professor in the Department of Mathematics at C.V. Raman Global University, Bhubaneswar, 752054, India. E-mail: (sivaprasad@cgu-odisha.ac.in).

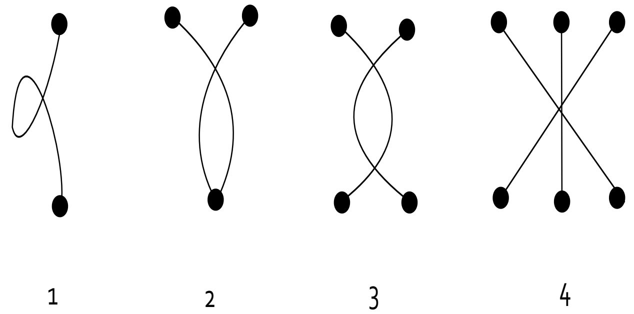


Fig. 1. Prohibited crossings in a good drawing.

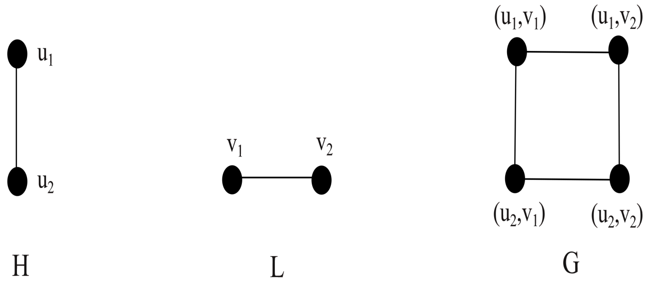


Fig. 2. $G = H \times L$.

A path with $m$ vertices $p_1, p_2, ..., p_m$, $m$ is odd, together with the edges $p_{2i-1}p_{2i+1}$, $1 \leq i \leq \frac{m-1}{2}$, is called the triangular snake graph, or $TS_m$ (Figure 3 for $m = 7$).

In 2012, Rajan et al. [2] investigated $cr$ of the join product of $TS_m$ with an empty graph $nK_1$, $P_n$ (on $n$ vertices) and $C_n$. They established $cr(TS_m + nK_1) = Z(m, n) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor$, $cr(TS_m + P_n) = Z(m, n) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ and $cr(TS_m + C_n) = Z(m, n) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor + 2 : m \geq 5$ and $n \leq 6$. In 2023, Alhajjar et al. [12] began to study a new case in relation to the strong product of path with $TS_n$ by proving that $cr(P_2 \boxtimes TS_m) = 3\lfloor\frac{m}{2}\rfloor$ for $m \geq 3$.

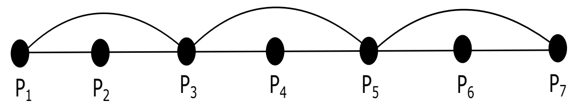


Fig. 3. The graph $TS_7$.

There are many applications of this concept, for instance, the process of creating integrated circuits using VLSI (very large scale integration) involves combining millions of transistors onto a single chip. In fact, VLSI allows integrated circuits to perform a wide range of functions on a single chip, whereas before VLSI, most integrated circuits had a more limited range of functions. In VLSI, the chip is represented as a graph, with vertices representing macro cells and edges representing wires connecting the vertices. The chip designer's task is to arrange macro cells and wires on a circuit board, with the aim of avoiding crossings between the wires because we have to change one of the wires' layers when two wires cross. Such a change in layer is known as a contact cut, and it is best to use as few contact cuts as possible because doing so forces the chip to use more area. In the context of the graph, this means fewer edge crossings overall, see [5]. Furthermore, Székely [9] and Pach et al. [7] provided multiple discrete geometry proofs based on crossing number theory findings. Purchase [8] clarified that the most significant aesthetic quality is the crossing number. To be more precise, graph drawings with as few crossings as possible are actually easier to understand. We may refer to [11] for extra information and terminology about crossing number theory.

## II. $cr(TS_m \times P_n)$

Let $P_n$ be the $Path$ on $n+1$ vertices, there are many studies regarding the class of the Cartesian product of path with other graphs, for example:

1) $cr(K_{1,m} \times P_n)$ was calculated by Bokal (2007) [3] using the following formula: $cr(K_{1,m} \times P_n)= (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$ for ($m \geq 3$ and $n \geq 1$).
2) The following conjecture was introduced by Zheng et. al. (2007) [17] regarding $cr(K_m \times P_n)$:
   **Conjecture:** $cr(K_m \times P_n) = \frac{1}{4}\lfloor\frac{m+1}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{m-2}{2}\rfloor(n\lfloor\frac{m+4}{2}\rfloor + \lfloor\frac{m-4}{2}\rfloor)$ ; $n \geq 1$.
   In 2014, it was proven by [16] for $m \leq 10$.
3) Wang et al. (2009) [6] considered the graph $W_m \times P_n$ and proved that: $cr(W_m \times P_n)= (n-1)\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + n + 1 : m \geq 3$ , $n \geq 1$.
4) In 2011, Zheng et al. [14] investigated $cr(W_{2,m} \times P_n)$, they established: $cr(W_{2,m} \times P_n)= 2n\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor + 2n : m \geq 3, n \geq 1$.
5) In 2022, Alhajjar et al. [10] started to study $cr(S_m \times P_n)$ where $S_m$ is a sunlet graph, they proved that: $cr(S_m \times P_2)= m : m \geq 3$, also they came up with the following conjecture:
   **Conjecture:** $cr(S_m \times P_n)= m(n-1) : m, n \geq 3$.

In this section, we extend this class of graphs by studying $cr(TS_m \times P_n)$. It is easier to label the vertex $w_{s,t}$ instead of $(p_s, v_t)$ where $s = 1, ..., m$, $t = 1, ..., n+1$ in $TS_m \times P_n$.

For $m = 3$, $TS_3 \times P_n$ is a planar graph (Figure 4).

**Theorem II.1.** $cr(TS_5 \times P_n) = 2(n-1)$ *for* $n \geq 1$.
**Proof:**
*Let us consider $D$ as a good drawing of $TS_5 \times P_n$ and $Q_t$ as a subgraph of $D$ induced by the vertices $\{w_{s,t}, w_{s,t+1}, w_{s,t-1} : s = 1,2,3,4,5\}$ for $t = 2, ..., n-1$. Hence, $TS_5 \times P_n = Q_2 \bigcup Q_3 \bigcup ... \bigcup Q_{n-1}$. Note that $Q_t$ has a subgraph which is a subdivision of $K_{3,4}$, thus $cr(Q_t) \geq 2$.*

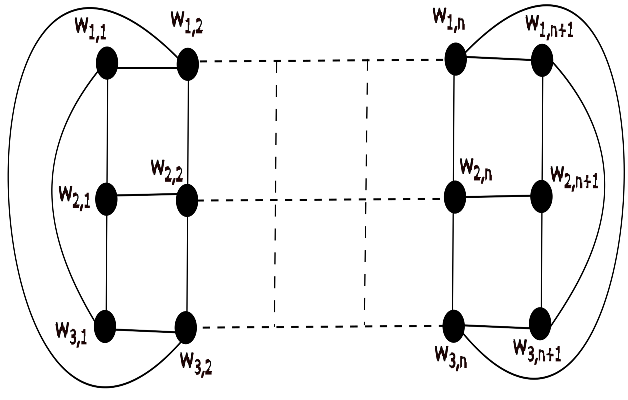


Fig. 4. The graph $TS_3 \times P_n$.

*Let $Q_t^{'}$ be the subgraph of $Q_t$ which is a subdivision of $K_{3,4}$, it is easy to check that $V(Q_t^{'}) = \{w_{3,t-1}, w_{3,t}, w_{3,t+1}\} \cup \{w_{1,t}, w_{2,t}, w_{4,t}, w_{5,t}\} \cup I(Q_t^{'})$ for $t = 2, ..., n-1$, where $I(Q_t^{'})$ is the set of two-degree inserted vertices in $Q_t^{'}$. (An example shown in Figure 5 for $j = 2$).*

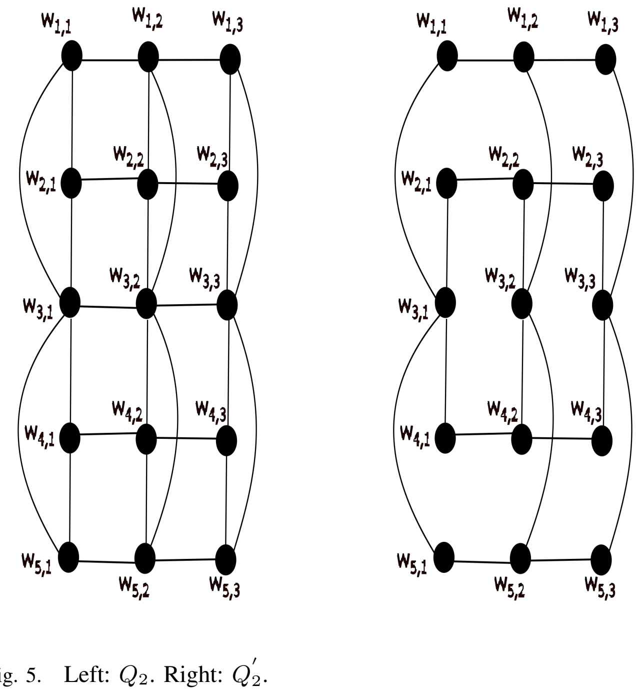


Fig. 5. Left: $Q_2$. Right: $Q_2^{'}$.

*The rest of the proof consists of demonstrating that each $Q_t^{'}$ has at least two crossings that do not occur in any other $Q_k^{'}$ $(t \neq k)$, this is known as the Counting Argument [4].*

*In $Q_t^{'}$, a crossing denoted by $r$ is formed by the intersection of two edges $e$ and $f$ where $e \neq f$ (essentially, $K_{2,2}$ forms every crossing), if $r$ occurs in more than one subgraph of $Q_t^{'}$, then the edges $e$ and $f$ must belong to each one of $Q_t^{'}$. Since the two-degree inserted vertices have no effect regarding the number of crossings in $Q_t^{'}$, then we can treat them as points from the edges of $Q_t^{'}$, therefor we can write:*

$cr_D(Q_t^{'}) = cr_D(w_{3,t-1}, w_{3,t}) + cr_D(w_{3,t-1}, w_{3,t+1}) +$

$cr_D(w_{3,t}, w_{3,t+1}) \geq 2 \qquad (1)$
*(In general, each drawing of $K_{m,n}$ can have a total of crossings counted in the following way: $\sum_{s=1}^{m-1}\sum_{k=s+1}^{m} cr(v_s, v_k)$, where $v_s, v_k$ belong to the set of vertices $m$).*

*The formula $(1)$ thus leads to the conclusion that changing the variable $t$ results in at least two crossings in each $Q_t^{'}$ that do not occur in any $Q_k^{'}$ $(t \neq k)$. As a result, there will be at least $2(n-1)$ crossings in $D$. Figure 6 exhibits a drawing of $TS_5 \times P_n$ with $2(n-1)$ crossings, it follows that $cr(TS_5 \times P_n) = 2(n-1)$.* □

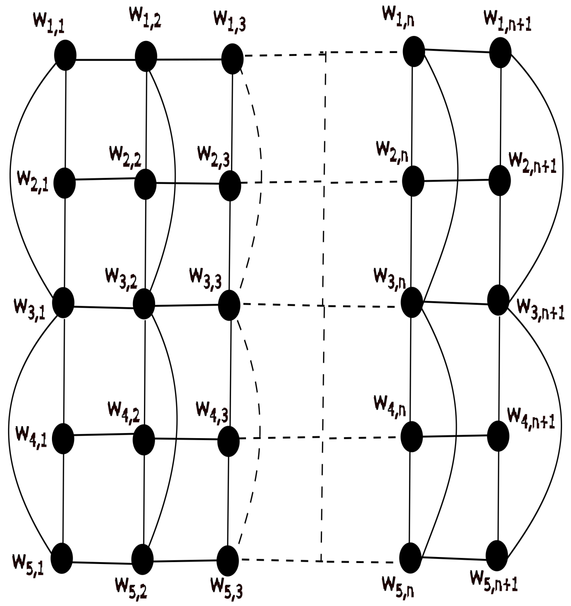


Fig. 6. A drawing of $TS_5 \times P_n$ with $2(n-1)$ crossings.

The natural extension obtained from Figure 6 exhibits a drawing of $TS_m \times P_n$ with $(n-1)\lfloor\frac{m}{2}\rfloor$ crossings, i.e. $cr(TS_m \times P_n) \leq (n-1)\lfloor\frac{m}{2}\rfloor$.

**Conjecture 2.1** $cr(TS_m \times P_n) = (n-1)\lfloor\frac{m}{2}\rfloor : m \geq 7$ and $n \geq 1$.

## III. $cr(TS_m \times C_n)$

Let $C_n$ be the $Cycle$ on $n$ vertices. The following theorem determines the value of $cr(TS_m \times C_n)$:

**Theorem III.1.** *$cr(TS_m \times C_n) = n\lfloor\frac{m}{2}\rfloor$ for $m, n \geq 3$ , $m$ is odd.*

**Proof:**
*The graph $TS_m$ consists of $\lfloor\frac{m}{2}\rfloor$ edge-disjoint subgraphs of $C_3$, therefore $cr(TS_m \times C_n) \geq n\lfloor\frac{m}{2}\rfloor$ for $m, n \geq 3$, $m$ is odd. (Recall that $cr(C_3 \times C_n) = n : n \geq 3$, [13]). Figure 7 exhibits a drawing of $TS_m \times C_n$ with $n\lfloor\frac{m}{2}\rfloor$ crossings, this applies $cr(TS_m \times C_n) = n\lfloor\frac{m}{2}\rfloor$ and the proof is completed.* □

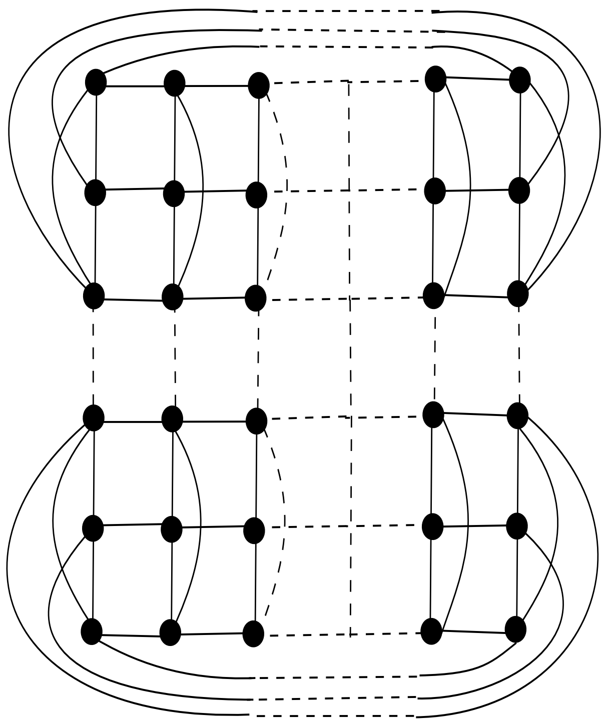

Fig. 7. The natural extension of $TS_m \times C_n$ with $n\lfloor\frac{m}{2}\rfloor$ crossings.

## IV. $cr(TS_m \times K_{1,n})$

The complete bipartite graph $K_{1,n}$ is known as the star graph. We only deal with $n \geq 3$ because $K_{1,1}$ and $K_{1,2}$ are isomorphic to $P_1$ and $P_2$ respectively. Let $v_0$ be the vertex with degree $n$ and $v_j$ be the vertex with degree 1 for $j = 1, ..., n$ in the graph $K_{1,n}$. In this section, we present new results related to $cr(TS_m \times K_{1,n})$, $m$ is odd and $n \geq 3$.

**Lemma IV.1.** $cr(TS_3 \times K_{1,n}) = \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor \ \forall \ n \geq 3$.

**Proof:**
*The subgraph $P_2$ is contained in $TS_3$, thus $cr(TS_3 \times K_{1,n}) \geq cr(P_2 \times K_{1,n}) = \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor \ \forall \ n \geq 3$, [3]. In fact, it is simple to observe that the graph $TS_3 \times K_{1,n}$ has a subgraph which is a subdivision of $K_{3,n}$. In Figure 8, there is the drawing of $TS_3 \times K_{1,n}$ with $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings. The claim follows.* □

**Theorem IV.1.** $cr(TS_5 \times K_{1,n}) = n(n-1) \ \forall \ n \geq 3$.

**Proof:**
*$TS_5$ contains $K_{1,4}$ as a subgraph, hence $cr(TS_5 \times K_{1,n}) \geq cr(K_{1,4} \times K_{1,n}) = n(n-1) \ \forall \ n \geq 3$, [15]. In the rest of the proof, we will show a drawing $D$ of $TS_5 \times K_{1,n}$ with $n(n-1)$ crossings exactly. Let us construct a drawing $D$ of $TS_5 \times K_{1,n}$ as follows:*

1) *For each $i = 1, ..., 5$, place the vertices $w_{i,0}$ on the $y-axis$ equally around the center.*
2) *If $j$ is even, then place the vertices $\{w_{1,j}, w_{2,j}, ..., w_{5,j}\}$ on the vertical line $x = \frac{j}{2}$ equally around the $x-axis$.*

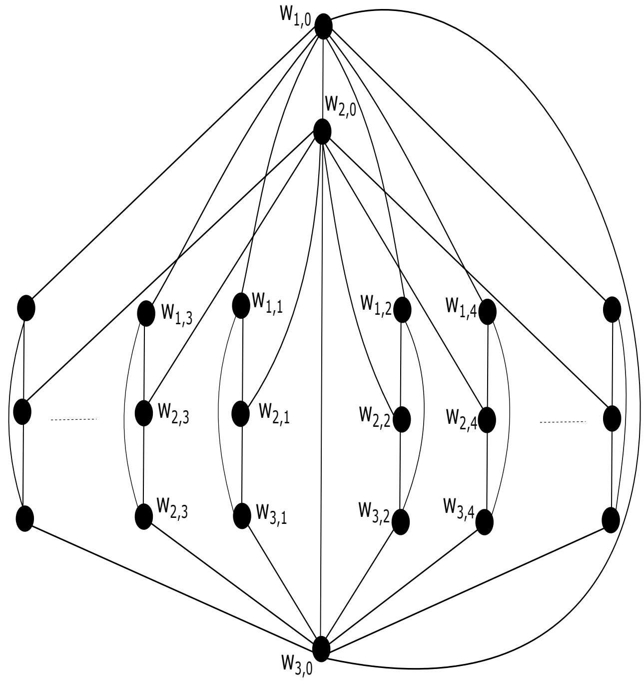


Fig. 8. A drawing of $TS_3 \times K_{1,n}$ with $\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings.

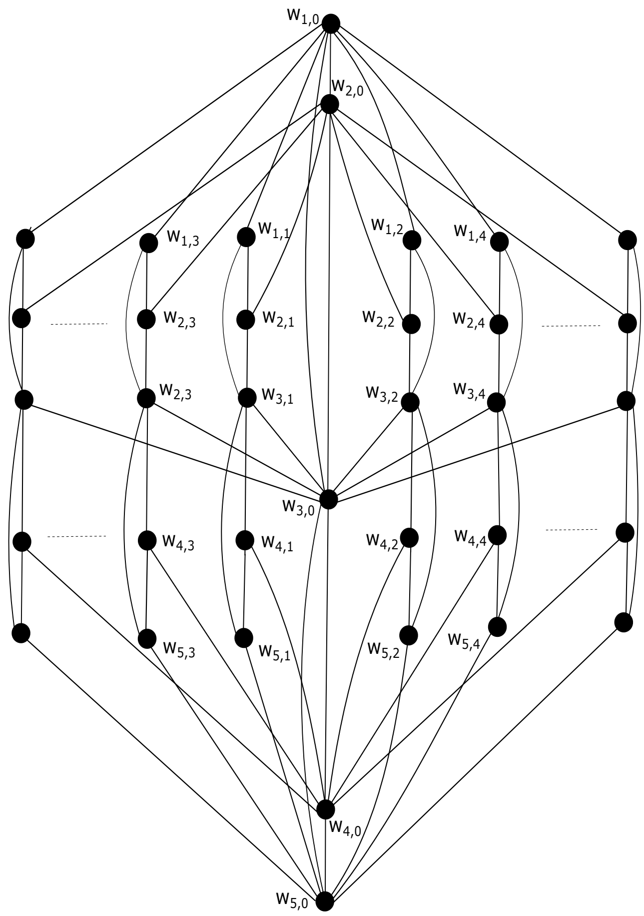


Fig. 9. A drawing of $TS_5 \times K_{1,n}$ with $n(n-1)$ crossings.

3) *If j is odd, then place the vertices* $\{w_{1,j}, w_{2,j}, ..., w_{5,j}\}$ *on the vertical line* $x = -\dfrac{j+1}{2}$ *equally around the* $x - axis$.

*Figure 9. Note that $D$ has a subgraph which is a subdivision of $K_{5,n}$ by deleting the set of the egdes $F = \{w_{1,j}w_{2,j}, w_{4,j}w_{5,j}, w_{1,0}w_{3,0}, w_{2,0}w_{3,0}, w_{3,0}w_{4,0}, w_{3,0}w_{5,0} : j = 0, 1..., n\}$. In fact, this forces $Z(5, n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings [4], moreover it is not difficult to find that the edges $w_{1,0}w_{3,0}$ and $w_{3,0}w_{5,0}$ contribute with $2\lfloor\frac{n}{2}\rfloor$ crossings in the presented drawing $D$ which are different from the crossings formed by the subdivision of $K_{5,n}$. Thus $cr_D(TS_5 \times K_{1,n}) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor = n(n-1)$, therefor $cr(TS_5 \times K_{1,n}) = n(n-1)$ for $n \geq 3$. The proof is completed.* □

The natural extension obtained from Figure 9 exhibits a drawing of $TS_m \times K_{1,n}$ with $4\lfloor\frac{m-2}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor = 4\lfloor\frac{m-2}{2}\rfloor Z(n) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ crossings, i.e. $cr(TS_m \times K_{1,n}) \leq 4\lfloor\frac{m-2}{2}\rfloor Z(n) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor$.

**Conjecture 4.1** $cr(TS_m \times K_{1,n}) = 4\lfloor\frac{m-2}{2}\rfloor Z(n) + \lfloor\frac{m}{2}\rfloor\lfloor\frac{n}{2}\rfloor$ for $m \geq 7$ and $n \geq 3$.

## V. $cr(TS_m \times 3 - vertex\ graphs)$

The path $P_2$ (on three vertices) and the cycle $C_3$ are the only two connected non-isomorphic graphs on 3 vertices. By Theorem III.1, $cr(TS_m \times C_3) = 3\lfloor\frac{m}{2}\rfloor$ for $m \geq 3$, $m$ is odd.

$Contraction$ the edge $cf$ in the graph $G$ is the process of merging it into a single vertex $w$ so that $w$ is adjacent to every vertex in $G$ that is adjacent to $c$ or $f$ in $G$, (Figure 10).

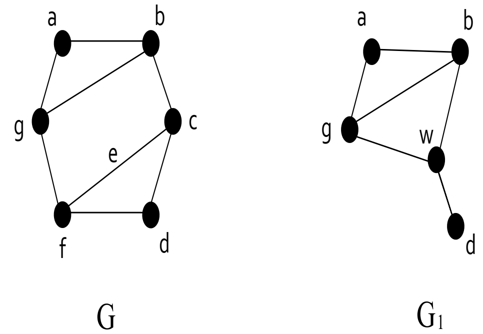


Fig. 10. Contraction the edge $e$ in the graph $G$ results in the graph $G_1$.

**Theorem V.1.** *$cr(TS_m \times P_2) = \lfloor\frac{m}{2}\rfloor$ for $m \geq 5$, $m$ is odd.*

**Proof:**

*Figure 11 exhibits a drawing of $TS_m \times P_2$ with $\lfloor\frac{m}{2}\rfloor$ crossings, thus $cr(TS_m \times P_2) \leq \lfloor\frac{m}{2}\rfloor$. Let us prove this inequality for the inverse direction.*

*By contracting the edges $(w_{i,1}, w_{i+1,1})$ as well as $(w_{i,3}, w_{i+1,3})$ for $i = 1, 2, ..., m-1$, we obtain a drawing of $TS_m + 2K_1$ which has $\lfloor\frac{m}{2}\rfloor$ crossings* [2], therefore: $cr(TS_m \times P_2) \geq cr(TS_m + 2K_1) = \lfloor\frac{m}{2}\rfloor$. The proof is completed. □

## VI. Conclusions

We introduced the conjecture in which $cr(TS_m \times P_n) = (n-1)\lfloor\frac{m}{2}\rfloor$ crossings and we proved that it is true when

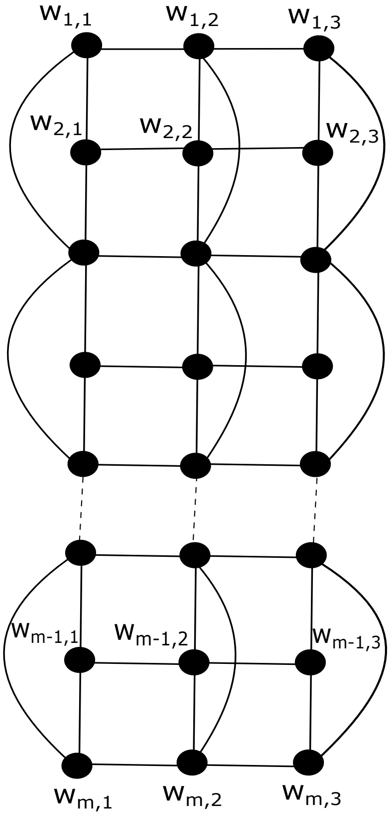


Fig. 11. A drawing of $TS_m \times P_2$ with $\lfloor \frac{m}{2} \rfloor$ crossings.

$m = 3, 5, n \geq 1$. Moreover, we could identify $cr(TS_m \times C_n)$ by proving that for a given $m, n \geq 3$, $m$ is odd: $cr(TS_m \times C_n) = n\lfloor \frac{m}{2} \rfloor$. In addition, we could introduce a conjecture that $cr(TS_m \times K_{1,n})$ equals $4\lfloor \frac{m-2}{2} \rfloor Z(n) + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor$ crossings, and we succeeded in proving that it is true for ($m = 5$ and $n \geq 3$). Finally, we extended the study to 3-vertex graphs by proving that $cr(TS_m \times P_2) = \lfloor \frac{m}{2} \rfloor$ for $m \geq 5$, $m$ is odd.

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