The Scalar Auxiliary Variable Method for the Extended Fisher-Kolmogorov Equation

Danchen Zhu, Qianfeng Qian*, Jing Jiang, Yanping Qiu

***Abstract*—In this paper, we study the numerical approximation of the extended Fisher-Kolmogorov (EFK) equation. First of all, a novel scheme is obtained by introducing the scalar auxiliary variables (SAV) method, which is to treat the nonlinear term, then we only need to solve an independent linear equation. The presented scheme is highly efficient and easy-to-implement. In addition, the energy stability is proved rigorously. Under the corresponding regularity assumption, we prove that the error estimate for Φ in the L^2-norm is able to achieve first-order convergence rate in time and the $O(h^r)(r \geq 1)$ in space. The non-local variable r also achieve the same convergence rate. In the last, the accuracy and energy stability of the proposed scheme are verified by numerical simulations.**

***Index Terms*—SAV method, Error estimate, First-order scheme, EFK equation.**

I. Introduction

THE EFK model has been widely used in the study of physical, material, and biological systems [3], [8], [16], [17], such as the propagation of magnetic region walls in liquid crystals [17] and the growth of certain types of primary brain tumors [3]. Two-dimensional extensions of the Fisher-Kolmogorov equation have also appeared in various applications, such as the formation of bistable systems, traveling waves in reactivity diffusion systems [1], [2] and mesoscale models of phase transitions in binary systems near Lipschitz points [8]. In particular, in phase transitions near critical points (Lipschitz points), the fourth derivative becomes important. In recent years, people began to pay attention to the steady-state equation of (2) in [14]. Consider the following free energy (Lyapunov) functional

$$E[\Phi] = \int_\Omega \left(\frac{Y}{2}|\Delta\Phi|^2 + \frac{1}{2}|\nabla\Phi|^2 + F(\Phi) \right) dx, \tag{1}$$

where the spatial domain is $\Omega = (0, L)^2 \subset \mathbb{R}^2$, the parameter $\gamma > 0$ is a positive constant, and $F(\Phi) = \frac{1}{4}\left(1 - \Phi^2\right)^2$ is the bistable type admitting two local minima. Mathematically, the governing system of the EFK model could be derived via an L^2 gradient flow associated with the energy functional $E[\Phi]$, that is,

$$\begin{cases} \partial_t\Phi = -\dfrac{\delta E}{\delta\Phi} = \Delta\Phi - \gamma\Delta^2\Phi - f(\Phi), \\ \Phi(x,0) := \Phi_0(x), \end{cases} \tag{2}$$

subjected to periodic boundary conditions , and the nonlinear bulk $f(v) = (v)^3 - v$. An inherent property of the EFK model is the law of energy dissipation,

$$\frac{dE}{dt} = \left(\frac{\delta E}{\delta\Phi}, \partial_t\Phi\right) = -\|\partial_t\Phi\|^2_{L^2} \leq 0, \tag{3}$$

in which $(f, g) := \int_\Omega fg dx$. The associated L^2 norm $\|f\|_{L^2} = \sqrt{(f, f)}$ for all $f, g \in L^2(\Omega)$, where $T = (0, t_{end})$ is a finite time interval and $\Omega \in \mathbb{R}^2$ is a polygonally bounded domain with boundary $\partial\Omega$ subdivided into Dirichlet $\partial\Omega_D$ and Neumann $\partial\Omega_N$ parts.

In fact, in recent years, there has been a lot of research on EFK in terms of computational research. For example, Pani and Danumjaya [5] first adopted the one-dimensional solution of OCSC formula to solve equation (2). Next, Danumjaya and Pani [6] used finite element method to consider the uniqueness and existence. Arora and Mittal [13] proposed a five-dollar B-spline allocation method to solve EFK equation. Ismail et al. [11] proposed the three-stage linearized difference method to solve equation (2). Later, Ismail et al. [10] proposed a third-order linearized high-order precision difference scheme for simulation, and wavelet collocation method [12] was used to solve two-dimensional EFK equation, but theoretical derivation and analysis were not carried out. It is worth noting that although there has been a lot of research on EFK equations, solving EFK models is still a challenge.

It is very challenging to construct efficient and easy-to-solve numerical scheme. Inspired by [9], this paper uses SAV method to deal with nonlinear term, and presents an efficient and easy to solve numerical scheme. The implicit-explicit Euler scheme is used for semi-discretization in time, while the spatial finite element method is used for full discretization, and the SAV method is used to linearize the nonlinear term. We strictly prove that the scheme satisfies the law of energy dissipation and error estimation.

This paper is organized as follows. In section II, we propose the scheme to solve the EFK equation and we strictly prove that the scheme satisfies the energy dissipation rate and that it can achieve first-order convergence in time and space. The numerical results in Section III demonstrate the effectiveness of the scheme for solving two-dimensional EFK equation. Some conclusions and future research are drawn in Section IV.

II. Numerical scheme

In this section, we give the equivalent system and a fully discrete scheme for the EFK model (2). We can define the

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Danchen Zhu is a graduate student of the Guangxi Normal University, Guilin Guangxi 541006, P.R. China (e-mail: zhudanchen@stu.gxnu.edu.cn).

Qianfeng Qian is a graduate student of the Guangxi Normal University, Guilin Guangxi 541006, P.R. China (Corresponding author to provide phone: 199-958-18673; fax: +86-020-17879531504; e-mail: 3380246257@qq.com).

Jing Jiang is a graduate student of the Guangxi Normal University, Guilin Guangxi 541006, P.R. China (e-mail:792057525@qq.com).

Yanping Qiu is a graduate student of the Guangxi Normal University, Guilin Guangxi 541006, P.R. China (e-mail:yanpqiu1618@163.com).

nonlocal variable $r(t)$,

$$r(t)=\sqrt{\int_\Omega F(\Phi)dx+C_0}:=\sqrt{E_1(\Phi)}, \quad (4)$$

where C_0 is a positive constant such that the radicand is positive. Then, we can define

$$H(\Phi)=\frac{f(\Phi)}{r(t)}. \quad (5)$$

Using the above nonlocal variables, we have a new but equivalent system as follows:

$$\begin{cases}\partial\Phi_t=\Delta\Phi-\gamma\Delta^2\Phi-H(\Phi)r(t),\\ \partial r_t=\frac{1}{2}\int_\Omega H(\Phi)\partial_t\Phi dx,\end{cases} \quad (6)$$

with the initial conditions as $\Phi(x,0):=\Phi_0(x)$.

A. Fully-discrete scheme

In this section, we give a fully discrete scheme for the EFK model. Assuming that the polygonal/polyhedral domain Ω is discretized by a conforming and shape regular triangulation/tetrahedron mesh $\mathcal{T}_h$ that is composed by open disjoint elements K such that $\bar{\Omega}=\bigcup_{K\in T_h}\bar{K}$. We use $\mathcal{P}_r$ to denote the space of polynomials of total degree at most r and $d=2,3$. For $r\geq 1$, we introduce several conforming finite element spaces as follows:

$$\Psi_h=\{\psi\in C(\Omega):\psi|_K\in\mathcal{P}_r(K),\forall K\in\mathcal{T}_h\}. \quad (7)$$

Let $N>0$ denote the total number of time steps, we define the uniform time step size as $\tau=\left[\frac{T}{N}\right]$ and set $t_n=n\Delta t$, $n=1,\ldots,N$. The L^2 inner product of any two functions $\phi(x)$ and $\psi(x)$ is denoted by $(\phi(x),\psi(x))=\int_\Omega\phi(x)\psi(x)dx$, and the L^2 norm of $\phi(x)$ is denoted by $\|\phi\|^2=(\phi,\phi)$. Let ψ^n be the numerical approximation to the function $\psi(\cdot,t)|_{t=t^n}$. Then, the semi-discrete numerical scheme for (2) reads as:

$$\begin{cases}\frac{\Phi^n-\Phi^{n-1}}{\Delta t}=\Delta\Phi^n-\gamma\Delta^2\Phi^n-r^nH\left(\Phi^{n-1}\right),\\ r^n-r^{n-1}=\frac{1}{2}\int_\Omega H\left(\Phi^{n-1}\right)\left(\Phi^n-\Phi^{n-1}\right)dx.\end{cases} \quad (8)$$

Then, the fully discrete numerical scheme for (2) reads as: find $\Phi_h^n\in\Psi_h$ and $r_h^n\in\mathbb{R}^d$ such that

$$\begin{cases}\left(\frac{\Phi_h^n-\Phi_h^{n-1}}{\Delta t},\psi_h\right)=-\left(\nabla\Phi_h^n,\nabla\psi_h\right)-\gamma\left(\Delta\Phi_h^n,\Delta\psi_h\right)\\ \qquad\qquad-\left(r^nH\left(\Phi_h^{n-1}\right),\psi_h\right),\\ r_h^n-r_h^{n-1}=\frac{1}{2}\int_\Omega H\left(\Phi_h^{n-1}\right)\left(\Phi_h^n-\Phi_h^{n-1}\right)dx.\end{cases} \quad (9)$$

In this paper, we assume that the solution of the EFK equation exists and satisfies the regularities.

Assumption 2.1. We assume the following regularity holds:

$$\begin{aligned}\Phi\in&H^2\left(0,T;L^2(\Omega)\right)\cap H^1\left(0,T;H^{r+1}(\Omega)\right)\\ &\cap C\left(0,T;W^{2,4}(\Omega)\right)\end{aligned} \quad (10)$$

and

$$\partial_{tt}\phi\in L^2\left(0,T;H^1(\Omega)\right),\qquad \partial_t\Phi\in L^2\left(0,T;H^{r+1}(\Omega)\right). \quad (11)$$

Assumption 2.2. Let $\Phi_h^{n+1}\in\Psi_h$ be the unique solution of (9). The following estimates hold for all $\tau,h>0$,

$$\max_{1\leq n\leq N}\left\|\Phi_h^n\right\|_{H^1}\leq C, \quad (12)$$

$$\left((\Delta t)\sum_{n=1}^N\left\|(\Delta t)^{-1}\left(\Phi_h^n-\Phi_h^{n-1}\right)\right\|_{L^2}\right)^{\frac{1}{2}}\leq C, \quad (13)$$

where C is a positive constant independent of Δt and h.

Let $R_h:H^1(\Omega)\to\Psi_h$ be a classic Ritz projection defined by [15],

$$\left(\nabla\left(\psi-R_h\Psi\right),\nabla\varphi_h\right)=0, \quad (14)$$

for all $\varphi_h\in\Psi_h$ with $\int_\Omega\left(\psi-R_h\psi\right)dx=0$.

Lemma 2.1: [4] Following finite element theory, it holds that

$$\|\psi-R_h\psi\|_{L^2}+h\|\psi-R_h\psi\|_{H^1}\leq Ch^r\|\psi\|_{H^r}, \quad (15)$$

$$\begin{aligned}&\left\|(\Delta t)^{-1}\left(\left(\psi^n-R_h\psi^n\right)-\left(\psi^{n-1}-R_h\psi^{n-1}\right)\right)\right\|\\ &\quad\leq Ch^r\left\|(\Delta t)^{-1}\left(\psi^n-\psi^{n-1}\right)\right\|_{H^{r+1}},\end{aligned} \quad (16)$$

where $r\geq 1$.

Lemma 2.2: The regularity hypothesis (10), then we have the following estimate

$$\left\|H\left(\Phi^n\right)-H\left(\Phi_h^{n-1}\right)\right\|^2\leq C\Delta t^2. \quad (17)$$

Proof. By direct calculation, we deduce

$$\begin{aligned}\left\|H\left(\Phi^{n+1}\right)-H\left(\Phi_h^*\right)\right\|&=\left\|\frac{f\left(\Phi^n\right)}{\sqrt{E_1\left(\Phi^n\right)}}-\frac{f\left(\Phi_h^{n-1}\right)}{\sqrt{E_1\left(\Phi_h^{n-1}\right)}}\right\|\\ \leq\left\|f\left(\Phi_h^{n-1}\right)\right\|&\frac{\left|E_1\left(\Phi_h^{n-1}\right)-E_1\left(\Phi^n\right)\right|}{\sqrt{E_1\left(\Phi^n\right)E_1\left(\Phi_h^{n-1}\right)}\left(E_1\left(\Phi^n\right)+E_1\left(\Phi_h^{n-1}\right)\right)}\\ &+\frac{\left\|f\left(\Phi^n\right)-f\left(\Phi_h^{n-1}\right)\right\|}{\sqrt{E_1\left(\Phi^n\right)}}\\ \leqslant C\Delta t,\end{aligned} \quad (18)$$

where together with (11), we get the desired result.

Lemma 2.3: $\partial_{tt}\left(\Phi_h^n\right)$, satisfy the regularity hypothesis (11), then we have the following estimate

$$\Delta t\left\|E_r^{n+1}\right\|_{L^2}^2\leqslant C(\Delta t)^2. \quad (19)$$

Proof. By using Taylor expansion, we have

$$\begin{aligned}r^{n-1}&=r^n-\Delta t\frac{\partial r^n}{\partial t}+\frac{(\Delta t)^2}{2}\frac{\partial^2r^n}{\partial t^2}+o\left((\Delta t)^3\right),\\ \left\|E_r^{n+}\right\|^2&=\left\|\partial_tr^n-\frac{r^n-r^{n-1}}{\Delta t}\right\|^2\\ &\leqslant C(\Delta t)^2\left\|\frac{\partial^2r^n}{\partial t^2}\right\|^2_{L^2\left(t^{n-1},t^n;H^1(\Omega)\right)}\\ &\leq C(\Delta t)^2.\end{aligned} \quad (20)$$

B. Energy dissipation law

Theorem 2.1: The system (8) satisfies the following discrete energy dissipation law:

$$E^n - E^{n-1} = \frac{1}{2}\|\Phi^n\|^2 - \frac{1}{2}\left\|\Phi^{n-1}\right\|^2 + 2\Delta t|r^n|^2 - 2\Delta t\left|r^{n-1}\right|^2 \leq -\frac{\Delta t}{2}\|\nabla\Phi^n\|^2 - \frac{\gamma\Delta t}{2}\|\Delta\Phi^n\|^2, \tag{21}$$

where $\|\cdot\|$ denotes the discrete L^2 norm in domain Ω.

Proof. Taking the first equation of (8) and $\Delta t\Phi^n$ by the inner product of L^2, we obtain

$$\begin{aligned}&\frac{1}{2}\|\Phi^n\|^2 - \frac{1}{2}\left\|\Phi^{n-1}\right\|^2 + \frac{1}{2}\left\|\left(\Phi^n - \Phi^{n-1}\right)\right\|^2 + \frac{\Delta t}{2}\|\nabla\Phi^n\|^2\\&+\frac{\gamma\Delta t}{2}\|\Delta\Phi^n\|^2 + \Delta t\left(H\left(\Phi^{n-1}\right)r^n, \Phi^n + \Phi^{n-1}\right)\\&= 0.\end{aligned} \tag{22}$$

Taking the second equation of (8) and r^n by the inner product of L^2, we have

$$\begin{aligned}&\frac{1}{2}|r^n|^2 - \frac{1}{2}\left|r^{n-1}\right|^2 + \frac{1}{2}\left|r^n - r^{n-1}\right|^2\\&= \frac{1}{2}\left(H\left(\Phi^{n-1}\right)r^n, \Phi^n - \Phi^{n-1}\right).\end{aligned} \tag{23}$$

By combining (22) with (23) together, we derive

$$\begin{aligned}&\frac{1}{2}\|\Phi^n\|^2 - \frac{1}{2}\left\|\Phi^{n-1}\right\|^2 + \frac{1}{2}\left\|\left(\Phi^n - \Phi^{n-1}\right)\right\|^2\\&+\frac{\Delta t}{2}\|\nabla\Phi^n\|^2 + \frac{\gamma\Delta t}{2}\|\Delta\Phi^n\|^2 + \Delta t|r^n|^2 - \Delta t\left|r^{n-1}\right|^2\\&+\Delta t\left|r^n - r^{n-1}\right|^2 = 0.\end{aligned} \tag{24}$$

It is easy to get

$$\begin{aligned}&\frac{1}{2}\|\Phi^n\|^2 - \frac{1}{2}\left\|\Phi^{n-1}\right\|^2 + \frac{\Delta t}{2}\|\nabla\Phi^n\|^2 + \frac{\gamma\Delta t}{2}\|\Delta\Phi^n\|^2\\&+2\Delta t|r^n|^2 - 2\Delta t\left|r^{n-1}\right|^2\\&\leq 0.\end{aligned} \tag{25}$$

Hence, we get the result of theorem 2.1.

C. Error analysis

To derive error estimates for the full-discrete formulation (2.7), we introduce some notations:
Let us define

$$e_\Phi^n = R_h\Phi^n - \Phi_h^n, \quad \theta_\Phi^n = R_h\Phi^n - \Phi^n \quad e_r^n = r^n - r_h^n,$$
$$\Phi^n - \Phi_h^n = \left(R_h\Phi^n - \Phi_h^n\right) - (R_h\Phi^n - \Phi^n) = e_\Phi^n - \theta_\Phi^n.$$

We subtract (9) from (8) to get the following error equation for e_Φ^{n+1},

$$\begin{aligned}&\left(\frac{e_\Phi^n - e_\Phi^{n-1}}{\Delta t}, \psi_h\right) - \left(\frac{\theta_\Phi^n - \theta_\Phi^{n-1}}{\Delta t}, \psi_h\right) + \left(\nabla e_\Phi^n, \nabla\psi_h\right)\\&-\left(\nabla\theta_\Phi^n, \nabla\psi_h\right) + \gamma\left(\Delta e_\Phi^n, \Delta\psi_h\right) - \gamma\left(\Delta\theta_\Phi^n, \Delta\psi_h\right)\\&+\left(r^nH\left(\Phi^{n-1}\right) - r_h^nH\left(\Phi_h^{n-1}\right), \psi_h\right) = 0,\end{aligned} \tag{26}$$

$$\begin{aligned}e_r^n - e_r^{n-1} =& \frac{1}{2}\left(H\left(\Phi^n\right) - H\left(\Phi_h^{n-1}\right), \Phi^n - \Phi^{n-1}\right)\\&-\frac{1}{2}\left(H\left(\Phi_h^{n-1}\right), \theta_\Phi^n - \theta_\Phi^{n-1}\right)\\&+\frac{1}{2}\left(H\left(\Phi_h^{n-1}\right), e_\Phi^n - e_\Phi^{n-1}\right) + \Delta tE_r^n,\end{aligned} \tag{27}$$

where $E_r^{n+1} = \frac{r^n - r^{n-1}}{\Delta t} - \partial_t r^n$.

Theorem 2.2: Assume that the system (1) has a unique solution Φ satisfying (6). The fully discrete system (9) yields a unique solution Φ_h^{n+1}. It satisfies the following error estimates:

$$\left\|e_\Phi^n\right\|^2 - \left\|e_\Phi^0\right\|^2 + \Delta t\left|e_r^n\right|^2 - \Delta t\left|e_r^0\right|^2 \leqslant C_\varepsilon\left(h^{2r} + (\Delta t)^2\right), \tag{28}$$

where C_ε is independent of n, τ and h, but dependent on ε, and ε is a sufficiently small constant.

Proof. By multiplying both sides of the second equation of equation (27) by $2\Delta t\left(e_r^n\right)$, we can deduce

$$\begin{aligned}&\Delta t\left|e_r^n\right|^2 - \Delta t\left|e_r^{n-1}\right|^2 + \Delta t\left|e_r^n - e_r^{n-1}\right|^2\\&= \Delta te_r^n\left(H\left(\Phi^n\right) - H\left(\Phi_h^{n-1}\right), \Phi^n - \Phi^{n-1}\right)\\&-\Delta te_r^n\left(H\left(\Phi_h^{n-1}\right), \theta_\Phi^n - \theta_\Phi^{n-1}\right)\\&+\Delta te_r^n\left(H\left(\Phi_h^{n-1}\right), e_\Phi^n - e_\Phi^{n-1}\right)\\&+\Delta te_r^n\Delta tE_r^n.\end{aligned} \tag{29}$$

Taking $\psi_h = \Delta t\left(e_\Phi^n - e_\Phi^{n-1}\right)$ and the Triangle inequality, we have

$$\begin{aligned}&\left\|e_\Phi^n\right\|^2 - \left\|e_\Phi^{n-1}\right\|^2 + \frac{\Delta t}{2}\left\|\nabla e_\Phi^n\right\|^2 - \frac{\Delta t}{2}\left\|\nabla e_\Phi^{n-1}\right\|^2\\&+\frac{\Delta t}{2}\left\|\nabla\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2 + \frac{\gamma\Delta t}{2}\left\|\Delta e_\Phi^n\right\|^2\\&-\frac{\gamma\Delta t}{2}\left\|\Delta e_\phi^{n-1}\right\|^2 + \frac{\gamma\Delta t}{2}\left\|\Delta\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2\\&+\Delta t\left(r^n\left(H\left(\Phi^{n-1}\right) - H\left(\Phi_h^{n-1}\right)\right), e_\Phi^n - e_\Phi^{n-1}\right)\\&+\Delta te_r^n\left(H\left(\Phi_h^{n-1}\right), e_\Phi^n - e_\Phi^{n-1}\right)\\&\leqslant\left(\theta_\Phi^n - \theta_\Phi^{n-1}, \left(\theta_\Phi^n - e_\Phi^{n-1}\right)\right) + \Delta t\left(\nabla\theta_\Phi^n, \nabla\left(e_\Phi^n - e_\Phi^{n-1}\right)\right)\\&+\Delta t\gamma\left(\Delta\theta_\Phi^n, \Delta\left(e_\Phi^n - e_\Phi^{n-1}\right)\right).\end{aligned} \tag{30}$$

By combining (29) with (30) and (14), we can get

$$\begin{aligned}&\left\|e_\Phi^n\right\|^2 - \left\|e_\Phi^{n-1}\right\|^2 + \frac{\Delta t}{2}\left\|\nabla e_\Phi^n\right\|^2 - \frac{\Delta t}{2}\left\|\nabla e_\Phi^{n-1}\right\|^2\\&+\frac{\Delta t}{2}\left\|\nabla\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2 + \frac{\gamma\Delta t}{2}\left\|\Delta e_\Phi^n\right\|^2\\&-\frac{\gamma\Delta t}{2}\left\|\Delta e_\phi^{n-1}\right\|^2 + \frac{\gamma\Delta t}{2}\left\|\Delta\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2\\&+\Delta t\left|e_r^n\right|^2 - \Delta t\left|e_r^{n-1}\right|^2\\&\leqslant\left(\theta_\Phi^n - \theta_\Phi^{n-1}, \left(e_\Phi^n - e_\Phi^{n-1}\right)\right)\\&-\Delta t\left(r^n\left(H\left(\Phi^{n-1}\right) - H\left(\Phi_h^{n-1}\right)\right), e_\Phi^n - e_\Phi^{n-1}\right)\\&+\Delta te_r^n\left(H\left(\Phi^n\right) - H\left(\Phi_h^{n-1}\right), \Phi^n - \Phi^{n-1}\right)\\&-\Delta te_r^n\left(H\left(\Phi_h^{n-1}\right), \theta_\Phi^n - \theta_\Phi^{n-1}\right)\\&+2(\Delta t)^2e_r^nE_r^n = \sum_{i=1}^{5}I_i^n.\end{aligned} \tag{31}$$

We estimate each term on the right hand side of equation (31) in turn. Thus each term of (31) can be estimated as follows:

$$\begin{aligned}I_1^{n+1} &= \Delta t\left((\Delta t)^{-1}\left(\theta_\Phi^n - \theta_\Phi^{n-1}\right), e_\Phi^n + e_\Phi^{n-1}\right)\\&\leqslant C\Delta t\left\|(\Delta t)^{-1}\left(\theta_\Phi^n - \theta_\Phi^{n-1}\right)\right\|_{H^{-1}}\left\|e_\Phi^n - e_\Phi^{n-1}\right\|_{H^1}\\&\leqslant C\varepsilon^{-1}\Delta th^{2r} + \varepsilon\Delta t\left\|\nabla\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2,\end{aligned} \tag{32}$$

where we have used lemma 2.1, Young inequality, Cauchy-Schwarz inequality and regularity hypothesis (3.3).

$$\begin{aligned} I_2^{n+1} &\leqslant C\Delta t|r^n| \left\| H\left(\Phi^{n-1}\right) - H\left(\Phi_h^{n-1}\right)\right\| \left\|e_\Phi^n - e_\Phi^{n-1}\right\| \\ &\leqslant C\varepsilon^{-1}(\Delta t)^3 + \varepsilon\Delta t\left\|e_\Phi^n - e_\Phi^{n-1}\right\|^2, \end{aligned} \tag{33}$$

where we have used lemma 2.2, Young inequality, Cauchy-Schwarz inequality.

$$\begin{aligned} I_3^{n+1} &\leqslant C\Delta t|e_r^n| \left\|\Phi^n - \Phi^{n-1}\right\|_{L^\infty} \left\| H\left(\Phi^{n-1}\right) - H\left(\Phi_h^{n-1}\right)\right\| \\ &\leqslant C\varepsilon^{-1}(\Delta t)^3 + \varepsilon\Delta t|e_r^n|^2, \end{aligned} \tag{34}$$

similarly, where we have used lemma 2.2, Young inequality, Cauchy-Schwarz inequality.

$$\begin{aligned} I_4^{n+1} &\leqslant C\Delta t|e_r^n| \left\| H\left(\Phi_h^{n-1}\right)\right\|_{L^\infty} \left\|\theta_\Phi^n - \theta_\Phi^{n-1}\right\| \\ &\leqslant C\varepsilon^{-1}(\Delta t)^3 + \varepsilon\Delta t|e_r^n|^2. \end{aligned} \tag{35}$$

It is obviously easy to see

$$I_5^{n+1} = 2\Delta t e_r^{n+1} E_r^{n+1} \leq C_\varepsilon \Delta t|e_r^n|^2 + C\varepsilon^{-1}(\Delta t)^3. \tag{36}$$

Finally, summing the time from t_1 to t_N for (31)-(33) and adding three inequalities, we obtain

$$\begin{aligned} &\left\|e_\Phi^n\right\|^2 - \left\|e_\Phi^0\right\|^2 + \frac{\Delta t}{2}\left\|\nabla e_\Phi^n\right\|^2 - \frac{\Delta t}{2}\left\|\nabla e_\Phi^0\right\|^2 \\ &\quad + \sum_{i=1}^{n} \frac{\Delta t}{2}\left\|\nabla\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2 + \frac{\gamma\Delta t}{2}\left\|\Delta e_\Phi^n\right\|^2 \\ &\quad - \frac{\gamma\Delta t}{2}\left\|\Delta e_\phi^0\right\|^2 + \sum_{i=1}^{n} \frac{\gamma\Delta t}{2}\left\|\Delta\left(e_\Phi^n - e_\Phi^{n-1}\right)\right\|^2 \\ &\quad + \Delta t\left|e_r^n\right|^2 - \Delta t\left|e_r^0\right|^2 + \sum_{i=1}^{n}\Delta t\left|e_r^n - e_r^{n-1}\right|^2 \\ &\leqslant C\varepsilon^{-1}\Delta t\left(h^{2r} + (\Delta t)^2\right), \end{aligned} \tag{37}$$

where ε is small enough and C is independent of n, Δt and h.

III. Numerical experiments

The efficiency of the proposed algorithm is shown in this section. The software FreeFem ++ developed by Hecht et al. [7] is used in our experiments. We verify the convergence of fully discrete scheme. In this section, we get some numerical results to certify the proposed method. We consider the following EFK equation with initial and boundary value problem (2) in a square domain $\Omega = \{(x_1, x_2) \in [0,\pi]\times[0,\pi]\}$ with Neumann boundary in $x_2 = 0, 1$. The exact solution is given as follows:

$$\Phi(x_1, x_2, 0) = \sin(x_1)\sin(x_2), \quad (x_1, x_2) \in \Omega. \tag{38}$$

$$\Phi = 0, \quad \Delta\Phi = 0, \quad (x_1, x_2, t) \in \partial\Omega \times (0, T]. \tag{39}$$

$$\begin{aligned} \Phi_t + \gamma\Delta^2\Phi - \Delta\Phi - \Phi + \Phi^3 &= F(x_1, x_2, t), \\ &\text{in } \Omega \times (0, T], \end{aligned} \tag{40}$$

where

$$\begin{aligned} F(x_1, x_2, t) &= 4\gamma\sin(x_1)\sin(x_2)\exp(-t) \\ &\quad + \left|\sin(x_1)\sin(x_2)\exp(-t)\right|^3. \end{aligned}$$

The exact solution of this example is

$$\Phi(x_1, x_2, 0) = \exp(-t)\sin(x_1)\sin(x_2). \tag{41}$$

In the numerical experiment, we compute the solution up to the final time $T = 1$ and taking $\gamma = 0.01$. P_1 finite element is used for Φ. To demonstrate the convergence order of our scheme, we first set refine the spatial grid size with $h = \frac{1}{256}$. The errors of the variable between the numerical solution and the exact solution at $T = 1$ with different time step size $\Delta t = 3^{-i}, i = 1, 2, 3, 4, 5, 6$ in Table I. We observe that our scheme presents first order accuracy in L^2 norm for Φ.

Table I. Errors and rates of convergence for the phase-field Φ.

τ	L^2-error	order	CPU time (s)
$\frac{1}{3}$	0.0285893	-	4.548
$\frac{1}{6}$	0.0118704	1.27	15.487
$\frac{1}{12}$	0.00551655	1.11	40.041
$\frac{1}{24}$	0.00279187	0.98	90.131
$\frac{1}{48}$	0.00142317	0.97	192.656
$\frac{1}{96}$	0.000721306	0.98	413.13

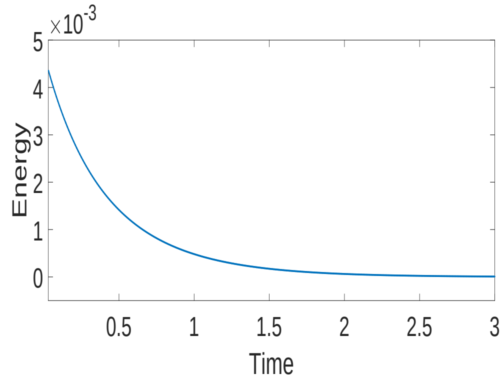

Fig. 1. Unconditional energy stability with T=3.

The change of total energy over time is shown in Figure 1, where $C_0 = 1$, $\gamma = 0.1$ and $T = 3$. It shows that the energy is monotonically decreasing, which proves the unconditional energy stability of the proposed scheme.

IV. Conclusion and outlook

In this paper, we present the SAV method for the EFK equation. We focus on the numerical approximation of the EFK equation. A scalar auxiliary variable (SAV) method is introduced to deal with nonlinear term. We construct a new efficient and easy-to-implement numerical scheme. In addition, the energy stability is proved rigorously. Under the corresponding regularity assumption, we prove that the error estimate for Φ in the L^2-norm is able to achieve first-order convergence rates in time and the $O(h^r)(r \geq 1)$ in space. The non-local variable r also achieve the same convergence rate. The accuracy, energy stability of the proposed scheme are verified by numerical examples. In the last, we will consider using the SAV method to solve more complex coupling models.

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Dan-chen Zhu was born in Fengcheng, Jiangxi Province, china, in 1999. The author is a graduate student in computational mathematics at Guangxi Normal University, Guilin, Guangxi Zhuang Autonomous Region, China, starting in 2022.

The author current research interests include numerical solution of partial differential equations and computing sciences, two-phase fluids, discontinuous Galerkin finite element method.

Qianfeng Qian was born in Qujing, Yunnan Province, china, in 2000. The author is a graduate student in computational mathematics at Guangxi Normal University, Guilin, Guangxi Zhuang Autonomous Region, China, starting in 2023.

The author current research interests include numerical solution of partial differential equations and computing sciences, fluids and fluid-fluid interaction problems, discontinuous Galerkin finite element method.

Jing Jiang was born in Guilin, Guangxi, China in 1996. The author started studying computational mathematics at Guangxi Normal University in Guilin, Guangxi Zhuang Autonomous Region, China, starting in 2023.

The author's current research directions include numerical solutions for partial differential equations and computational science, physical information neural networks, and finite difference methods.

Yan-ping Qiu was born in Yulin, Guangxi Province, china, in 1999. The author is a graduate student in computational mathematics at Guangxi Normal University, Guilin, Guangxi Zhuang Autonomous Region, China, starting in 2022.

The author current research interests include network control systems.