Direct Product of Ternary Semigroups and Characteristics of its Generators

Abin Sam Tharakan and G. Sheeja*

Abstract—**In this paper, we introduce the concept of a Free Ternary Semigroup and explore several key properties. We establish the essential conditions under which the direct product of two infinite semigroups can be finitely generated and provide an upper bound for its rank. Additionally, we determine the necessary and sufficient criteria for the external direct product of two free ternary semigroups to be finitely generated.**

Index Terms—**Free Ternary Semigroup, Ternary Generating Set, Finitely Generated, Rank, Complete generating set.**

I. Introduction

M. L. Santiago [1] and Sribala [2][3] developed the theory of Ternary semigroup. Lehmer introduced the theory of ternary Semigroup [4] in 1932. Robertson et.al. (1998) [5] [6] examined the direct product of semigroups and established the specific condition under which the direct product of semigroups can be considered finitely generated. Also, if both ternary semigroups are finite, then their direct product is finitely generated. Here, the focus is on the direct product of an infinite ternary semigroup. Free semigroup is the important tool for presentation of semigroup which was analogously introduced by J.M Howie [7].

It can be noted if $\mathbb{Z}^+ = \{1, 2, ...\}$ is the additive ternary semigroup with generators $\{1, 2\}$. But $\mathbb{Z}^+ \times \mathbb{Z}^+$ is finitely generated.

In this paper, we introduce the concept of a free ternary semigroup and prove the homomorphism theorem, which states that for any ternary semigroup, it is possible to find a free ternary semigroup. We prove another homomorphism theorem that gives the relationship between the quotient free ternary semigroup and ternary semigroup and illustrate an example for this theorem. We also establish the necessary condition for the direct product of ternary semigroups to be finitely generated and show that the converse may not be true. Furthermore, we provide a bound for the rank of the direct product of two ternary semigroups and prove the necessary and sufficient criteria for the external direct product of two free ternary semigroups to be finitely generated. Finally, we introduce the idea of a complete generating set and establish the necessary and sufficient condition for a generating set for a ternary semigroup to be complete. We also prove this same condition for the direct product of two ternary semigroups.

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II. Preliminaries

Definition II.1. *[2] A ternary semigroup is a set $\mathcal{T}$ that is non empty and has a ternary operation $(e, f, g) \to [efg]$ that satisfies the associative law of the first kind.*
That is, the equation $([efg]hi) = (e[fgh]i) = (ef[ghi])$ holds for all values of e, f, g, h, and i in the set $\mathcal{T}$.

Example II.1.*(i) $\mathcal{T}_1 = \{i, -i\}$ under multiplication.*
(ii) $\mathcal{T}_2 = \mathbb{Z}^-$ under multiplication.

Definition II.2. *[2] A non-empty set $\mathcal{E}$ can be termed as a generating set for the ternary semigroup $\mathcal{T}$ if it is capable of generating the entire $\mathcal{T}$.*

Definition II.3. *A ternary semigroup is finitely generated if its generating set is finite.*

Definition II.4. *Let $\mathcal{T}$ be a ternary semigroup. Then $\mathcal{T}^1$ is either a ternary semigroup with the neutral element or adjoining a neutral element to the $\mathcal{T}$ if and only if it is derived from a binary semigroup [8]. An element u is said to be a neutral element of $\mathcal{T}$ if $[auu] = [uau] = [uua] = a$ for all $a \in \mathcal{T}$.*

Definition II.5. *[2] A non-empty subset $\mathcal{A}$ of $\mathcal{T}$ is said to be a right ideal of $\mathcal{T}$ if $[\mathcal{A}\mathcal{T}\mathcal{T}] \subseteq \mathcal{A}$*

Definition II.6. *$\{a\} \cup [a\mathcal{T}\mathcal{T}]$ is called the right ideal generated by a.*

III. Free Ternary Semigroup

Definition III.1. *Consider a non-empty set $\mathcal{E}$. Define $\mathcal{T}_\mathcal{E}$ as the set of all non-empty finite words with odd length $e_1, e_2, ..., e_m$ for any m that are odd numbers, where e_i belongs to the alphabet $\mathcal{E}$. A ternary operation is defined as the combination of words*

$$(e_1, e_2, ...e_m)(f_1, f_2, ...f_m)(g_1, g_2, ...g_m) = e_1....e_m f_1....f_m g_1.....g_m \ for\ all$$
$$(e_1, e_2, ...e_m), (f_1, f_2, ...f_m), (g_1, g_2, ...g_m) \in \mathcal{T}_\mathcal{E}$$

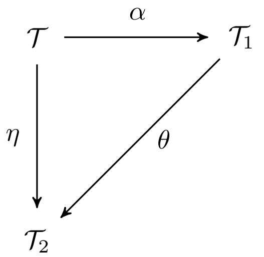
The ternary semigroup $\mathcal{T}_\mathcal{E}$ is defined on the ternary operation of concatenation and is referred to as a Free ternary semigroup. Here, $\mathcal{E}$ is referred to as a generating set for $\mathcal{T}_\mathcal{E}$. The rank of $\mathcal{T}_\mathcal{E}$ is the number of elements of $\mathcal{E}$.

Example III.1.

$$Let\ \mathcal{E} = \{a, b\}\ Then,\ \mathcal{T}_\mathcal{E} = \{a, b, aaa, bbb, aba, aab, ...\}$$

Definition III.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two ternary semigroups. Homomorphism from $\mathcal{T}_\infty$ to $\mathcal{T}_\in$ is the mapping ϕ from $\mathcal{T}_1$ to $\mathcal{T}_2$ such that for all $u, v, w \in \mathcal{T}_1$*
$\phi(uvw) = \phi(u)\phi(v)\phi(w)$.

Theorem III.1. *Let α and η be a homomorphism of a ternary semigroup $\mathcal{T}$ upon ternary semigroup $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively* *such that $\alpha \circ \alpha^{-1} \subseteq \eta \circ \eta^{-1}$. Then, there is a unique homomorphism θ of $\mathcal{T}_1$ upon $\mathcal{T}_2$ such that $\alpha\theta = \eta$.*



Proof: Define $\alpha : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ by $(e\alpha)\theta = e\eta$ for all $e \in \mathcal{T}$.
Let $f \in \mathcal{T}_1$.Then, $e\alpha = f$.
So, $f\theta = e\eta$.
If $g\eta = f$, then $(e, g) \in \alpha \circ \alpha^{-1} \subseteq \eta \circ \eta^{-1}$.
So, $e\eta = g\eta$.
Clearly, θ is well defined.

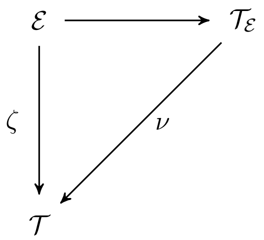
$$\begin{aligned}[(e\alpha)(f\alpha)(g\alpha)]\theta &= [(efg)\alpha]\theta \\ &= (efg)\eta \\ &= (e\eta)(f\eta)(g\eta) \\ &= (e\eta)\theta(f\eta)\theta(g\eta)\theta\end{aligned}$$

So, θ is a homomorphism. ■

Corollary III.1. *If ρ_1 and ρ_2 are congruences on a ternary semigroup $\mathcal{T}$ such that $\rho_1 \subseteq \rho_2$. Then, $\mathcal{T}/\rho_2$ is the homomorphic image of $\mathcal{T}/\rho_1$.*

Proof: Let $\mathcal{T}_1 = \mathcal{T}/\rho_1$, $\mathcal{T}_2 = \mathcal{T}/\rho_2$.
Since, $\rho_1 = \alpha \circ \alpha^{-1}$ and $\rho_2 = \eta \circ \eta^{-1}$.
By the **Theorem**, there is a homomorphism from $\mathcal{T}/\rho_1$ to $\mathcal{T}/\rho_2$. ■

Theorem III.2. *Consider a nonempty set $\mathcal{E}$ and a ternary semigroup $\mathcal{T}$. If $\zeta : \mathcal{A} \rightarrow \mathcal{T}$ is any mapping, then there exist a unique homomorphism $\nu : \mathcal{T}_{\mathcal{E}} \rightarrow \mathcal{T}$ that satisfies $\zeta = \nu$ and the following diagram commutes.*



Proof:
Define

$$\begin{aligned}&\nu : \mathcal{T}_{\mathcal{E}} \rightarrow \mathcal{T} \ by \\ &\nu(e_1, e_2, ...e_m) = \zeta(e_1)\zeta(e_2)....\zeta(e_m) \\ &\qquad = [\zeta(e_1)\zeta(e_2)\zeta(e_3)].......\zeta(e_m). \\ &\qquad Let \ \ e_1e_2.....e_m = f_1f_2.....f_m \\ Then, \ &\nu(e_1, e_2, ...e_m) = [\zeta(e_1)\zeta(e_2)\zeta(e_3)]....\zeta(e_m) \\ &\qquad = [\zeta(f_1)\zeta(f_2)\zeta(f_3)].......\zeta(f_m) \\ &\qquad = \nu(f_1, f_2, ...f_m)\end{aligned}$$

So, mapping is well defined.

$$Let \ \ e_1e_2.....e_m, f_1f_2.....f_n, g_1, g_2, ...g_m \in \mathcal{T}_{\mathcal{E}}.$$

Then,

$$\begin{aligned}&\nu(e_1, e_2, \ldots, e_m.f_1, f_2, \ldots, f_n.g_1, g_2, \ldots, g_o) \\ &\quad = \zeta(e_1)\zeta(e_2)\ldots\zeta(e_m) \\ &\quad\ \ \zeta(f_1)\zeta(f_2)\ldots\zeta(f_n) \\ &\quad\ \ \zeta(g_1)\zeta(g_2)\ldots\zeta(g_o) \\ &\quad = \nu(e_1, e_2, \ldots, e_m) \\ &\quad\ \ \nu(f_1, f_2, \ldots, f_n) \\ &\quad\ \ \nu(g_1, g_2, \ldots, g_o)\end{aligned}$$

So, ν is an homomorphism. ■

Definition III.3. *Consider $\mathcal{T}_{\mathcal{E}}$, which is a free ternary semigroup. Let ρ be an equivalence relation on $\mathcal{T}_{\mathcal{E}}$. We can define $\mathcal{T}_{\mathcal{E}}/\rho$ as the collection of equivalence classes of ρ on $\mathcal{T}_{\mathcal{E}}$.*

To define a ternary operation on $\mathcal{T}_{\mathcal{E}}/\rho$, we can do so in a natural way by stating that

$$[(a\rho)(b\rho)(c\rho)] = [abc]\rho \forall a, b, c \in \mathcal{T}_{\mathcal{E}}$$

Lemma III.1. *Let $\mathcal{T}_{\mathcal{E}}$ be a free ternary semigroup. Let ρ be an equivalence relation on $\mathcal{T}_{\mathcal{E}}$. Then, $\mathcal{T}_{\mathcal{E}}/\rho$ defined as the collection of equivalences classes of ρ on $\mathcal{T}_{\mathcal{E}}$. Define a ternary operation on $\mathcal{T}_{\mathcal{E}}/\rho$ in a natural way as*

$$[(a\rho)(b\rho)(c\rho)] = [abc]\rho \forall a, b, c \in \mathcal{T}_{\mathcal{E}}$$

Then, $\mathcal{T}_{\mathcal{E}}$ ρ is a ternary semigroup under the above ternary operation.

Proof: Let $\mathcal{T}_{\mathcal{E}}$ be a free ternary semigroup.
Let ρ be an equivalence relation on $\mathcal{T}_{\mathcal{E}}$. Clearly, the ternary operation defined above is closed under $\mathcal{T}_{\mathcal{E}}/\rho$.
Now, we have to prove the ternary operation is associative.
Let $a\rho, b\rho, c\rho, d\rho, e\rho \in \mathcal{T}_{\mathcal{E}}/\rho$.

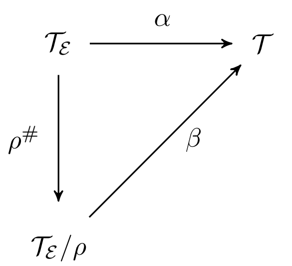
$$\begin{aligned}[(a\rho)(b\rho)(c\rho)](d\rho)(e\rho) &= [abc]de\rho \\ &= a[bcd]e\rho \\ &= (a\rho)[(b\rho)(c\rho)(d\rho)](e\rho) \\ &= ab[cde]\rho \\ &= (a\rho)(b\rho)[(c\rho)(d\rho)(e\rho)]\end{aligned}$$

So, Ternary operation on $\mathcal{T}_{\mathcal{E}}$ ρ is associative.
Therefore, $\mathcal{T}_{\mathcal{E}}$ ρ under the above ternary operation is a ternary semigroup. ■

Definition III.4. *Consider $\mathcal{T}_{\mathcal{E}}$, a free ternary semigroup. Let ρ be a congruence on $\mathcal{T}_{\mathcal{E}}$. We can define $\mathcal{T}_{\mathcal{E}}/\rho$ as the collection of congruence classes of ρ on $\mathcal{T}_{\mathcal{E}}$.*
To define a ternary operation on $\mathcal{T}_{\mathcal{E}}/\rho$, we can do so in a natural way by stating that

$$[(a\rho)(b\rho)(c\rho)] = [abc]\rho \forall a, b, c \in \mathcal{T}_{\mathcal{E}}$$

Theorem III.3. *Let $\mathcal{T}_{\mathcal{E}}$ be the free ternary semigroup. Let ρ_0 be any relation on $\mathcal{T}_{\mathcal{E}}$ and let ρ be the smallest congruence containing ρ_0. Let $\rho^{\#}$ be the natural homomorphism from $\mathcal{T}_{\mathcal{E}}$ to $\mathcal{T}_{\mathcal{E}}/\rho$.*
Let $\mathcal{T}$ be any ternary semigroup. Let α be a homomorphism from $\mathcal{T}_{\mathcal{E}}$ to $\mathcal{T}$ such that $u\alpha = v\alpha \forall u, v \in \rho_0$. Then, there exist a homomorphism β from $\mathcal{T}_{\mathcal{E}}/\rho$ to $\mathcal{T}$ such that $\rho^{\#}\beta = \alpha$.



Proof: We first show that if $w, w^{'} \in \mathcal{T}_{\mathcal{E}}$ such that $w\rho w^{'}$, then $w\alpha = w^{'}\alpha$.
By hypothesis, $(u, v) \in \rho \implies u\alpha = v\alpha$.
So, $\rho^0 \subseteq \alpha \circ \alpha^{-1}$.
Since, ρ is the smallest congruence on T containing ρ^0 and $\alpha \circ \alpha^{-1}$ is a congruence.
Therefore, $\rho \subseteq \alpha \circ \alpha^{-1}$.
So, $(w, w^{'}) \in \rho \implies w\alpha = w^{'}\alpha$.
Define a mapping $\beta : \mathcal{T}_{\mathcal{E}}/\rho \to \mathcal{T}$ by
$(w\rho^{\#})\beta = w\alpha \forall w \in \mathcal{T}_{\mathcal{E}}$. Prove that mapping defined above is well defined.
Let $w\rho^{\#}, w^{'}\rho^{\#} \in \mathcal{T}_{\mathcal{E}}/\rho$.
Suppose $w\rho^{\#} = w^{'}\rho^{\#}$. That is, $(w, w^{'}) \in \rho \implies w\alpha = w^{'}\alpha$.
So, it is well-defined.
It is evident that $\rho^{\#}\beta = \alpha$.
So, we have to show that β is a homomorphism.
Let $w, w^{'}, w^{''} \in \mathcal{T}_{\mathcal{E}}$. Then,

$$\begin{aligned}\beta([(w\rho^{\#})(w^{'}\rho^{\#})(w^{''}\rho^{\#})]) &= \beta([ww^{'}w^{''}]\rho^{\#}) \\ &= \alpha([ww^{'}w^{''}]) \\ &= \alpha(w)\alpha(w^{'})\alpha(w^{''}) \\ &= \beta(w\rho^{\#})\beta(w^{'}\rho^{\#})\beta(w^{''}\rho^{\#})\end{aligned}$$

Therefore, β is a Homomorphism. ■

Example III.2. *Tricyclic Semigroup $\mathcal{C}$ to be the ternary semigroup generated by a 3 element set $\{x_1, x_2, x_3\}$.*
Let ρ_0 be the relation $[x_1x_2x_3] = 1$.
Let $\mathcal{T}_{\mathcal{E}}^{'}$ be the free ternary semigroup with identity generated by $\mathcal{E} = \{x_1, x_2, x_3\}$.
Take ρ as the smallest congruence on $\mathcal{T}_{\mathcal{E}}^{'}$ generated by ρ^0.
Then, $\mathcal{C} = \mathcal{T}_{\mathcal{E}}^{'}/\rho$ is generated by congruence class $p = x_1\rho^{\#}, q = x_2\rho^{\#}, r = x_3\rho^{\#}$ satisfying the relation $[pqr] = 1$

Theorem III.4. *Let $\mathcal{T}_{\mathcal{E}}$ be a free ternary semigroup and let $\mathcal{R} \neq \mathcal{T}_{\mathcal{E}}$ be a proper right ideal. If $\mathcal{R}$ is finitely generated then it is not free.*

Proof: Since $\mathcal{R} \neq \mathcal{T}_{\mathcal{E}}$ there exists $a \in \mathbf{E}$ such that $a \notin \mathcal{R}$.
Suppose that $a^i \in \mathcal{R}$ for all $i \geq 1$ and i in odd numbers.
Let $r \in R$ of minimal length.
Then ra^i, $i \geq 1$ and i in odd numbers. since $\mathcal{R}$ is a right ideal, but ra^i is not a product of three elements of R.
Therefore, each generating set of $\mathcal{R}$ contains all the words ra^i, $i \geq 1$ and i in odd numbers, and $\mathcal{R}$ is not finitely generated, a contradiction.
Thus $\mathcal{R}$ contains some power of a. Let i be the minimal such power; obviously $i \geq 1$ and i in odd numbers. The word a^{i+2} belongs to $\mathcal{R}$ since $\mathcal{R}$ is a right ideal, but a^{i+2} is not a product of three elements of $\mathcal{R}$. since $i \geq 1$; hence each generating set for $\mathcal{R}$ contains both a^i and a^{i+2}.
Since a^i and a^{i+2} satisfy the non-trivial relation $a^ia^{i+2} = a^{i+2}a^i$, $\mathcal{R}$ cannot be free. ■

Example III.3. *$\mathcal{T}_{\mathcal{E}}$ be a free ternary semigroup generated by $\mathcal{E} = \{a, b\}$.*
$\mathcal{R} = \{a\} \cup [a\mathcal{T}_{\mathcal{E}}\mathcal{T}_{\mathcal{E}}]$ be the right ideal generated by a.
The set $\{ab^ib^j : i, j \geq 0, i, j \ is \ odd \ numbers\}$ is the minimal generating set for $\mathcal{R}$
Therefore, $\mathcal{R}$ is free.

Definition III.5. *An arbitrary ternary semigroup $\mathcal{T}$ is said to be free if it is isomorphic to a free ternary semigroup $\mathcal{T}_{\mathcal{E}}$.*

Example III.4. *$\mathbb{Z}^+$ under addition is free with free ternary semigroup $\{a, b, aaa, aba, , ...\}$.*

Definition III.6. *Consider $\mathcal{T}_{\mathcal{E}}$, a free ternary semigroup. A set $\mathcal{T}_{\mathcal{E}}$ is said to be finitely generated if it either contains a finite number of generators or if it has a finite generating set.*

Example III.5. *Let $\mathcal{E} = \{a, b\}$.*
Then, $\mathcal{T}_{\mathcal{E}} = \{a, b, aaa, aba, ...\}$ is finitely generated.

Definition III.7. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two ternary semigroups. Let $\mathcal{T}_1 \times \mathcal{T}_2 = \{(x, y) : x \in \mathcal{T}_1 \& y \in \mathcal{T}_2\}$ and the ternary operation is defined as*
$(x_1, y_1)(x_2, y_2)(x_3, y_3) = (x_1x_2x_3, y_1y_2y_3)$.
$\mathcal{T}_1 \times \mathcal{T}_2$ is a ternary semigroup under the above ternary operation and is called the direct product of ternay semigroup.

Lemma III.2. *Consider $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{F}}$ as two free ternary semigroups.Then, the direct product of $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{T}_{\mathcal{F}}$ forms a ternary semigroup.*

Proof:
Let (v_1, w_1) and $(v_2, w_2) \in \mathcal{T}_{\mathcal{E}} \times \mathcal{T}_{\mathcal{F}}$.
Then, $(v_1, w_1)(v_2, w_2) = (v_1v_2, w_1w_2) \in \mathcal{T}_{\mathcal{E}} \times \mathcal{T}_{\mathcal{F}}$. Since, $v_1v_2 \in \mathcal{T}_{\mathcal{E}}, w_1w_2 \in \mathcal{T}_{\mathcal{F}}$.
Let (v_1, w_1), (v_2, w_2), (v_3, w_3), (v_4, w_4) and (v_5, w_5) $\in \mathcal{T}_{\mathcal{E}} \times \mathcal{T}_{\mathcal{F}}$

$$\begin{aligned}&[(v_1, w_1)(v_2, w_2)(v_3, w_3)](v_4, w_4)(v_5, w_5) = \\ &= [(v_1v_2v_3, w_1w_2w_3)](v_4, v_4)(v_5, w_5) \\ &= (v_1v_2v_3v_4v_5, w_1w_2w_3w_4w_5) \\ &Since, \ \mathcal{T}_{\mathcal{E}} \ and \ \mathcal{T}_{\mathcal{F}} \ are \ free \ ternary \ semigroup. \\ &= (v_1, w_1)[(v_2v_3v_4, w_2w_3w_4)](v_5, w_5) \\ &= (v_1, w_1)(v_2, w_2)[(v_3v_4v_5, w_3w_4w_5)]\end{aligned}$$

■

IV. Direct Product of Ternary Semigroups

Here, we provide the precise condition that is both necessary and sufficient for the direct products of two ternary semigroups to be finitely generated. Additionally, we have successfully demonstrated the necessary and sufficient condition for the direct products of two free ternary semigroups to be finitely generated.

Definition IV.1. *Given a ternary semigroup $\mathcal{T}$. Then, $\mathcal{T}$ is decomposable if there exists an element $t \in \mathcal{T}$ such that t can be expressed as the product of three elements t_1, t_2, and t_3, where $t_1, t_2, t_3 \in \mathcal{T}$. The set of all decomposable elements in $\mathcal{T}$ is represented by $\mathcal{T}^3$. In other words, $\mathcal{T}^3$ is the set $\{t_1t_2t_3 : t_1, t_2, t_3 \in \mathcal{T}\}$.*
The collection of ternary semigroups that cannot be decomposed is represented by $\mathcal{T}/\mathcal{T}^3$.

Example IV.1. *$\mathbb{Z}$ under addition is a decomposible set.*

Definition IV.2. *Consider $\mathcal{T}_\mathcal{E}$ as a free ternary semigroup. If there exists a word $t \in \mathcal{T}_\mathcal{E}$ that can be expressed as the concatenation of three subwords t_1, t_2, and t_3, where $t_1t_2t_3 \in \mathcal{T}_\mathcal{E}$, Then $\mathcal{T}_\mathcal{E}$ is decomposable. The set of all decomposable words in $\mathcal{T}_\mathcal{E}$ is represented by ${\mathcal{T}_\mathcal{E}}^3$. This set is defined as the product of three copies of $\mathcal{T}_\mathcal{E}$, denoted as $\mathcal{T}_\mathcal{E}\mathcal{T}_\mathcal{E}\mathcal{T}_\mathcal{E}$. In other words, ${\mathcal{T}_\mathcal{E}}^3$ consists of all words of the form $t_1t_2t_3$, where t_1, t_2, and t_3 are elements of $\mathcal{T}_\mathcal{E}$.*

The collection of free ternary semigroup that is not decomposible is denoted by $\mathcal{T}_\mathcal{E}/{\mathcal{T}_\mathcal{E}}^3$

Example IV.2. *Let $\mathcal{E} = \{a, b\}$ $\mathcal{T}_\mathcal{E} = \{a, b, aaa, aba, ...\}$ only set that is not decomposible in $\mathcal{T}_\mathcal{E}$ are a,b.*

Lemma IV.1. *Consider two ternary semigroups denoted by $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $\kappa : \mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}_1$ denote the natural projection. If $\mathcal{E}$ is a set that generates $\mathcal{T}_1 \times \mathcal{T}_2$, then the set $\kappa(\mathcal{E})$ generates $\mathcal{E}$. If the Cartesian product of $\mathcal{T}_1$ and $\mathcal{T}_2$ is finitely generated, then $\mathcal{T}_1$ is also finitely generated.*

Proof: Define natural projection $\kappa : \mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}_1$ by

$$\kappa(t_1, t_2) = t_1$$

Clearly, this mapping is an epimorphism. Let $\mathcal{E}$ be a generating set for $\mathcal{T}_1 \times \mathcal{T}_2$.
So, let $(s_1, s_2) \in \mathcal{E}$.
Then, s_1 will be the element in generating set for $\mathcal{T}_1$ and $\kappa(\mathcal{E})$ becomes the generating set for $\mathcal{T}_1$.Since, κ is an onto morphism.
It is evident that the direct product of $\mathcal{T}_1$ and $\mathcal{T}_2$, denoted as $\mathcal{T}_1 \times \mathcal{T}_2$, is finitely generated. Therefore, we may conclude that $\mathcal{T}_1$ is also finitely generated. ∎

Lemma IV.2. *Consider a ternary semigroup $\mathcal{T}$ satisfying the property $\mathcal{T}^3 = \mathcal{T}$. Let $\mathcal{E} = \{e_i : i \in \Lambda\}$ be a set that generates $\mathcal{T}$. Then, there are elements t_i and r_i in $\mathcal{T}$, where i belongs to Λ. Further, there is a mapping η from Λ to Λ such that $e_i = e_{\eta(i)}t_ir_i$.*

Proof: Given that $\mathcal{T}^3 = \mathcal{T}$, it can be concluded that $\mathcal{T}$ does not possess any indecomposable elements. Each element e_i can be expressed as a product $e_{i_1}e_{i_2}...e_{i_p}$ of generators, where p is greater than or equal to 3. Let $\eta(i)$ be defined as i_1 and

$$t_i = \prod_{l=2}^{\frac{p-1}{2}+1} a_{i_l}$$

$$r_i = \prod_{m=\frac{p-1}{2}+2}^{p} a_{i_m}$$

∎

Proposition IV.1. *Consider two ternary semigroups denoted as $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\mathcal{T}_1$ satisfies the condition $\mathcal{T}_1^3 = \mathcal{T}_1$ and $\mathcal{T}_2$ satisfies the condition $\mathcal{T}_2^3 = \mathcal{T}_2$.Let $\mathcal{E}$ be the set of elements e_i for all i in Λ, and let $\mathcal{F}$ be the set of elements b_j for all j in Λ. These sets serve as generating sets for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Select elements $t_i, r_i : i \in \Lambda$ from the set $\mathcal{T}_1$, and elements $s_i, u_i : i \in \Lambda$ from the set $\mathcal{T}_2$. Also, choose a mapping $\eta : \Lambda \to \Lambda$ such that $e_i = e_{\eta(i)}t_ir_i$ for all $i \in \Lambda$. Additionally, select a mapping $\gamma : \Gamma \to \Gamma$ such that $f_j = f_{\gamma(j)}s_ju_j$ for all $j \in \Gamma$. Then the set $\mathcal{T}_1 \cup \{t_ir_i : i \in \Lambda\} \times \mathcal{T}_2 \cup \{s_ju_j : j \in \Gamma\}$ generates $\mathcal{T}_1 \times \mathcal{T}_2$.*

Proof: Consider an arbitrary element t_1 belonging to the set $\mathcal{T}_1$. Assume that t_1 may be expressed as a product of m generators from $\mathcal{T}_1$. By iteratively substituting an arbitrary generator a_i with the product $e_i = e_{\eta(i)}t_ir_i$, we observe that for every $n \geq m$, the element t_1 may be represented as a composition of n elements from the set $\mathcal{T}_1 \cup \{t_ir_i : i \in I\}$.
Let t_2 be an arbitrary element of $\mathcal{T}_2$, and suppose that t_2 can be expressed as a product of m generators from $\mathcal{T}_2$. By iteratively substituting an arbitrary generator b_j with the product $b_j = a_{\gamma(j)}s_ju_j$, we observe that for every $n \geq m$, the element t_2 can easily be represented as a product of n items from the set $\mathcal{T}_2 \cup \{s_ju_j : j \in J\}$.Let t_1 belong to $\mathcal{T}_1$ and t_2 belong to $\mathcal{T}_2$, where t_1 and t_2 are arbitrary. Let's assume that t_1 can be expressed as the multiplication of n_1 generators from $\mathcal{E}$, and that t_2 can be expressed as the multiplication of n_2 generators from $\mathcal{F}$.
Let k be the maximum of n_1 and n_2. Then,

$$t_1 = \rho_1\rho_2.....\rho_k$$
$$t_2 = \sigma_1\sigma_2....\sigma_k$$

of k elements from $\mathcal{T}_1 \cup \{t_ir_i : i \in \Lambda\}$ and $\mathcal{T}_2 \cup \{s_ju_j : j \in \Gamma\}$ respectively.
We may express (t_1, t_2) as a multiplication of elements from $\mathcal{T}_1 \cup \{t_ir_i : i \in I\} \times \mathcal{T}_2 \cup \{s_ju_j : j \in J\}$. Therefore,

$$(t_1, t_2) = (\rho_1, \gamma_1)(\rho_2, \gamma_2).......(\rho_k, \gamma_k)$$

∎

Corollary IV.1. *Consider two infinite ternary semigroups denoted as $\mathcal{T}_1$ and $\mathcal{T}_2$. It is given that $\mathcal{T}_1^3 = \mathcal{T}_1$ and $\mathcal{T}_2^3 = \mathcal{T}_1$. Then*

$$rank(\mathcal{T}_1 \times \mathcal{T}_2) \leq 9rank(\mathcal{T}_1)rank(\mathcal{T}_2).$$

Proof: If we choose the generating sets $\mathcal{E}$ and $\mathcal{F}$ for $\mathcal{T}_1$ and $\mathcal{T}_2$ to have cardinalities equal to the rank of $\mathcal{T}_1$ and the rank of $\mathcal{T}_2$, respectively, then the generating set for $\mathcal{T}_1 \times \mathcal{T}_2$, as established in Proposition 1, will have a cardinality that is at most 9 times the product of the ranks of $\mathcal{T}_1$ and $\mathcal{T}_2$. ∎

Theorem IV.1. *Consider two infinite ternary semigroups denoted by $\mathcal{T}_1$ and $\mathcal{T}_2$. If both $\mathcal{T}_1$ and $\mathcal{T}_2$ are finitely generated, and $\mathcal{T}_1^3 = \mathcal{T}_1$ & $\mathcal{T}_2^3 = \mathcal{T}_2$, then The Cartesian product of $\mathcal{T}_1$ and $\mathcal{T}_2$, denoted as $\mathcal{T}_1 \times \mathcal{T}_2$, is finitely generated.*

Proof: This theorem is the immediate consequence of the above **Lemma** IV.1, **Lemma**IV.2 and **Proposition**IV.1. ∎

a) Remark:: Converse may not be true.
For example, $\mathbb{Z}^+ = \{1, 2, ...\}$ is the additive ternary semigroup with generators $1, 2$ and $\mathbb{Z}^{+^3} \neq \mathbb{Z}^+$. But $\mathbb{Z}^+ \times \mathbb{Z}^+$ is finitely generated.

b) Remark:: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two infinite ternary semigroups satisfying the conditions $\mathcal{T}_1^3 = \mathcal{T}_1$ and $\mathcal{T}_2^3 = \mathcal{T}_1$. It follows that $(\mathcal{T}_1 \times \mathcal{T}_2)^3 = \mathcal{T}_1 \times \mathcal{T}_2$, whereas the converse is not universally true.
If each $\mathcal{T}_i$ (where $1 \leq i \leq p$) is finitely generated and each $\mathcal{T}_i^3 = \mathcal{T}_i$ (where $1 \leq i \leq p$), then the direct product of $\mathcal{T}_1, \mathcal{T}_2, ...\mathcal{T}_p$ is also finitely generated.

Theorem IV.2. *Let $\mathcal{T}_\mathcal{E}$ and $\mathcal{T}_\mathcal{F}$ be two free ternary semigroups. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two infinite ternary semigroups. If $\alpha : \mathcal{T}_1 \cup \{t_i r_i : i \in I\} \times \mathcal{T}_2 \cup \{s_j u_j : j \in J\} \to \mathcal{T}_1 \times \mathcal{T}_2$ is an arbitrary mapping. Then there exist a homomorphism $\beta : \mathcal{T}_\mathcal{E} \times \mathcal{T}_\mathcal{F} \to \mathcal{T}_1 \times \mathcal{T}_2$ such that $\alpha = \beta$ and the direct product of $\mathcal{T}_\mathcal{E}$ and $\mathcal{T}_\mathcal{F}$ is finitely generated if and only if the direct product of $\mathcal{T}_1$ and $\mathcal{T}_2$ is finitely generated.*

Proof: Homomorphisms of $\mathcal{T}_\mathcal{E} \times \mathcal{T}_\mathcal{F}$ and $\mathcal{T}_1 \times \mathcal{T}_2$ can be proved from **Lemma**IV.1.
The necessary and sufficient conditions for the direct product mentioned above are established by **Theorem**IV.1. ∎

Definition IV.3. *A generating set $\mathcal{E}$ of a ternary semigroup $\mathcal{T}$ is said to be complete if $\mathcal{E} \subseteq \mathcal{E}^3$. That is, every generator of $\mathcal{E}$ can be expressed as a product of three generators of $\mathcal{E}$.*

Proposition IV.2. *A ternary semigroup $\mathcal{T}$ has a complete generating set $\mathcal{E}$ if and only if $\mathcal{T}^3 = \mathcal{T}$.*

Proof: (Necessary Part) Assume $\mathcal{E}$ is a complete generating set.
Then, every element of $\mathcal{E}$ is decomposible.
So, $\mathcal{T}$ has no indecompossible elements. Therefore, $\mathcal{T}^3 = \mathcal{T}$.
(Sufficiency Part) Let $\mathcal{T}^3 = \mathcal{T}$.
Take $\mathcal{E}_0 = \{e_i : i \in \Lambda\}$.
Each e_i is decomposible, So

$$e_i = e_{\tau(i,1)} e_{\tau(i,2)} ......e_{\tau(i,p_i)} \tag{1}$$

where $p_i \geq 3$ and $\tau(i, j) \in \Lambda$ for all j $(1 \leq j \leq p_i)$. For all i and j $(i \in \Lambda, 1 \leq j \leq p_i - 1)$ define

$$\begin{aligned} \kappa_{i,j} &= e_{\tau(i,j+1)} ......e_{\tau(i,\frac{p_i-1}{2})} \\ \beta_{i,j} &= e_{\tau(i,\frac{p_i-1}{2}+1)} ......e_{\tau(i,p_i)} \end{aligned} \tag{2}$$

Then the below set is a generating set for $\mathcal{T}$.

$$\mathcal{E} = \mathcal{E}_0 \cup \{\kappa_{i,j} : i \in \Lambda, 1 \leq j \leq \frac{p_i - 1}{2} - 1\} \cup \{\beta_{i,j} : i \in \Lambda, \frac{p_i - 1}{2} \leq j \leq p_i - 1\}$$

Combining (1) and (2), we get $\mathcal{E}$ is complete. ∎

V. Applications

Free Ternary semigroups, fundamental in mathematics and computer science, find diverse applications across various domains. In automata theory, they underpin the theory of regular languages, aiding in the construction of finite automata and defining regular expressions. Moreover, in formal language theory[9], free ternary semigroups serve as the cornerstone, enabling the representation of strings over given alphabets and defining operations like concatenation and Kleene closure. Combinatorics on words benefits greatly from free ternary semigroups, as they facilitate the study of finite or infinite sequences of symbols, crucial for tasks like word enumeration and pattern matching. In algorithm design [10], particularly in string processing and text compression, free semigroups play a pivotal role, enabling the development of efficient algorithms for tasks such as searching and indexing. Furthermore, in coding theory, they provide a mathematical framework for analyzing error-correcting codes and designing encoding and decoding algorithms. In semigroup actions, symbolic dynamics, and semigroup presentations, free ternary semigroups offer insights into the structure and behavior of discrete systems, adding depth to the study of these areas.
The direct product of semigroups serves as a powerful tool in various mathematical contexts and practical applications. In algebraic structures, such as group theory, the direct product of semigroups provides a way to combine multiple semigroups into a single structure, preserving their individual properties. This concept finds application in the study of systems with parallel or independent components, where the behavior of each component can be analyzed separately before considering their combined effect. In computer science and engineering [11], the direct product of semigroups is used in modeling and analyzing concurrent systems, distributed computing, and communication protocols. By representing each component of a system as a semigroup, their direct product allows for the systematic study of interactions and dependencies among these components. Moreover, in cryptography [12] and coding theory [13], the direct product of semigroups can be utilized to construct error-correcting codes and cryptographic protocols with enhanced security and reliability.

VI. Conclusion

We have provided a clear definition of the concept of a free ternary semigroup and have also proven the mapping theorem of homomorphisms for this type of semigroup.We proved another homomorphism theorem that gives the relationship between the quotient free ternary semigroup and ternary semigroup and illustrated an example for this theorem. We have proven the essential requirements for the direct product of two infinite ternary semigroups to be finitely generated. In addition, we determined an upper limit for the rank of the direct product of two infinite semigroups. The necessary and sufficient condition for the direct product of two free ternary semigroups has been conclusively proven.

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