Analysis of Convergence and Fixed Point Computation in Graphical Generalized Metric Spaces with Applications to Cantilever Beam Problem

A. Bharathi, M. V. R. Kameswari, M. Madhuri and P. Padmavathi

***Abstract*—This study focuses on establishing the existence of fixed points in graphical $b_v(s)$ spaces using generalized contractions based on Reich and Edelstein-type conditions. The proposed framework extends classical contraction principles, allowing broader applications. Examples are provided to illustrate the results and enhance their clarity. The study further explores the solution of a fourth-order differential equation modeling the deformation of a cantilever beam under a uniformly distributed load, offering insights into its mechanical behavior.**

***Index Terms*—Cantilever beam problem, Fixed points, Graphical $b_v(s)$ metric spaces, Graphical Edelestein contractions, Graphical Reich contractions.**

I. Introduction

Recent research underscores the growing importance of graph theory across various disciplines, particularly in the realm of metric fixed point theory. A significant contribution by Jachymski [6] proposed a novel interpretation of the Banach contraction principle, highlighting the critical role of graph structures rather than merely relying on the traditional order structure of metric spaces. This idea has sparked further developments and generalizations in the field. For instance, Shukla et al. [9] introduced the notion of graphical metric spaces, where the triangle inequality is applicable only to elements connected within the graph, rather than universally across the entire space. This concept has been expanded to include several related structures, such as graphical b-metric spaces [4], graphical rectangular metric spaces [1], and graphical dislocated b-metric spaces [12]. Further studies exploring fixed point results within graph-based frameworks can be found in [2-4, 9-16]. Recently, Baradol et al. [3] advanced this field by extending the concept of $b_v(s)$ metric spaces [7] into the context of graph structures, thereby introducing graphical $b_v(s)$ metric spaces. They successfully formulated a graphical adaptation of the Banach contraction principle within this extended framework.

Manuscript received November 10, 2023; revised September 20, 2024.

A. Bharathi is a Research scholar in the Department of Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, Andhra Pradesh, India. (email:balla1@gitam.in)

M. V. R. Kameswari is an Associate Professor in the Department of Mathematics, School of Science, GITAM Deemed to be University ,Rushikonda, Visakhapatnam-530045, Andhra Pradesh, India. (Corresponding Author; e-mail: kmukkavi@gitam.edu)

M. Madhuri is a Research scholar of Department in the Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, Andhra Pradesh, India. (email: mmudunur@gitam.in)

P. Padmavathi is a Research scholar of Department in the Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, Andhra Pradesh, India. (e-mail:pboddapu@gitam.in)

In our current study, inspired by these graph-based approaches and the existing body of work, we propose a method to establish the existence of fixed points in graphical $b_v(s)$ metric spaces using the contraction frameworks developed by Reich [8] and Edelstein [5]. To demonstrate the breadth of our results, we provide examples that illustrate fixed point convergence, supported by both graphical analysis and numerical iteration. Our findings not only extend the work presented in [3] but also incorporate and generalize several key results from the current literature. Additionally, we apply our theoretical findings to solve a fourth-order differential equation related to the Cantilever beam problem under a uniform load distribution

II. Preliminaries

Before outlining our key findings, we start this section with some commonly used terminology and conclusions that are crucial to our findings.

Consider a non-empty set $\mathcal{J}$ that contains the diagonal Δ. We define a graph $\mathcal{G}$ as an ordered pair $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \Xi(\mathcal{G}))$, where $\mathcal{V}(\mathcal{G})$ is the set of vertices of $\mathcal{G}$ and $\Xi(\mathcal{G}) \subseteq \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$ is the set of edges of $\mathcal{G}$ i.e., $\mathcal{V}(\mathcal{G}) = \mathcal{J}$ and the set of edges $\mathcal{G}$ all the self loops on each vertex, that is $\Xi(\mathcal{G}) \supseteq \Delta$. In a graph $\mathcal{G}$. A directed path of length from x to y is a sequence of $(n+1)$ distinct vertices $\{x_i\}_{i=0}^{n}$ such that $x_0 = x, x_n = y$ and $(x_{i-1}, x_i) \in \Xi(\mathcal{G})$, for all $i = 1, 2, ..., n$, shortly $(xPy)_{\mathcal{G}}$ is used to represent a path from x to y in a graph $\mathcal{G}$. $z \in (xPy)_{\mathcal{G}}$ means that z lies on the path from x to y. A sequence $\{x_n\}$ is said to be $\mathcal{G}$-termwise connected ($\mathcal{G}$-TWC) if $(x_n P x_{n+1})_{\mathcal{G}}$, for all $n \in N$. For a mapping $\mathcal{L} : \mathcal{J} \times \mathcal{J}$, a sequence $\{x_n\}$ said to be $\mathcal{L}$-picard sequence if $\mathcal{L}x_n = x_{n+1}$ and $x_n \neq x_{n+1}$, for all $n \in N$. For $n \in N$, $[x]_{\mathcal{G}}^n = \{y \in \mathcal{J} : (xPy)_{\mathcal{G}}$ of length $n\}$.

Definition II.1:([9]) Assume that $\mathcal{G}$ is a graph associated with a non-empty set $\mathcal{J}$. A *graphical metric* on $\mathcal{J}$ is mapping $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ that satisfies the following criteria : for all $\jmath, \ell, c \in \mathcal{J}$ such that

$(\mathcal{G}_1)$ $\wp(\jmath, \ell) = 0$ iff $\jmath = \ell$.

$(\mathcal{G}_2)$ $\wp(\jmath, \ell) = \wp(\ell, \jmath)$.

$(\mathcal{G}_3)$ $(\jmath P \ell)_{\mathcal{G}}$ and $c \in (\jmath P \ell)_{\mathcal{G}} \Rightarrow \wp(\jmath, \ell) \leq \wp(\jmath, c) + \wp(c, \ell)$,

the pair $(\mathcal{J}, \wp)$ is a graphical metric space.

Definition II.2:([3]) Assume that $\mathcal{G}$ is a graph associated with a non-empty set $\mathcal{J}$. A *graphical b- metric* on $\mathcal{J}$ is a mapping $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ that satisfies the following criteria : for all $\jmath, \ell,\ c \in \mathcal{J}$ such that

$(\mathcal{G}_b(1))$ $\wp_b(\jmath, \ell) = 0$ iff $\jmath = \ell$.

$(\mathcal{G}_b(2))$ $\wp_b(\jmath, \ell) = \wp_b(\ell, \jmath)$.

$(\mathcal{G}_b(3))$ $(\jmath P\ell)_{\mathcal{G}}$ and $c \in (\jmath P\ell)_{\mathcal{G}}$

$$\Rightarrow \wp_b(\jmath, \ell) \leq s[\wp_b(\jmath, c) + \wp_b(c, \ell)],$$

the pair $(\mathcal{J}, \wp_b)$ is termed as graphical b-metric space.

Definition II.3:([2,11]) Let $\mathcal{G}$ be a graph associated with a non-empty set $\mathcal{J}$. A *graphical rectangular metric* on $\mathcal{J}$ is a mapping $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ that satisfies the following criteria : for all $\jmath, \ell \in \mathcal{J},\ c, d \in \mathcal{J}/\{\jmath, \ell\}$ such that

$(\mathcal{G}r(1))$ $\wp_{rms}(\jmath, \ell) = 0$ iff $\jmath = \ell$.

$(\mathcal{G}r(2))$ $\wp_{rms}(\jmath, \ell) = \wp_{rms}(\ell, \jmath)$,

$(\mathcal{G}r(3))$ $\wp_{rms}(\jmath, \ell) \leq s[\wp_{rms}(\jmath, c) + \wp_{rms}(c, d) + \wp(d, \ell)]$.

Then the pair $(\mathcal{J}, \wp_{rms})$ is termed as graphical rectangular metric space.

Definition II.4:([3]) Let $\mathcal{G}$ be a graph associated with a non-empty set $\mathcal{J}$. A mapping $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ that satisfies the following criteria is a *graphical* $b_v(s)$ *metric*: for $v \in N$ and for all $\jmath, \ell \in \mathcal{J}$ such that

$(\mathcal{G}b_v(s)(1))$ $\wp_{b_v(s)}(\jmath, \ell) = 0$ iff $\jmath = \ell$.

$(\mathcal{G}b_v(s)(2))$ $\wp_{b_v(s)}(\jmath, \ell) = \wp_{b_v(s)}(\ell, \jmath)$,

$(\mathcal{G}b_v(s)(3))$ for all distinct $p_1, p_2, ..., p_v \in (\jmath P\ell)_{\mathcal{G}}$ and a real number $s \geq 1$ holds

$$\wp_{b_v(s)}(\jmath, \ell) \leq s[\wp_{b_v(s)}(\jmath, p_1) + \wp_{b_v(s)}(p_1, p_2) + ... + \wp_{b_v(s)}(p_v, \ell)].$$

The pair $(\mathcal{J}, \wp)$ is termed as graphical $b_v(s)$ metric space.

By giving precise values for v and s, it becomes evident that we can draw the following conclusions [3].

(i) A graphical metric space is a graphical $b_1(1)$-metric space.

(ii) A graphical b-metric space with coefficient s is known as a graphical $b_1(s)$-metric space.

(iii) A graphical rectangular metric space is known as a graphical $b_2(1)$-metric space

(iv) A graphical rectangular b with coefficient s is known as a graphical $b_2(s)$-metric space.

Note II.5: From the definitions $b_v(s)$ metric space (Mitrovic and Radenovic) and graphical $b_v(s)$ metric space (Baradol et. al.,), it is observed that every graphical $(\mathcal{G}b_v(s)(1))$ metric space is $b_v(s)$ metric space. However, it is important be note that the converse is not necessarily true, as illustrated in the following examples.

Example II.6: Let $\mathcal{J} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$ and let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ be an undirected graph, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected components with

$V(\mathcal{G}_1) = \{a_1, a_2, a_3, a_4, a_5\}$,

$\Xi(\mathcal{G}_1) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and

$V(\mathcal{G}_2) = \{a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}\}$,

$\Xi(\mathcal{G}) = \{e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}\}$

encompassing a graph as shown in Fig 3 .

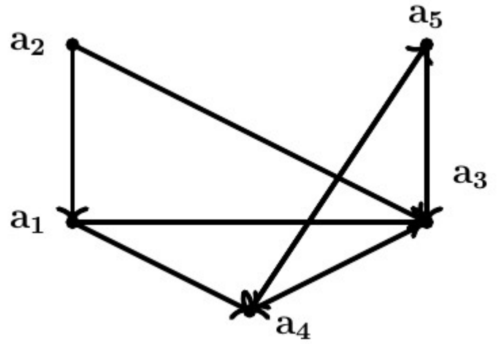

Fig. 1. $\mathcal{G}_1$

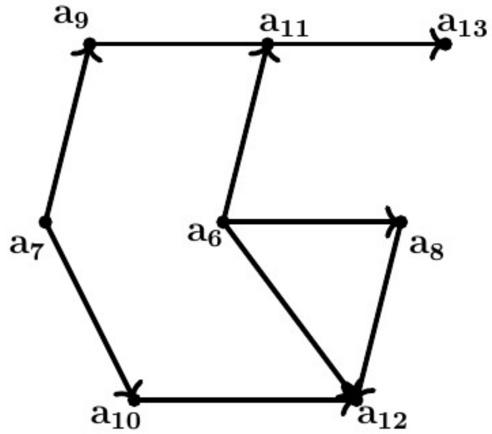

Fig. 2. $\mathcal{G}_2$

Fig. 3. $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$

Let $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ be a mapping defined in the following way:

$$\wp(a_i, a_j) = \begin{cases} o & if\ a_i = a_j \\ d(a_i, a_j) & if a_i, a_j \in \mathcal{G}_l, l = \{1, 2\} \\ \frac{1}{5} & otherwise. \end{cases}$$

where $d(a_i, a_j)$ = shortest distance between a_i and a_j.

Then clearly, $(\mathcal{J}, \wp)$ is a graphical $b_3(1)$ metric space but not $b_3(1)$ metric space. For, let $x = a_7, y = a_{13}$

$$\wp(a_7, a_{13}) = 3 > \wp(a_7, a_2) + \wp(a_2, a_1) + \wp(a_1, a_4) + \wp(a_4, a_{13}) = \tfrac{1}{5} + 1 + 1 + \tfrac{1}{5} = \tfrac{12}{5}.$$

Example II.7: Let $\mathcal{J} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \cup \{1, 0, 5\}$. Let $\mathcal{G}$ be a directed graph defined by $\mathcal{V}(\mathcal{G}) = \mathcal{J}$ and

$$\Xi(\mathcal{G}) = \{(0, 1), (1, \tfrac{1}{3}), (\tfrac{1}{3}, 5), (5, \tfrac{1}{2}), (0, \tfrac{1}{2}), (\tfrac{1}{4}, \tfrac{1}{2}), (\tfrac{1}{4}, 5), (1, \tfrac{1}{4}), (\tfrac{1}{3}, 0), (\tfrac{1}{3}, 1), (\tfrac{1}{4}, 0), (\tfrac{1}{4}, \tfrac{1}{3})\}.$$

We define $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ by $\wp(x, y) = 0$ implies $x = y$

$\wp(0, \frac{1}{2}) = 4 = \wp(\frac{1}{2}, 0)$, $\wp(\frac{1}{4}, \frac{1}{2}) = \wp(\frac{1}{2}, \frac{1}{4}) = 10$,

$\wp(\frac{1}{4}, 1) = \wp(1, \frac{1}{4}) = 2$, $\wp(\frac{1}{3}, 0) = \wp(0, \frac{1}{3}) = 1$,

$\wp(1, \frac{1}{3}) = \wp(\frac{1}{3}, 1) = 1$, $\wp(\frac{1}{4}, 0) = \wp(0, \frac{1}{4}) = 5$,
$\wp(\frac{1}{4}, \frac{1}{3}) = \wp(\frac{1}{3}, \frac{1}{4}) = 3$
$\wp(5, 1) = \wp(1, 5) = 1$, $\wp(0, 1) = \wp(1, 0) = 2$,
$\wp(x, y) = \frac{4}{5}$ otherwise, endowed with the graph as shown in the following Fig 4.

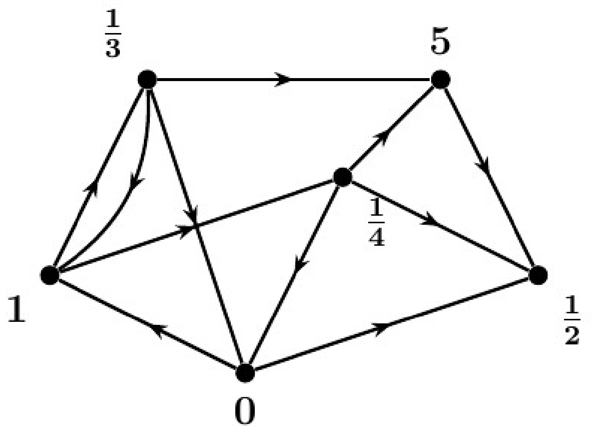

Fig. 4. **Graph $\mathcal{G}$ associated with $\mathcal{J}$**

Then $(\mathcal{J}, \wp)$ is a graphical $b_4(1)$ metric space. It is important to note that $(\mathcal{J}, \wp)$ is not a $b_4(1)$ metric space since,

$$\wp(\tfrac{1}{4}, \tfrac{1}{2}) = 10 > \wp(\tfrac{1}{4}, 1) + \wp(1, 5) + \wp(5, \tfrac{1}{3}) + \wp(\tfrac{1}{3}, 0) + \wp(0, \tfrac{1}{4}) = \tfrac{49}{5}.$$

Definition II.8:([3]) Let $(\mathcal{J}, \wp)$ denote a graphical $b_v(s)-$ metric space associated with a graph $\mathcal{G}$ and consider a sequence $\{y_n\}$ in $\mathcal{J}$, $\{y_n\}$ is a cauchy sequence if for each $\epsilon > 0$, there exists $m \in N$ such that $\wp(y_k, y_l) < \epsilon$, for all $k, l \geq m$. That is $\lim_{k,l\to\infty} \wp(y_k, y_l) = 0$, the sequence $\{y_n\}$ converges to $z \in \mathcal{J}$ if for each $\epsilon \geq 0$, there exists $m \in N$ such that $\wp(y_k, z) < \epsilon$, for all $k \geq m$ i.e., $\lim_{k\to\infty} \wp(y_k, z) = 0$.

Definition II.9: ([3]) A graphical $b_v(s)$-metric space is called $\mathcal{G}$-complete, if every term-wise connected (shortly $\mathcal{G}$-TWC), Cauchy sequence converges in $\mathcal{J}$.

Definition II.10:([3]) Consider a graphical $b_v(s)$-metric space $(\mathcal{J}, \wp)$. A graphic Banach contraction ($\mathcal{G}$BC) on $\mathcal{J}$ is a mapping $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ such that:

($\mathcal{G}$BC-I) $(\mathcal{L}_\jmath, \mathcal{L}_\ell) \in \Xi(\mathcal{G})$, whenever $(\jmath, \ell) \in \Xi(\mathcal{G})$.

($\mathcal{G}$BC-II) For all $(\jmath, \ell) \in \Xi(\mathcal{G})$, there exists $\eta \in (0, 1)$ such that $\wp(\mathcal{L}_\jmath, \mathcal{L}_\ell) \leq \eta\wp(\jmath, \ell)$.

Here it is observed that any Banach contraction on a non-empty set $\mathcal{J}$ is a graphic Banach contraction on $\mathcal{J}$ after considering the set of edges is equal to $\mathcal{J} \times \mathcal{J}$. But, its converse may not hold, as stated in Remark 2.3 ([3]).

Definition II.11:([3]) A graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \Xi(\mathcal{G}))$ is said to have property $(\mathcal{S})$ if, for each convergent $\mathcal{G}$-TWC, the $\mathcal{L}$-Picard sequence $\{y_n\}$ has a limit ρ in $\mathcal{J}$ such that, $(y_k, \rho) \in \Xi(\mathcal{G})$ or $(\rho, y_k) \in \Xi(\mathcal{G})$, for all $k \geq m$.

Definition II.12:([3]) Consider a complete graphical $b_v(s)$ -metric space $(\mathcal{J}, \wp)$ and let $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ be an injective $\mathcal{G}$-TWC on $\mathcal{J}$. Suppose that the following conditions are satisfied:

(i) There exists $x_o \in \mathcal{J}$ with $\mathcal{L}^p_{x_o} \in [x_o]^{r_p}_{\mathcal{G}}$ for $p = 1, 2, ..., v$, where $r_p = m_p v + 1$ and $m_p \in N \cup \{0\}$.

(ii) $\mathcal{G}$ has property $(\mathcal{S})$.

Then for initial term $x_o \in \mathcal{J}$, the $\mathcal{L}$-Picard sequence $\{x_n\}$ in $\mathcal{G}$-TWC and converges to both ρ^* and $\mathcal{L}\rho^*$ in $\mathcal{J}$.

Definition II.13:([3]) Let $\mathcal{G}$ be a graph associated with graphical $b_v(s)$ -metric space $(\mathcal{J}, \wp)$. A graphical Reich contraction($\mathcal{G}$RC) on $\mathcal{J}$ is a mapping $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ such that

($\mathcal{G}$RC-I) for all $(p, q) \in \Xi(\mathcal{G}) \Rightarrow (\mathcal{L}p, \mathcal{L}q), (p, \mathcal{L}p), (q, \mathcal{L}q) \in \Xi(\mathcal{G})$.

($\mathcal{G}$RC-II) for all $(p, q) \in \Xi(\mathcal{G})$ then there exists non-negative integers a, b, c such that $a + b + c < 1$ and

$$\wp(\mathcal{L}p, \mathcal{L}q) \leq a\wp(p, \mathcal{L}p) + b\wp(q, \mathcal{L}q) + c\wp(p, q).$$

Remark II.14: By giving precise values for a, b , c , v and s, we can draw the following conclusions, highlighting the extensive applicability and versatility of Definition II.3.

(i) When we consider $a = b = 0$ in $(\mathcal{G}RC)$, in Definition II.13, we obtain graphic Banach contraction $(\mathcal{G}BC)$[3].

(ii) When we consider $a = b = \lambda$, where $\lambda \in [0, \frac{1}{s+1})$ and $v = 2$ in Definition II.13, we obtain Kannan $\mathcal{G}$-contraction [1].

(ii) When we consider $v = 2$, in Definition II.13, we obtain Reich $\mathcal{G}$-contraction [1].

Example II.15: Let $\mathcal{J} = \{0, \frac{1}{2}, \frac{1}{4}, 1, 5\}$. Let $\mathcal{G}$ be a directed graph defined by $\mathcal{V}(\mathcal{G}) = \mathcal{J}$ and

$$\Xi(\mathcal{G}) = \{(0, 0), (0, 1), (1, 0), (0, \tfrac{1}{2}), (\tfrac{1}{2}, 5), (0, 5), (5, 1), (\tfrac{1}{2}, 1), (5, 0), (0, \tfrac{1}{4}), (\tfrac{1}{4}, 1), (5, \tfrac{1}{4}), (1, 1)\}.$$

We define $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ by $\wp(\jmath, \ell) = 0$ implies $\jmath = \ell$
$\wp(0, \frac{1}{2}) = 2 = \wp(\frac{1}{2}, 0)$, $\wp(\frac{1}{2}, 5) = \wp(5, \frac{1}{2}) = 6$
$\wp(\frac{1}{4}, 1) = \wp(1, \frac{1}{4}) = \wp(\frac{1}{2}, 1) = \wp(1, \frac{1}{2})$
$= \wp(5, 0) = \wp(0, 5) = 3$,
$\wp(\frac{1}{4}, \frac{1}{2}) = \wp(\frac{1}{2}, \frac{1}{4}) = 5$,$\wp(0, 1) = \wp(1, 0) = 1$,
$\wp(5, 1) = \wp(1, 5) = 2$, $\wp(\jmath, \ell) = \frac{4}{5}$ otherwise, endowed with the graph as shown in the following Fig 5.

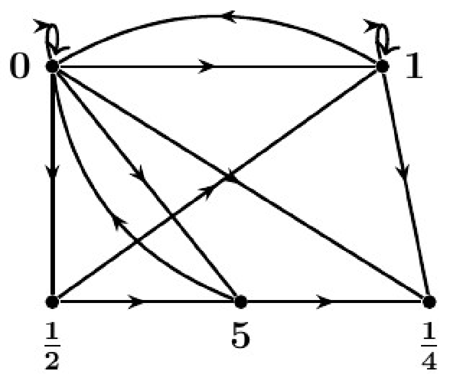

Fig. 5. **Graph $\mathcal{G}$ associated with $\mathcal{J}$**

Then $(\mathcal{J},\wp)$ is a graphical $b_3(1)$ metric space. It is important to note that $(\mathcal{J},\wp)$ is not a $b_3(1)$ metric space, since,

$$\wp(\frac{1}{2},5)=6>\wp(\frac{1}{2},1)+\wp(1,0)+\wp(0,\frac{1}{4})+\wp(\frac{1}{4},5)=\frac{28}{5}.$$

We now define $\mathcal{L}:\mathcal{J}\to\mathcal{J}$ as

$$\mathcal{L}(\jmath) = \begin{cases} 0 & if\ \jmath\in\{0,1\} \\ 1 & if\ \jmath\in\{5,\frac{1}{4}\} \\ 5 & \jmath=\frac{1}{2} \end{cases}$$

Clearly, $\mathcal{L}$ is a graphical Reich contraction on $\mathcal{J}$ with $a=\frac{1}{3}, b=\frac{1}{4}$ and $c=\frac{5}{13}$.

It is important note that when considering $\jmath=\frac{1}{2}$ and $\ell=1$, there does not exist any constant $c\in(0,1)$, for which graphic Banach contraction $(\mathcal{G}BC)$ holds, indeed

$$\wp(\mathcal{L}\jmath,\mathcal{L}\ell)=\wp(\mathcal{L}\frac{1}{2},\mathcal{L}1)=3>c(3)=c\wp(\jmath,\ell).$$

Therefore, it is of interest to us to ascertaining the existence and uniqueness of fixed points for graphical Reich contractions.

Following on the similar lines of Lemma 2 and Lemma 3 of [3] , we have the following lemmas.

Lemma II.16: Let $\mathcal{G}$ be a graph associated with graphical $b_v(s)$ -metric space $\mathcal{L}:\mathcal{J}\to\mathcal{J}$ such that $\mathcal{L}$ is $\mathcal{G}$RC. If $(p,\mathcal{L}p)\in\Xi(\mathcal{G})$ then $\wp(\mathcal{L}^n p,\mathcal{L}^{n+1}p)\le\beta^n\wp(p,\mathcal{L}p)$, where $\beta=\frac{a+c}{1-b}$.

Lemma II.17: Let $(\mathcal{J},\wp)$ be a graphically $b_v(s)$ -metric space associated with a graph $\mathcal{G}$ and $\mathcal{L}:\mathcal{J}\to\mathcal{J}$ is a $\mathcal{G}$RC. If $(p,q)\in\Xi(\mathcal{G})$ then $\wp(\mathcal{L}^n p,\mathcal{L}^n q)\to 0$ as $n\to\infty$.

Definition II.18: Let $(\mathcal{J},\wp)$ denote a graphical $b_v(s)$ -metric space associated with a graph $\mathcal{G}$ and $\mathcal{L}$ is a $\mathcal{G}$RC contraction on $\mathcal{J}$. The quadruple $(\mathcal{J},\wp,\mathcal{G},\mathcal{L})$ have property$(\mathcal{R}^*)$, if each $\mathcal{G}$-TWC $\mathcal{L}$-Picard sequence $\{x_n\}$ in $\mathcal{J}$ has the unique fixed point.

III. Fixed point results on graphical Reich contractions

We now prove our main result.

Theorem III.1: Consider an injective map $\mathcal{L}:\mathcal{J}\to\mathcal{J}$ where $(\mathcal{J},\wp)$ is a $\mathcal{G}$-complete $b_v(s)$ -metric space. Suppose that $\mathcal{L}$ is a $\mathcal{G}$RC defined on $\mathcal{J}$ satisfying:

(i) there exists $x_o\in\mathcal{J}$ with $\mathcal{L}^p_{x_o}\in[x_o]^{r_p}_{\mathcal{G}}$ for $p=1,2,...,v$, where $r_p=m_pv+1$ and $m_p\in N\cup\{0\}$

(ii) $\mathcal{G}$ has property $(\mathcal{S})$

(iii) graphical $b_v(s)$ -metric space is continuous.

Then the Picard sequence $\{x_n\}$, for any $x_o\in\mathcal{J}$ is $\mathcal{G}$-TWC and converges to ρ^* in $\mathcal{J}$.

Proof: For $p=1,2,...,v$, under the assumption (i), suppose that $x_o\in\mathcal{J}$ such that

$\mathcal{L}^p_{x_o}\in[x_o]^{r_p}_{\mathcal{G}}$, where $r_p=m_pv+1$ and $m_p\in N\cup\{0\}$.

Then there exists a path $\{e^1_k\}^{r_1}_{k=0}$ such that $x_o=e^1_0$, $\mathcal{L}^1_{x_o}=e^1_{r_1}$ and $(e^1_{k-1},e^1_k)\in\Xi(\mathcal{G})$, for all $k=1,2,...,r_1$.

Since $(e^1_{k-1},e^1_k)\in\Xi(\mathcal{G})$ by ($\mathcal{G}$RC-I), we have

$$(\mathcal{L}e^1_{k-1},\mathcal{L}e^1_k),(e^1_{k-1},\mathcal{L}e^1_{k-1}),(e^1_k,\mathcal{L}e^1_k)\in\Xi(\mathcal{G}),$$

for all $k=1,2,3,...,\ r_1$.

Therefore $\{\mathcal{L}e^1_k\}^{r_1}_{k=0}$ is a path from $\mathcal{L}e^1_0=\mathcal{L}x_o=x_1$ to $\mathcal{L}e^1_{r_1}=\mathcal{L}^2x_o=x_2$ of length r_1.

Continuing this process, for all $n\in N$, we obtain $\{\mathcal{L}^ne^1_k\}^{r_1}_{k=0}$ a path from $\mathcal{L}^ne^1_0=\mathcal{L}^nv_0=x_n$ to $\mathcal{L}^ne^1_{r_1}$ $=\mathcal{L}^n\mathcal{L}x_o=x_{n+1}$ of length r_1, which leads $\{x_n\}$ is $\mathcal{G}$-TWC sequence.

On using Lemma II.14, we have

$$\begin{aligned}\wp(x_n,x_{n+1})&=\wp(\mathcal{L}^nx_0,\mathcal{L}^nx_1)=\wp(\mathcal{L}^ne^p_0,\mathcal{L}^ne^1_{r_1})\\ &\le s[\wp(\mathcal{L}^ne^1_0,\mathcal{L}^ne^1_1)+\wp(\mathcal{L}^ne^1_1,\mathcal{L}^ne^1_2)+...\\ &\quad+\wp(\mathcal{L}^ne^1_{v-1},\mathcal{L}^ne^1_v)]+s^2[\wp(\mathcal{L}^ne^1_v,\mathcal{L}^ne^1_{v+1})\\ &\quad+\wp(\mathcal{L}^ne^1_{v+1},\mathcal{L}^ne^1_{v+2})+..\\ &\quad+\wp(\mathcal{L}^ne^1_{2v-1},\mathcal{L}^ne^1_{2v})+...\\ &\quad+s^{m_1}[\wp(\mathcal{L}^ne^1_{(m_1-1)v},\mathcal{L}^ne^1_{(m_1-1)v+1})+...\\ &\quad+\wp(\mathcal{L}^ne^1_{r_1-1},\mathcal{L}^ne^1_{r_1})] \qquad (1)\\ &\quad\to 0 \text{ as } n\to\infty.\end{aligned}$$

Similarly, for $p=1,2,...,v$, Lemma II.16 and (1), we have

$$\begin{aligned}\wp(x_n,x_{p+n})&=\wp(\mathcal{L}^nx_0,\mathcal{L}^nx_p)=\wp(\mathcal{L}^ne^p_0,\mathcal{L}^ne^p_{r_p})\\ &\le s[\wp(\mathcal{L}^ne^p_0,\mathcal{L}^ne^p_1)+\wp(\mathcal{L}^ne^p_1,\mathcal{L}^ne^p_2)\\ &\quad+...+\wp(\mathcal{L}^ne^p_{v-1},\mathcal{L}^ne^p_v)]\\ &\quad+s^2[\wp(\mathcal{L}^ne^p_v,\mathcal{L}^ne^p_{v+1})\\ &\quad+\wp(\mathcal{L}^ne^p_{v+1},\mathcal{L}^ne^p_{v+2})+...\\ &\quad+\wp(\mathcal{L}^ne^p_{2v-1},\mathcal{L}^ne^p_{2v})]+..\\ &\quad+s^{m_p}[\wp(\mathcal{L}^ne^p_{(m_p-1)v},\mathcal{L}^ne^p_{(m_p-1)v+1})+...\\ &\quad+\wp(\mathcal{L}^ne^p_{r_p-1},\mathcal{L}^ne^p_{r_p})]\to 0 \text{ as } n\to\infty,\end{aligned}$$

therefore $\{x_n\}$ is a Cauchy sequence in $\mathcal{J}$. (2)

Consequently, by $\mathcal{G}$-completeness of $\mathcal{J}$ implies $x_n\to\rho^*$, for some $\rho^*\in N$. According to the property $(\mathcal{S})$, there exists $k\in N$ such that $(x_n,\rho^*)\in\Xi(\mathcal{G})$ or $(\rho^*,x_n)\in\Xi(\mathcal{G}),\forall n>k$.

Assume that for all $n>k,(x_n,\rho^*)\in\Xi(\mathcal{G})$. Then by ($\mathcal{G}$RC-I), $(\mathcal{L}x_n,\mathcal{L}\rho^*)\in\Xi(\mathcal{G})$. By ($\mathcal{G}$RC-II), we have

$$\begin{aligned}\wp(\mathcal{L}x_n,\mathcal{L}\rho^*)&\le a\wp(x_n,\mathcal{L}x_n)+b\wp(\rho^*,\mathcal{L}\rho^*)+c\wp(x_n,\rho^*)\\ &=a\wp(x_n,x_{n+1})+b\wp(\rho^*,\mathcal{L}\rho^*)+c\wp(x_n,\rho^*).\end{aligned}$$

Taking limits as $n\to+\infty$, using continuity of $\wp$, employing (1) and (2), we obtain

$$\wp(\rho^*,\mathcal{L}\rho^*)\le b\wp(\rho^*,\mathcal{L}\rho^*)<\wp(\rho^*,\mathcal{L}\rho^*),$$

which leads to a contradiction. Therefore $x_{n+1}\to\mathcal{L}\rho^*$. Hence, $\mathcal{L}\rho^*$ is also limit of the sequence $\{x_n\}$.

Similarly, we can prove the case when $(\rho^*,x_n)\in\Xi(\mathcal{G})$, for $n>k$.

It is obvious that a sequence in graphical b$_v$(s) metric space may converge to more than one limit[3].

Theorem III.2: Along with the hypotheses of Theorem III.1, if the quadruple $(\mathcal{J}, \wp, \mathcal{G}, \mathcal{L})$ has property $(\mathcal{R}^*)$, then $\mathcal{L}$ has a fixed point.

Proof: From the proof of Theorem III.1 and the Property $(\mathcal{R}^*)$, we have $\mathcal{L}\rho^* = \rho^*$.

Theorem III.3: Along with the conditions of Theorem III.2, suppose that for all $\rho^*, \eta^* \in fix(\mathcal{L})$F there exists a path $(\rho^* P \eta^*)_{\mathcal{G}}$ between ρ^* and η^* of length l, where $l = 1$ or $l = mv + 1$ for $m \in N \cup \{0\}$. Then $\mathcal{L}$ has a unique fixed point.

Proof: Case(i). If $l = 1$, then $(\rho^*, \eta^*) \in \Xi(\mathcal{G})$.

By ($\mathcal{G}$RC-I), $(\mathcal{L}\rho^*, \mathcal{L}\eta^*) \in \Xi(\mathcal{G})$.

Again, by ($\mathcal{G}$RC-II), we have

$$\wp(\mathcal{L}\rho^*, \mathcal{L}\eta^*) \leq a\wp(\rho^*, \mathcal{L}\rho^*) + b\wp(\eta^*, \mathcal{L}\eta^*) + c\wp(\rho^*, \eta^*)$$

implies $\wp(\rho^*, \eta^*) \leq c\wp(\rho^*, \eta^*)$,

hence, $\rho^* = \eta^*$.

Case(ii). If $l = mv + 1, m \in N \cup \{0\}$ and let $\{e_i\}_{j=0}^{l}$ be the path from ρ^* to η^*, so that $e_0 = \rho^*$ and $e_k = \eta^*$. Then

$$\begin{aligned}\wp(\rho^*, \eta^*) &= \wp(\mathcal{L}^n\rho^*, \mathcal{L}^n\eta^*)\\ &\leq s[\wp(\mathcal{L}^n e_0, \mathcal{L}^n e_1) + \wp(\mathcal{L}^n e_1, \mathcal{L}^n e_2) + ...\\ &+\wp(\mathcal{L}^n e_{v-1}, \mathcal{L}^n e_v)] + s^2[\wp(\mathcal{L}^n e_v, \mathcal{L}^n e_{v+1})\\ &+\wp(\mathcal{L}^n e_{v+1}, \mathcal{L}^n e_{v+2}) + ... + \wp(\mathcal{L}^n e_{2v-1}, \mathcal{L}^n e_{2v})]\\ &+ ... + s^m[\wp(\mathcal{L}^n e_{(m-1)v}, \mathcal{L}^n e_{(m-1)v+1}) + ...\\ &+\wp(\mathcal{L}^n e_{l-1}, \mathcal{L}^n e_l)] \to 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore $\rho^* = \eta^*$.

Example III.4: Let $A = \{\frac{1}{3^n} : n \in N\}$, $B = \{0, 1, 2\}$ and $\mathcal{J} = A \cup B$ associated with the graph $\mathcal{G} = (V(\mathcal{G}, \Xi(\mathcal{G}))$ such that $V(G) = \mathcal{J}$ and $\Xi = \Delta \cup \{(0, \frac{1}{3^n}) : n \in N\} \cup \{(r, s) \in A \times A : r \leq s\}$. We define a function $\wp : \mathcal{J} \times \mathcal{J} \to [0, \infty)$ such that

$$\wp(0, \frac{1}{3^n}) = \wp(\frac{1}{3^n}, 0) = 0$$

$$\wp(\frac{1}{3^m}, \frac{1}{3^n}) = \wp(\frac{1}{3^n}, \frac{1}{3^m}) = \frac{1}{3^{l-1}}$$

,

$$\wp(a, b) = \wp(b, a) = 0,$$

where $a \in A$, $b \in B$ and $l = max\{m, n\}$. Then $(\mathcal{J}, \wp)$ is a complete $b_4(3)$ metric space. We define $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ by

$$\mathcal{L}(\jmath) = \begin{cases} \frac{\jmath^2}{3^2} & if\ x \in [0, 1) \\ \frac{\jmath}{2} & otherwise, \end{cases}$$

then $\mathcal{L}$ is an injective $\mathcal{G}$RC with $a = \frac{1}{2},\ b = \frac{1}{3},\ c = \frac{1}{5}$. For initial point $x_o = \frac{1}{3}$, the sequence $x_n = \frac{1}{3^n}$ is a $\mathcal{G}$-TWC $\mathcal{L}$ -picard sequence. Since, for some fixed $t \in N$ there exists $n_o \in N$ such that

$$0 \leq (\frac{1}{3})^n (3)^t < 1$$

for all $n > n_0$, using this inequality, we can prove that $\{x_n\}$ is a Cauchy sequence. Hence $\mathcal{L}$ satisfies all the conditions of Theorem III.2, with 0 is the unique common fixed point of $\mathcal{L}$.

Example III.5: Let $\mathcal{J} = [0, 1]$ be endowed with a $b_v(s)$ metric space defined by
$\wp(x, \ell) = (x - \ell)^3$, for $x, \ell \in \mathcal{J}$. Obviously, $\mathcal{J}$ is graphical $b_3(1)$ metric space. Consider a mapping from $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ by

$$\mathcal{L}(x) = \begin{cases} \frac{x}{x^2+7} & if\ \jmath \in [0, 1) \\ \frac{1}{4} & otherwise. \end{cases}$$

We define $\mathcal{G} = (\mathcal{V}(\mathcal{G})), \Xi(\mathcal{G})$ by $\mathcal{V}(\mathcal{G}) = \mathcal{J}$ and $\Xi = \{(x, \ell) \in \mathcal{J} \times \mathcal{J} : x \leq \ell, x, \ell \in [0, 1)\}$. Clearly, $\mathcal{L}$ is an injective mapping. We now verify the inequality $\mathcal{G}RC - II$ with $a = \frac{1}{2},\ b = \frac{1}{5},\ c = \frac{1}{8}$,

$$\begin{aligned}\wp(\mathcal{L}x, \mathcal{L}\ell) &= |\tfrac{x}{x^2+7} - \tfrac{\ell}{\ell^2+7}|^3 = |x - \ell|^3 (\tfrac{|7 - x\ell|}{(\ell^2+7)(x^2+7)})^3\\ &\leq \tfrac{1}{8}|x - \ell|^3\\ &\leq \tfrac{1}{2}|x - \tfrac{x}{x^2+7}|^3 + \tfrac{1}{5}|\ell - \tfrac{\ell}{\ell^2+7}|^3 + \tfrac{1}{8}|x - \ell|^3.\end{aligned}$$

Clearly, '0' is the unique fixed point of $\mathcal{L}$.

We now present numerical calculations for the approximate fixed point of $\mathcal{L}$, as shown in Table I. The convergence behavior of these iterations is also analyzed, with results displayed in Fig 6.

TABLE I

Iteration	x_0= 0.3	x_0=0.5	x_0=0.7	x_0= 0.9
x_1	0.0423	0.0690	0.0935	0.1152
x_2	0.0060	0.0098	0.0133	0.0164
x_3	0.0009	0.0014	0.0019	0.0023
x_4	0.0001	0.0002	0.0003	0.0003
x_5	0.0000	0.0000	0.0000	0.0000
x_6	0.0000	0.0000	0.0000	0.0000

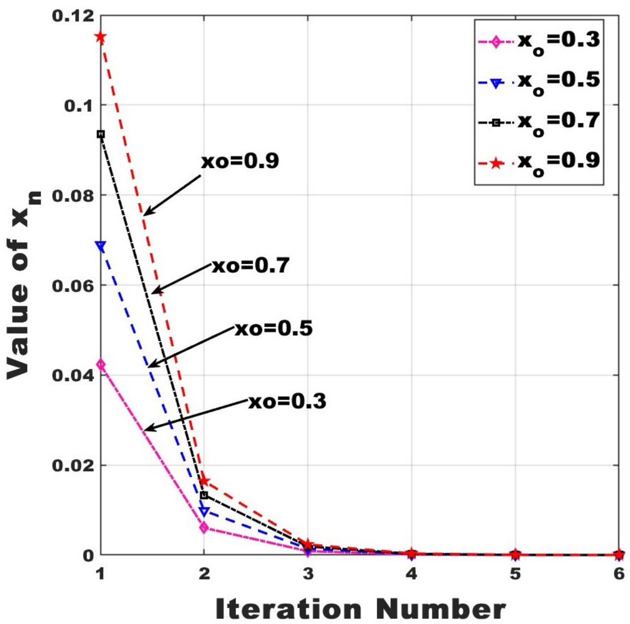

Fig. 6. Graph of convergence behaviour of x_n

IV. Fixed point theorems on graphical Edlestein contraction maps

Let $\mathcal{G}$ be a graph associated with b_v(s)-metric space $(\mathcal{J}, \wp)$. A path $\{\hbar_i\}_{i=0}^{r_p}$ from x to y, such that $\hbar_0 = e_0^p = x$, $\hbar_{r_p} = e_{r_p}^p = y$ is said to be ϵ-chainable if $\wp(\hbar_{i-1}, \hbar_i) < \epsilon$, $i = 1, 2, ..., r_p$ i.e., denote $\Xi_\epsilon(\mathcal{G}) = \{(p, q) \in \Xi(\mathcal{G}) : \wp(p, q) < \epsilon\}$.

Definition IV.1: Let $\mathcal{G}$ be a graph associated with graphical b_v(s)-metric space $(\mathcal{J}, \wp)$. An $\mathcal{G}$ -Edelestein contraction on $\mathcal{J}$ is a mapping $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ such that

($\mathcal{G}$ETC-I) for each $(\jmath, \ell) \in \mathcal{G}$ implies $(\mathcal{L}\jmath, \mathcal{L}\ell) \in \Xi(\mathcal{G})$

($\mathcal{G}$ETC-II) there exists $\Lambda \in [0, 1)$ such that for all $\jmath, \ell \in \Xi_\epsilon(\mathcal{G})$ implies

$$\wp(\mathcal{L}\jmath, \mathcal{L}\ell) < \Lambda\wp(\jmath, \ell) \tag{3}$$

Theorem IV.2: Let $\mathcal{G}$ be a graph associated with b_v(s)-metric space and $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ be an injective ($\mathcal{G}$ETC-II). Suppose that :

(i) there exist $x_o \in \mathcal{J}$ with $\mathcal{L}_{x_o}^p \in [x_o]_{\mathcal{G}}^{r_p}$ for

$p = 1, 2, ..., v$,where $r_p = m_p v + 1$ and $m_p \in N \cup \{0\}$.

(ii) there exists $x_o \in \mathcal{J}$ such that there is an ϵ- chainable

path from x_o to $\mathcal{L}x_o$ in the graph $\mathcal{G}$.

(iii) If a $\mathcal{G}$-TWC Picard sequence $\{x_n\}$ converges in $\mathcal{J}$,

there exists $l \in N$ and $\zeta^* \in \mathcal{J}$ such that

$(x_n, \zeta^*) \in \Xi_\epsilon(\mathcal{G})$ or $(\zeta^*, x_n) \in \Xi_\epsilon(\mathcal{G})$ for all $n > l$.

Then for initial term $x_o \in \mathcal{J}$, the picard sequence $\{x_n\}$ is $\mathcal{G}$TWC and converges to ζ^* in $\mathcal{J}$.

Proof: For $p = 1, 2, 3, ...v$, let $x_o \in \mathcal{J}$, $\mathcal{L}^p x_o \in [x_o]_{\mathcal{G}}^{r_p}$, where $r_p = m_p v + 1$ and $m_p \in N \cup \{0\}$. Then there exists path $\{e_k^1\}_{k=0}^{r_1}$ such that $x_o = e_0^1$, $\mathcal{L}^1 x_o = e_{r_1}^1$ and $(e_{k-1}^1, e_k^1) \in \Xi(\mathcal{G})$ for all $k = 1, 2, ..., r_1$ since $(e_{k-1}^1, e_k^1) \in \Xi(\mathcal{G})$ by ($\mathcal{G}$ETC-I), we have $(\mathcal{L}e_{k-1}^1, \mathcal{L}e_k^1)$. Therefore $\{\mathcal{L}e_k^1\}_{k=0}^{r_1}$ is an ϵ-chainable path from $\mathcal{L}e_o^1 = \mathcal{L}x_o = x_1$ to $\mathcal{L}e_{r_1}^1 = \mathcal{L}^2 x_o = x_2$ of length r_1. Continuing this process, for all $n \in N$, we obtain $\{\mathcal{L}^n e_{k-1}^1\}_{k=0}^{l_1}$ a path from $\mathcal{L}^n e_o^1 = \mathcal{L}^n x_o = x_n$ to $\mathcal{L}^n e_{r_1}^1 = \mathcal{L}^n \mathcal{L}x_o = x_{n+1}$ of length r_1. Thus $\{x_n\}$ is $\mathcal{G}$TWC sequence.

Since $\Xi(\mathcal{L}^n e_k, \mathcal{L}^n e_k) \in \Xi_\epsilon(\mathcal{G})$ for all $n \in N$ and for $k = 1, 2, ..., r_p$, and $p = 1, 2, ..., v$, we have

$$\wp(\mathcal{L}^n e_k^p, \mathcal{L}^n e_{k-1}^p) < \Lambda\wp(\mathcal{L}^{n-1} e_k^p, \mathcal{L}^{n-1} e_{k-1}^p) < ....$$
$$< \Lambda^n \wp(e_k^p, e_{k-1}^p) < \Lambda^n \epsilon. \tag{4}$$

By condition (iii) of Definition of $b_v(s)$ metric space and for $p = 1, 2, ...v$, we have

$$\begin{aligned}
\wp(x_0, x_p) &= \wp(e_0^p, e_{r_p}^p) \\
&\le s[\wp(e_0^p, e_1^p) + \wp(e_1^p, e_2^p) + ... + \wp(e_{v-1}^p, e_v^p)] \\
&+ s^2[\wp(e_v^p, e_{v+1}^p) + \wp(e_{v+1}^p, e_{v+2}^p) + ... + \wp(e_{2v-1}^p, e_{2v}^p)] \\
&+ ... + s^{m_p}[\wp(e_{(m_p-1)v}^p, e_{(m_p-1)v+1}^p) + ... + \wp(e_{r_p-1}^p, e_{rp}^p)] \\
&< s[\epsilon + \epsilon + ... + \epsilon] + s^2[\epsilon + \epsilon + ... + \epsilon] + ... \\
&+ s^{m_p}[\epsilon + \epsilon + ... + \epsilon] \\
&= sv + s^2 v + ... + s^{m_p}(v + 1)]\epsilon = M r_p.
\end{aligned} \tag{5}$$

On using (4) and (5), we have

$$\begin{aligned}
\wp(x_n, x_{n+P}) &= \wp(\mathcal{L}^n e_0^p, \mathcal{L}^n e_{r_p}^p) \\
&\le s[\wp(\mathcal{L}^n e_0^p, \mathcal{L}^n e_1^p) + \wp(\mathcal{L}^n e_1^p, \mathcal{L}^n e_2^p) + ... \\
&+ \wp(\mathcal{L}^n e_{v-1}^p, \mathcal{L}^n e_v^p)] + s^2[\wp(\mathcal{L}^n e_v^p, \mathcal{L}^n e_{v+1}^p) \\
&+ \wp(\mathcal{L}^n e_{v+1}^p, \mathcal{L}^n e_{v+2}^p) + ... \\
&+ \wp(\mathcal{L}^n e_{2v-1}^p, \mathcal{L}^n e_{2v}^p)] + ... \\
&+ s^{m_p}[\wp(\mathcal{L}^n e_{(m_p-1)v}^p, \mathcal{L}^n e_{(m_p-1)v+1}^p) + ... \\
&+ \wp(\mathcal{L}^n e_{m_p-1}^p, \mathcal{L}^n e_{m_p}^p)] \\
&< s[\Lambda^n \epsilon + \Lambda^n \epsilon + ... + \Lambda^n \epsilon] \\
&+ s^2[\Lambda^n \epsilon + \Lambda^n \epsilon + ... + \Lambda^n \epsilon] + ... \\
&+ s^{m_p}[\Lambda^n \epsilon + \Lambda^n \epsilon + ... + \Lambda^n \epsilon] \\
&= \Lambda^n[sv + s^2 v + ... + s^{m_p}(v + 1)]\epsilon \\
&= \Lambda^n M r_p \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Hence, by $\mathcal{G}$-completeness of $\mathcal{J}$ implies $x_n \to \rho^*$, for some $\rho^* \in N$. By condition(iii) of our assumption, there exists $k \in N$ such that $(x_n, \rho^*) \in \Xi(\mathcal{G})$ or $(\rho^*, x_n) \in \Xi(\mathcal{G}), \forall n > k$.

Assume that for all $n > k, (x_n, \rho^*) \in \Xi(\mathcal{G})$. Then from (4), we have

$\wp(\mathcal{L}x_n, \mathcal{L}\rho^*) < \Lambda\wp(x_n, \rho^*)$.

Taking limits as $n \to \infty$, we have $\wp(\mathcal{L}x_n, \mathcal{L}\rho^*) \to 0$

Therefore $x_n = \mathcal{L}\rho^*$. Hence $\mathcal{L}\rho^*$ is also limit of the sequence $\{x_n\}$.

Similarly, if $(\rho^*, x_n) \in \Xi(\mathcal{G})$ implies $\wp(\mathcal{L}x_n, \mathcal{L}\rho^*) \to 0$.

Therefore $x_n = \mathcal{L}\rho^*$. Hence $\mathcal{L}\rho^*$ is also limit of the sequence $\{x_n\}$.

Theorem IV.3: Let the conditions of Theorem IV.2 holds. Suppose that $(\mathcal{J}, \wp, \mathcal{G}, \mathcal{L})$ has property $\mathcal{R}^*$, then $\mathcal{L}$ has a fixed point.

Proof: From the proof of Theorem IV.2 and property($\mathcal{R}^*$), we have $\mathcal{L}\rho^* = \rho^*$.

Theorem IV.4: Let the conditions of Theorem IV.3 holds and suppose that for all $\rho^*, \eta^* \in fix(\mathcal{L})$, there exists an ϵ-chainable path $(\rho^* P \eta^*)_{\mathcal{G}}$ between ρ^* and η^* of length $p > 1$, where $p = mv + 1$ for $m \in N \cup \{0\}$. Then $\mathcal{L}$ has a unique fixed point.

Proof: In lieu of Theorem IV.3, $\mathcal{L}$ has a fixed point. Suppose $\rho^*, \eta^* \in$fix$(\mathcal{L})$. If $p = mv + 1, m \in N \cup \{0\}$ and let $\{e_i\}_{i=0}^p$ be the path from ρ^* to η^*, so that $e_0 = \rho^*$ and $e_k = \eta^*$. Then

$$\wp(\rho^*, \eta^*) = \wp(\mathcal{L}^n \rho^*, \mathcal{L}^n \eta^*)$$

$$\leq s[\wp(\mathcal{L}^n e_0, \mathcal{L}^n e_1) + \wp(\mathcal{L}^n e_1, \mathcal{L}^n e_2) + ...$$
$$+\wp(\mathcal{L}^n e_{v-1}, \mathcal{L}^n e_v)] + s^2[\wp(\mathcal{L}^n e_v, \mathcal{L}^n e_{v+1})$$
$$+\wp(\mathcal{L}^n e_{v+1}, \mathcal{L}^n e_{v+2}) + ... + \wp(\mathcal{L}^n e_{2v-1}, \mathcal{L}^n e_{2v})]$$
$$+... + s^m[\wp(\mathcal{L}^n e_{(m-1)v}, \mathcal{L}^n e_{(m-1)v+1}) + ...$$
$$+\wp(\mathcal{L}^n e_{l-1}, \mathcal{L}^n e_l)] \to 0 \text{ as } n \to \infty.$$

Therefore $\rho^* = \eta^*$.

V. Application to Cantilever beam problem

Consider the following fourth order two point boundary value problem which is an example of beam problem when uniform load is distributed, called Cantilever Beam problem

$$\frac{d^4 g}{dt^4} = \mathcal{K}(t, g(t)),\ 0 < t < 1 \tag{6}$$

$g(0) = g^{'}(0) = g^{''}(1) = g^{'''}(1) = 0$
with $I = [0,1]$ and $\mathcal{K} \in \mathcal{C}([0,1] \times R, R^+)$. This problems is equivalent to the integral equation

$$g(t) = \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r, g(r))dr \tag{7}$$

for $t \in I$, where $\mathcal{F} : I \times I \to [0, \infty)$ is the Green's function given

$$\mathcal{F}(t,r) = \begin{cases} \frac{r^2(3t-r)}{6} & if\ 0 \leq r \leq t \leq 1 \\ \frac{t^2(3t-r)}{6} & if\ 0 \leq t \leq r \leq 1. \end{cases}$$

Consider the set $\mathcal{J} = \mathcal{C}([0,1], R)$ be the set of all real-valued continuous functions defined on $[0,1]$. Let us define

$$\mathcal{U} = \{\dagger \in \mathcal{J} : inf_{t\in[0,1]}\dagger(t) > 0\ and\ \dagger(t) \leq 1, t \in [0,1]\}.$$

Now, to define graph structure $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \Xi(\mathcal{G}))$ on $\mathcal{J}$,

let $\mathcal{V}(\mathcal{G}) = \mathcal{J}$ and

$\Xi(\mathcal{G}) = \Delta \cup \{(p,q) \in u \times u : p(t) \leq q(t)\ \forall\ t \in [0,1]\}$

$= \{(p,p) : p \in \mathcal{J}\} \cup \{(p,q) \in \mathcal{U} \times \mathcal{U} : p(t) \leq q(t)\ \forall\ t \in [0,1]\}$.

Define a mapping $\wp : \mathcal{J} \times \mathcal{J} \to R$ as

$\wp(p,q) = sup_{0\leq t\leq 1}|p(t) - q(t)|^3$ for $p, q \in \mathcal{J}$. Then $(\mathcal{J}, \wp)$ is a complete $b_3(1)$ metric space.

Theorem V.1: Consider an injective function $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ defined

$$\mathcal{L}g(t) = \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r, g(r))dr \tag{8}$$

Suppose the following assumptions holds.

(i) The function $\mathcal{K} : I \times R \to R^+$ is increasing on $[0,1]$. In addition $\mathcal{K}(r,1) = r$ and $inf_{t\in[0,1]}\mathcal{F}(t,r) \geq 0$.

(ii) For every $t \in [0,1]$,

$$|\mathcal{K}(r, g(r)) - \mathcal{K}(r, v(r))| \leq |g(r) - v(r)|^3.$$

(iii) $\varphi \in \mathcal{C}([0,1], R)$ is the lower solution of equation (7) *i.e.*,

$$\varphi(s) \leq \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r, \varphi(r))dr.$$

Then the existence solution for equation (8) provides the solution for (6).

Proof: Clearly, $\mathcal{L} : \mathcal{J} \to \mathcal{J}$ is well defined. Let $(u,v) \in \Xi(\mathcal{G})$ i.e., $u, v \in \mathcal{U}$ and $g(t) \leq v(t)$, for all $t \in [0,1]$.

Now

$\mathcal{L}g(t) = \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r, g(r))dr$

$\leq \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r,1)dr = \int_0^1 \mathcal{F}(t,r)rdr \leq \frac{79}{360} \leq 1.$

Hence, from condition (i), $inf_{t\in[0,1]}\mathcal{L}(g(t)) > 0$ which implies $\mathcal{L}g(t) \in \mathcal{U}$. Similarly, $\mathcal{L}v(t) \in \mathcal{U}$. Since $\mathcal{K} : I \times R \to R^+$ is increasing on $[0,1]$, we have

$\mathcal{L}g(t) = \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r, g(r))dr$

$\leq \int_0^1 \mathcal{F}(t,r)\mathcal{K}(r, v(r))dr$

$= \mathcal{L}v(t),$
which implies $(\mathcal{L}g(t), \mathcal{L}v(t)) \in \Xi(\mathcal{G})$.

Now for $t \in [0,1]$, we have

$|\mathcal{L}g(t) - \mathcal{L}v(t)|$

$= |\int_0^1[\mathcal{F}(t,r)\mathcal{K}(s, g(s)) - \mathcal{F}(t,r)\mathcal{K}(s, v(s))]||s$

$\leq \int_0^1 \mathcal{F}(t,r)|\mathcal{K}(s, g(s)) - \mathcal{K}(s, v(s))|ds$

$\leq |g(s) - v(s)|^3 \int_0^1 \mathcal{F}(t,r)ds$

$= |g(s) - v(s)|^3 \int_0^1 \frac{r^2(3t-r)}{6}dr + \int_0^t \frac{t^2(3t-r)}{6}$

$\leq |g(s) - v(s)|^3[\frac{t^4}{8} + \frac{1}{12}(3t^2 - 2t^3 - t^4)]$

$\leq |g(s) - v(s)|^3[\frac{1}{8} + \frac{6}{12}]$

$= |g(s) - v(s)|^3[\frac{1}{8} + \frac{6}{12}]$

$= \frac{5}{8}|g(s) - v(s)|^3$

$\leq \frac{5}{8}sup_{t\in[0,1]}|g(s) - v(s)|^3$

$= \frac{5}{8}\wp(u,v)$

This implies

$$\wp(\mathcal{L}g(t), \mathcal{L}v(t)) \leq c\wp(u,v) + a\wp(p, \mathcal{L}p) + b\wp(q, \mathcal{L}q),$$

where $c = \frac{5}{8}$, for any a, b with $a + b + \frac{5}{8} < 1$. Thus $\mathcal{L}$ is $\mathcal{G}RC$ on $\mathcal{J}$. From condition (iii), there exist a solution $\varphi(s) \in \mathcal{J}$ such that $\mathcal{L}^p[\varphi(s)]^l_{\mathcal{G}}$ for each $p = 1, 2, 3, ...v$, so that the condition (i) of Theorem III.1 is satisfied. Also, it is easy to see that property $(\mathcal{R}^*)$ is satisfied. Hence, Theorem III.2 assures that $\mathcal{L}$ has a unique fixed point and hence the integral equation (8) has solution in $\mathcal{J}$ which confirms that the existence of the solution of Cantilever beam problem.

VI. Conclusions

We established the existence of fixed points for both Reich contractions [8] and Edelstein contractions [5] within the context of graphical $b_v(s)$ metric spaces, focusing on an injective mapping. The graphical $b_v(s)$ metric space is particularly noteworthy and compact because the triangle inequality is only enforced for the connected elements within the graph structure, rather than across the entire space. Our theorems not only extend but also generalize the well-known results from [3]. Additionally, our findings are supported by examples and have practical applications in solving fourth-order differential equations. In particular, we applied our results to solve the Cantilever beam problem under a uniformly distributed load.

References

[1] P.Baradol, D. Gopal, and S. Radenović, "Computational fixed points in graphical rectangular metric spaces with application", J. Comput. Appl. Math, vol. 375, 2020, Article ID 112805, pp. 12. pages.

[2] P.Baradol, J. Vujaković, D. Gopal and S. Radenović, " On some new results in graphical rectangular b-metric spaces", Mathematics, vol.8 , 2020, Article ID 488, pp. 17 .

[3] P. Baradol, D. Gopal and N. Damljanovic " A new fixed point results in graphical $b_v(s)$ metric space space with applications to differential equations", Kragujevac Journal of Mathematics 48(2024), pp. 441–451.

[4] N. Chuensupantharat, P. Kumam, V.Chauhan, D. Singh, and R. Menon, "Graphic contraction mappings via graphical b-metric spaces with applications", Bull. Malays. Math. Sci. Soc, vol. 42, 2019, pp. 3149–3165.

[5] M.Edelstein,, "An extension of Banach's contraction principle", Proc. of the Amer. Math. Soc, vol. 12, 1961,pp. 7-10.

[6] J. Jachymski, " The contraction principle for mappings on a metric space with a graph", Proc Amer Math Soc, vol. 136 2008, pp.1359-1373.

[7] Z. Mitrović and S. Radenović, "The Banach and Reich contractions in $b_v(s)$-metric spaces", J. Fixed Point Theory Appl, vol. 19, 2017, pp. 3087–3095.

[8] S. Reich, " Some remarks concerning contraction mappings", Canadian Mathematical Bulletin, vol. 14, 1971, pp. 121-124.

[9] S. Shukla, S. Radenović, and C. Vetro " Graphical metric space: a generalized setting in fixed point theory", Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat, vol. 111, 2017, pp. 641–655.

[10] S.Shukla,N. Mlaiki and H. Aydi, "On (G, G0)-Prešić–Ćirić operators in graphical metric spaces", Mathematics, vol. 7, 2019, Article ID 445, pp. 12.

[11] M. Younis, D. Singh and A. Goyal, "A novel approach of graphical rectangular b-metric spaces with an application to the vibrations of a vertical heavy hanging cable", J. Fixed Point Theory Appl, vol. 21, 2019, Article ID 33, pp.17 .

[12] M.Younis, D. Singh and A. Afrah, N. Abdou, "A fixed point approach for tuning circuit problem in dislocated b-metric spaces", Math. Meth. Appl. Sci, vol. 45, 2022, pp. 2234-2253.

[13] M. Younis, H. Ahmad, L. Ch. Maoan Han, " Computation and convergence of fixed points in graphical spaces with an application to elastic beam deformations", Journal of Geometry and Physics, vol. 192, 2023, 104955, pp. 13¿

[14] M. Younis, D. Singh, I. Altun and V. Chauhan, " Graphical structure of extended b-metric spaces: an application to the transverse oscillations of a homogeneous bar", Int. J. Non. Sci Numer. Simul, vol. 23, 2022, pp. 1239-1252.

[15] M. Younis and D. Bahuguna, " A unique approach to graph-based metric spaces with an application to rocket ascension", Compu. and Appl. Math, vol. 42,2023, pp.19 .

[16] M. Younis and A.Petruşel, Applications of graph Kannan mappings to the damped spring-mass system and deformation of an elastic beam, Discrete Dyn. Nat. Soc, vol. 2019, 2019, Article 1315387, pp. 9.

A Bharathi was born in Andhra Pradesh, India. She graduated in Master of Science (2004) from Govt. Degree College for Women at Visakhapatnam, under Andhra University in Andhra Pradesh. She is a Research scholar in the Department of Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, India. She is currently working as an Associate Professor in the Department of Mathematics, Raghu Engineering college, Visakhapatnam, Andhra Pradesh, India. Her research interests lie in fixed-point theory.

M V R Kameswari was born in Andhra Pradesh ,India. She is an Associate Professor in the Department of Mathematics, School of Science, GITAM Deemed to be University, Rushikonda, Visakhapatnam, Andhra Pradesh, India. She received Ph.D. Degree at Andhra University, Visakhapatnam, India. The results of her research have been published in several National Journals, International Proceedings, and International Journals Indexed by Scopus. Her research interest inculdes fixed points and apporximation of fixed points.

M Madhuri was born in Andhra Pradesh, India. She graduated in Master of Science (2011) from M R P G college at Vizianagaram, under Andhra University in Andhra Pradesh. She is a Research scholar in the Department of Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, India. She is currently working as an Assistant Professor in the Department of Mathematics, Lendi Institute of Engineering and Technology, Vizianagaram, Andhra Pradesh, India. Her research interests lie in fixed-point theory.

B Padmavathi was born in Andhra Pradesh, India. She graduated in Master of Science (2001) from Dr Lankapalli Bullayyya college, at Visakhapatnam, under Andhra University in Andhra Pradesh. She is a Research scholar in the Department of Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, India. She is currently working as Lecturer in the Department of Mathematics, Dr Lankapalli Bullayyya college, Visakhapatnam, Andhra Pradesh, India. Her research interests lie in fixed-point theory.