On Farey Edge Graceful Labeling

Ajay Kumar, Neeraj Gupta, Ajendra Kumar, Suraj Tyagi, Vipin Kumar *

Abstract— **In this paper, we introduce a new concept of Farey edge graceful labeling and Farey edge strength of graphs. We prove that the symmetric hairy cycle, helm, flower, wheel, T_k-graph and subdivision of star graphs are Farey edge graceful and we also determine Farey edge strength of these graphs.**

Index Terms: Farey sequence, Farey edge graceful labeling, Farey edge strength, symmetric hairy cycle, wheel graph, helm graph, flower graph, T_k-graph

I. INTRODUCTION

The idea of Farey edge graceful labeling has come from the observation of irregularity strength of graphs, which was introduced by Chartrand et. al. [2] and then proved that the irregularity strength of complete graph K_n is 3, path graph P_n is $\frac{n}{2}$ if $n \equiv 0$ $(mod\ 4)$, $\frac{n+1}{2}$ if $n \equiv 1, 3$ $(mod\ 4)$, $\frac{n+2}{2}$ if $n \equiv 2$ $(mod\ 4)$. Consider a graph $G(V, E)$ that lacks a K_2 component and with at most one isolated vertex. A mapping $f : E(G) \longrightarrow \mathbf{Z}^+$, such that each vertex is uniquely labeled by the sum of all incident edge weights, referred as irregular labeling [4]. The maximum edge weight in this context is termed as the strength of the graph G and denoted by $s(f)$. Kumar [4] determined the strengths of the helm graph, web graph $W(n, 4)$, flower graph, triangular snake graph, bi-wheel graph, double triangular snake graph. Packiam et. al. [5] established that $s(C_n \odot mK_1) = mn$, $s(C_n \odot K_2) = n+1$, $s(C_n \odot K_3) = n+1$, $s(P_n \odot K_2) = n+1$, $s(P_n \odot K_3) = n+1$.

Before going to the next section, we wish to emphasize on the following definitions.

*Manuscript received April 27, 2024; revised October 4, 2024.

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Definition: If l copies of complete graph K_1 are connected by an edge to each vertex of C_m then resulting graph is called symmetric hairy cycle and is denoted as $C_m \odot lK_1$.

Definition: The r-subdivision of star graph is obtained by replacing each edge of the star graph with a path of length r.

Definition: A wheel graph is obtained by connecting each vertex of a cycle to a vertex $w \notin V(C_n)$ by an edge.

Definition: The helm graph is obtained from wheel graph by connecting a pendant vertex to each vertex of a cycle by an edge.

Definition: If each pendant vertices of helm graph is connected to the central vertex by an edge then we get flower graph.

Definition: If we place one more vertex between each edge of cycle of helm graph H_k then we get T_k-graph.

Definition: A Farey sequence [1], [6] of order n, denoted as F_n, is a set of ascending rational numbers in the interval $[0, 1]$. It is defined by fractions of the form $\frac{n_1}{n_2}$ where $0 \leq n_1 \leq n_2 \leq n$ and $gcd(n_1, n_2) = 1$.

Kumar et al. [3] laid down the foundations of Farey graceful labeling and illustrated Farey gracefulness of caterpillars, hairy cycles, cycles, and paths. With the help of Farey edge graceful labeling, we can injectively label the edges and vertices of such a graph whose degree is very high. By Farey edge graceful labeling, we can label both the vertices and edges of the graph injectively. Whereas in irregular labeling, we can only label the vertex injectively.

In the next section, we define Farey edge graceful labeling and its strengths. We will also prove some theorems based on this labeling.

II. RESULTS

Definition 0.1. *In a simple, connected and undirected graph $G(V, E)$, if each edge is uniquely labeled by F_n such that $f(e_i) = \frac{a_i}{b_i} \in F_n$ $\forall$ $e_i \in E(G)$ and for any vertex $v_i \in V(G)$, the vertex weight is given as:*

$$w_f(v_i) = \sum_{e_i^j = v_i y_j \in E(G)} (b_i - a_i)$$

and the resulting weights are distinct. Then f is called Farey edge graceful labeling and G is called Farey edge graceful graph.

Definition 0.2. *Let f be a Farey edge graceful labeling on $G(V, E)$. Then Farey edge strength $s(f)$ is given as:*

$$s(f) = \max_{\forall e_i \in E(G)} (b_i - a_i)$$

where $f(e_i) = \frac{a_i}{b_i} \in F_n$.

Theorem 0.3. *All symmetric hairy cycles* $C_m \odot lK_1$ *are Farey edge graceful* $\forall\ m, l \in \mathbf{N}$ *and its Farey edge strength of* $C_m \odot lK_1$ *is* ml.

Proof. Let $x_1,\ x_2,\ ...,\ x_m$ are vertices of cycles and x_i^j represent the pendant vertices adjacent to x_i for $i = 1, 2, ..., m,\ j = 1, 2, ..., l$ in $C_m \odot lK_1$. Define a mapping $f : E(C_m \odot lK_1) \longrightarrow F_{m^2l+ml+1}$ as:

$$f(x_i x_{i+1}) = \frac{mli+1}{ml(i+1)+1} \quad , \quad 1 \leq i \leq (m-1)$$

$$f(x_m x_1) = \frac{m^2l+1}{m^2l+ml+1}$$

$$f(x_i x_i^j) = \frac{1}{m(j-1)+i+1} \quad , \quad 1 \leq i \leq m,\ 1 \leq j \leq l$$

By using the Definition 0.1, we can observe that vertex weights are:

$$w_f(x_i) = \frac{ml(l+3)}{2} + il \quad , \quad 1 \leq i \leq m$$

$$w_f(x_i^j) = m(j-1) + i \quad , \quad 1 \leq i \leq m,\ 1 \leq j \leq l$$

All of the vertex weights mentioned above are distinct. Hence, $C_m \odot lK_1$ is Farey edge graceful graph.

To determine Farey edge strength of the graph, we subtract the numerator from the denominator of each edge label. Consequently, we get the label of each edge from the set $\{1, 2, ..., ml\}$. Hence, Farey edge strength of $C_m \odot lK_1$ is ml. (see Fig. 1) □

Theorem 0.4. *The wheel graph* W_{m+1} *is Farey edge graceful and its Farey edge strength is* m.

Proof. If each vertex of C_m is connected to vertex $v \notin V(C_m)$ by an edge then we get the wheel graph W_{m+1}. Define a mapping $f : E(W_{m+1}) \longrightarrow F_{m+1}$ as:

$$f(x_i x_{i+1}) = \begin{cases} \frac{0}{1} & , \quad i = 1 \\ \frac{i}{i+1} & , \quad 2 \leq i \leq m-1 \end{cases}$$

$$f(x_m x_1) = \frac{m}{m+1}$$

$$f(v x_i) = \frac{1}{i+1} \quad , \quad 1 \leq i \leq m$$

By using the Definition 0.1, we can observe that the vertex weights are:

$$w_f(x_i) = (i+2) \quad , \quad 1 \leq i \leq m$$

$$w_f(v) = \frac{m(m+1)}{2}$$

All the vertex weights mentioned above are distinct. Hence, W_{m+1} is Farey edge graceful.

To determine Farey edge strength of the graph, we subtract the numerator from the denominator of each edge label. Consequently, we get the label of each edge from the set $\{1, 2, ..., m\}$. Hence, Farey edge strength of W_{m+1} is m. (see Fig. 2) □

Theorem 0.5. *The helm graph* H_k *is Farey edge graceful graph and its Farey edge strength is* k.

Proof. Let $x_1,\ x_2,\ ...,\ x_k$ are the vertices of cycle and $z_1,\ z_2,\ ...,\ z_k$ are pendant vertices in which each x_r is adjacent to z_r for $1 \leq r \leq k$ and v is the central vertex of H_k. We define a labeling $f : E(H_k) \longrightarrow F_{k^2+k+1}$ such as:

$$f(z_r x_r) = \frac{1}{r+1} \quad , \quad 1 \leq r \leq k$$

$$f(x_1 x_2) = \frac{0}{1}$$

$$f(x_k x_1) = \frac{k}{k+1}$$

$$f(x_r x_{r+1}) = \frac{r}{r+1} \quad , \quad 2 \leq r \leq k-1$$

$$f(v x_r) = \frac{rk+1}{rk+k+1} \quad , \quad 1 \leq r \leq k$$

By using the Definition 0.1, we can observe that the vertex weights are:

$$\begin{aligned} w_f(z_r) &= r & , \quad 1 \leq r \leq k \\ w_f(x_r) &= k + r + 2 & , \quad 1 \leq r \leq k \\ w_f(v) &= k^2 \end{aligned}$$

All these vertex weights mentioned above are distinct. Hence, H_k is a Farey edge graceful.

Each edge is labeled by Farey fraction and to determine Farey edge strength of the graph, we subtract the numerator from the denominator. Consequently, we get the label of each edge from the set $\{1, 2, ..., k\}$. Hence, Farey edge strength of H_k is k. (see Fig. 3) □

Theorem 0.6. *The flower graph* F_k *is Farey edge graceful and its Farey edge strength is* $\lceil \frac{k+1}{2} \rceil$.

Proof. Let H_k be the helm graph defined in Theorem 0.5. If we joined each vertex $z_1,\ z_2,\ ...,\ z_k$ to the central

we denote the ceiling function as $\lceil x \rceil$.

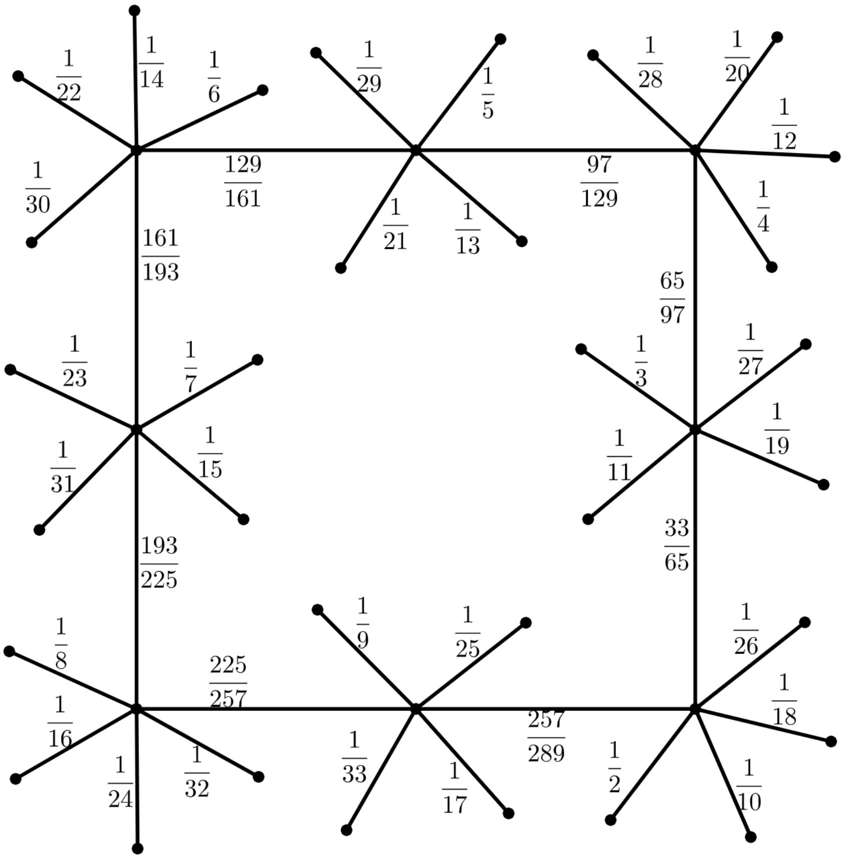

Fig. 1: Farey edge graceful labeling of $C_8 \odot 4K_1$

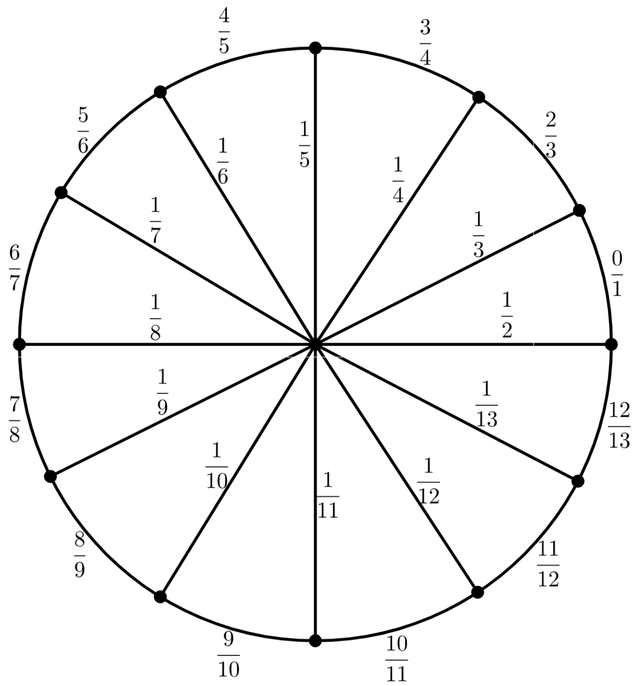

Fig. 2: Farey edge graceful labeling of W_{13}

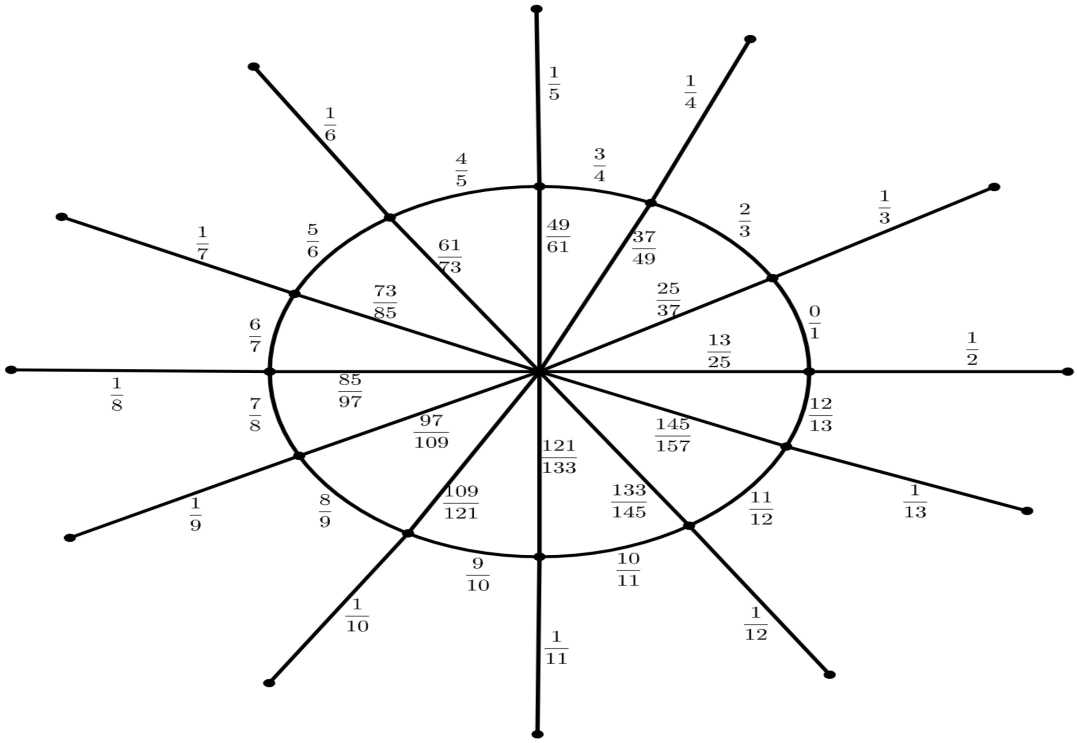

Fig. 3: Farey edge graceful labeling of H_{12}

vertex v by an edge then we get the graph $\digamma_k$. We define a labeling $f : E(\digamma_k) \longrightarrow F_{\lceil \frac{k+1}{2} \rceil (6+k)+1}$ as:

$$f(z_r x_r) = \begin{cases} \frac{1}{\frac{(r+1)}{2}+1} & , \quad 1 \le r \le k \text{ and } r \text{ is odd} \\ \frac{\frac{r}{2}+1}{r+1} & , \quad 1 \le r \le k \text{ and } r \text{ is even} \end{cases}$$

$$f(vz_r) = \begin{cases} \frac{r+2}{\frac{3(r+1)}{2}+1} & , \quad 1 \le r \le k \text{ and } r \text{ is odd} \\ \frac{7}{9} & , \quad r = 2 \\ \frac{\frac{3r}{2}+4}{(2r+5)} & , \quad 4 \le r \le k \text{ and } r \text{ is even} \end{cases}$$

$$f(vx_r) = \begin{cases} \frac{4}{5} & , \quad r = 1 \\ \frac{2(r-1)+5}{\frac{5(r-1)}{2}+6} & , \quad r \text{ is odd and } r \ge 3 \\ \frac{\frac{5r}{2}+6}{(3r+7)} & , \quad r \text{ is even and } r \ge 2 \end{cases}$$

$$f(x_r x_{r+1}) = \frac{\lceil \frac{k+1}{2} \rceil (6+r) - \lceil \frac{k+1}{2} \rceil + 1}{\lceil \frac{k+1}{2} \rceil (6+r)+1} \quad , \quad 1 \le r \le (k-1)$$

$$f(x_k x_1) = \frac{\lceil \frac{k+1}{2} \rceil (6+k) - \lceil \frac{k+1}{2} \rceil + 1}{\lceil \frac{k+1}{2} \rceil (6+k)+1}$$

By using the Definition 0.1, we can observe that the vertex weights are:

$$w_f(u_r) = (r+1) \quad , \quad 1 \le r \le k$$

$$w_f(x_r) = 2\lceil \tfrac{k+1}{2} \rceil + r + 1 \quad , \quad 1 \le r \le k$$

$$w_f(v) = \begin{cases} \frac{k^2+4k-1}{4} & , \quad \text{if } k \text{ is odd} \\ \frac{k^2+4k}{2} & , \quad \text{if } k \text{ is even} \end{cases}$$

All of the vertex weights mentioned above are distinct. Hence, $\digamma_k$ is Farey edge graceful.

To determine Farey edge strength of the graph, we subtract the numerator from the denominator. Consequently, we get the label of each edge by the set $\{1, 2, ..., \lceil \frac{k+1}{2} \rceil\}$. Hence, Farey edge strength of $\digamma_k$ is $\lceil \frac{k+1}{2} \rceil$. (see Fig. 4) □

Theorem 0.7. *The graph T_k is Farey edge graceful for $k \ge 4$ and its Farey edge strength is k.*

Proof. Let H_k be a helm graph defined in Theorem 0.5. We introduce one vertex on each edge (say w_1, w_2, ...,

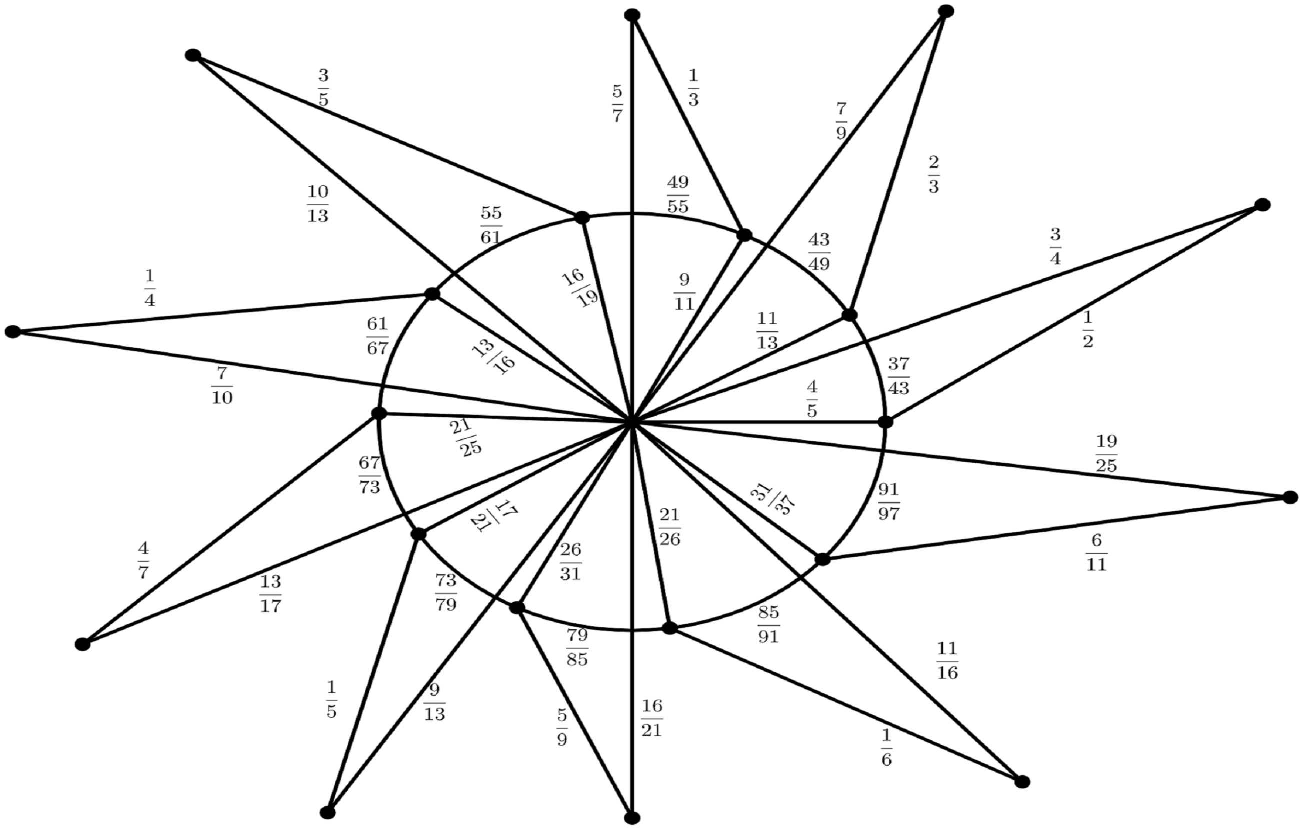

Fig. 4: Farey edge graceful labeling of $\mathcal{F}_{10}$

w_k) of the cycle and we will get the resulting graph T_k. We define a labeling $f : E(T_k) \longrightarrow F_{k^2+4k+1}$ as:

$$f(x_r z_r) = \frac{1}{r+1} \quad , \quad 1 \le r \le k$$

$$f(x_r w_r) = \begin{cases} \frac{\lceil \frac{k+r-1}{2} \rceil + 1}{2\lceil \frac{k+r-1}{2} \rceil + 1} & , \quad 1 \le r \le k \text{ and } r \text{ is odd} \\ \\ \frac{2\lceil \frac{k+r-1}{2} \rceil + 1}{3\lceil \frac{k+r-1}{2} \rceil + 1} & , \quad 1 \le r \le k \text{ and } r \text{ is even} \end{cases}$$

$$f(w_r x_{r+1}) = \begin{cases} \frac{4\lceil \frac{k+r}{2} \rceil + 1}{5\lceil \frac{k+r}{2} \rceil + 1} & , \quad 1 \le r \le k \text{ and } r \text{ is odd} \\ \\ \frac{3\lceil \frac{k+r}{2} \rceil + 1}{4\lceil \frac{k+r}{2} \rceil + 1} & , \quad 1 \le r \le k \text{ and } r \text{ is even} \end{cases}$$

$$f(x_1 w_k) = \begin{cases} \frac{4k+1}{5k+1} & , \quad \text{when } k \text{ is odd} \\ \\ \frac{3k+1}{4k+1} & , \quad \text{when } k \text{ is even} \end{cases}$$

$$f(x_1 v) = \frac{5\lceil \frac{k}{2} \rceil + 1}{6\lceil \frac{k}{2} \rceil + 1}$$

$$f(x_r v) = \frac{(r+3)m+1}{(r+4)m+1} \quad , \quad 2 \le r \le k$$

By using the Definition 0.1, we can observe that the weights assigned to the vertices are:

$$w_f(z_r) = r \quad , \quad 1 \le r \le k$$

$$w_f(x_1) = \begin{cases} 2k+1 & , \quad k \text{ is even} \\ \\ 2k+2 & , \quad k \text{ is odd} \end{cases}$$

$$w_f(x_r) = r + 2\lceil \tfrac{k-1+r}{2} \rceil + k \quad , \quad 2 \le r \le k$$

$$w_f(w_r) = k + r \quad , \quad 1 \le r \le k$$

$$w_f(v) = k(k-1) + \lceil \tfrac{k}{2} \rceil$$

All of the vertex weights mentioned above are distinct. Hence, T_k is a Farey edge graceful graph.

To determine Farey edge strength of the graph, we subtract the numerator from the denominator. Consequently, we get the label of each edge from the set $\{1, 2, ..., k\}$. Hence, Farey strength of T_k is k. (see Fig. 5) □

Theorem 0.8. *The r-subdivision of star graph (S_m^r) is Farey edge graceful for $m \ge 3$ and its Farey edge strength*

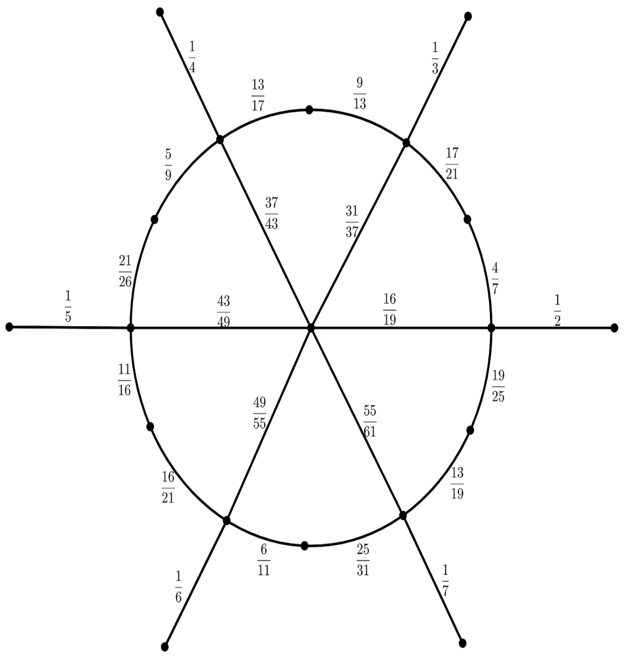

Fig. 5: Farey edge graceful labeling of T_6

is

$$s(f) = \begin{cases} \frac{mr}{2} & , \quad r \text{ is even} \\ \\ \frac{m(r+1)}{2} & , \quad r \text{ is odd} \end{cases}$$

Proof. Let $\{v_i^j\}$ be the set of vertices of the path P_m^r where $i = 1, 2, ..., m$ and $j = 1, 2, ..., r$. If we connect all the vertices of one end of the P_m^r, namely v_1^r, v_2^r, ..., v_m^r to a central vertex v with an edge, while leaving the other end of the vertices of P_m^r, we obtain the graph S_m^r. Define a mapping $f : E(S_m^r) \longrightarrow F_k$ $\left(\text{where } k = \begin{cases} \frac{(m+1)(r-1)m+2}{2} & , \quad r \text{ is odd} \\ \\ \frac{m(m+1)r+2}{2} & , \quad r \text{ is even} \end{cases}\right)$ as:

$$f(v_i^j v_i^{j+1}) = \begin{cases} \frac{1}{\frac{(j-1)m}{2}+i+1} & , \quad 1 \leq j \leq (r-1),\ 1 \leq i \leq m \text{ and } j \text{ is odd} \\ \\ \frac{1}{\frac{(j-1)m}{2}+i+1} & , \quad 1 \leq j \leq (r-1),\ 1 \leq i \leq m \text{ and } j \text{ is even} \end{cases}$$

$$f(v_i^r v) = \begin{cases} \frac{1}{\frac{(r-1)m}{2}+i+1} & , \quad r \text{ is odd and } 1 \leq i \leq m \\ \\ \frac{\frac{imr}{2}+1}{\frac{(i+1)mr}{2}+1} & , \quad r \text{ is even and } 1 \leq i \leq m \end{cases}$$

By using the Definition 0.1, we can observe that vertex weights are:

$$w_f(v_i^j) = (j-1)m + i \quad , \quad 1 \leq i \leq m,\ 1 \leq j \leq r$$

$$w_f(v) = \begin{cases} \frac{rm^2}{2} & , \quad r \text{ is even} \\ \\ \frac{m(mr+1)}{2} & , \quad r \text{ is odd} \end{cases}$$

All the vertex weights mentioned above are distinct. Hence, S_m^r is Farey edge graceful.

To determine Farey edge strength of the graph, we subtract the numerator from the denominator of each edge label. Consequently, we get the label of each edge from the set $\{1, 2, ..., \frac{mr}{2}\}$ when r is even or $\{1, 2, ..., \frac{m(r+1)}{2}\}$ when r is odd. Hence, Farey edge strength of S_m^r is

$$s(f) = \begin{cases} \frac{mr}{2} & , \quad r \text{ is even} \\ \\ \frac{m(r+1)}{2} & , \quad r \text{ is odd} \end{cases}$$

(see Fig. 6) □

III. APPLICATION

Farey edge graceful labeling technique can be used to reduce interference in communication networks by assigning frequencies (labels) to transmitters (nodes) in such a way that no two transmitters within a certain range share the same frequency. We can also determine the structure of a crystal using graph labeling techniques. To do this, we consider each molecule in the crystal as a vertex and each bond between molecules as an edge. It can also be studied in fuzzy set theory and probability theory as the numbers used in this labeling belong to the range $[0, 1]$.

IV. CONCLUSION

This paper presents an innovative concept of Farey edge graceful labeling and Farey edge strength for graphs. We also proved that all symmetric hairy cycle, helm, flower, wheel, T_k-graph and r-subdivision of star graphs are Farey edge graceful and we have calculated Farey edge strength of these graphs.

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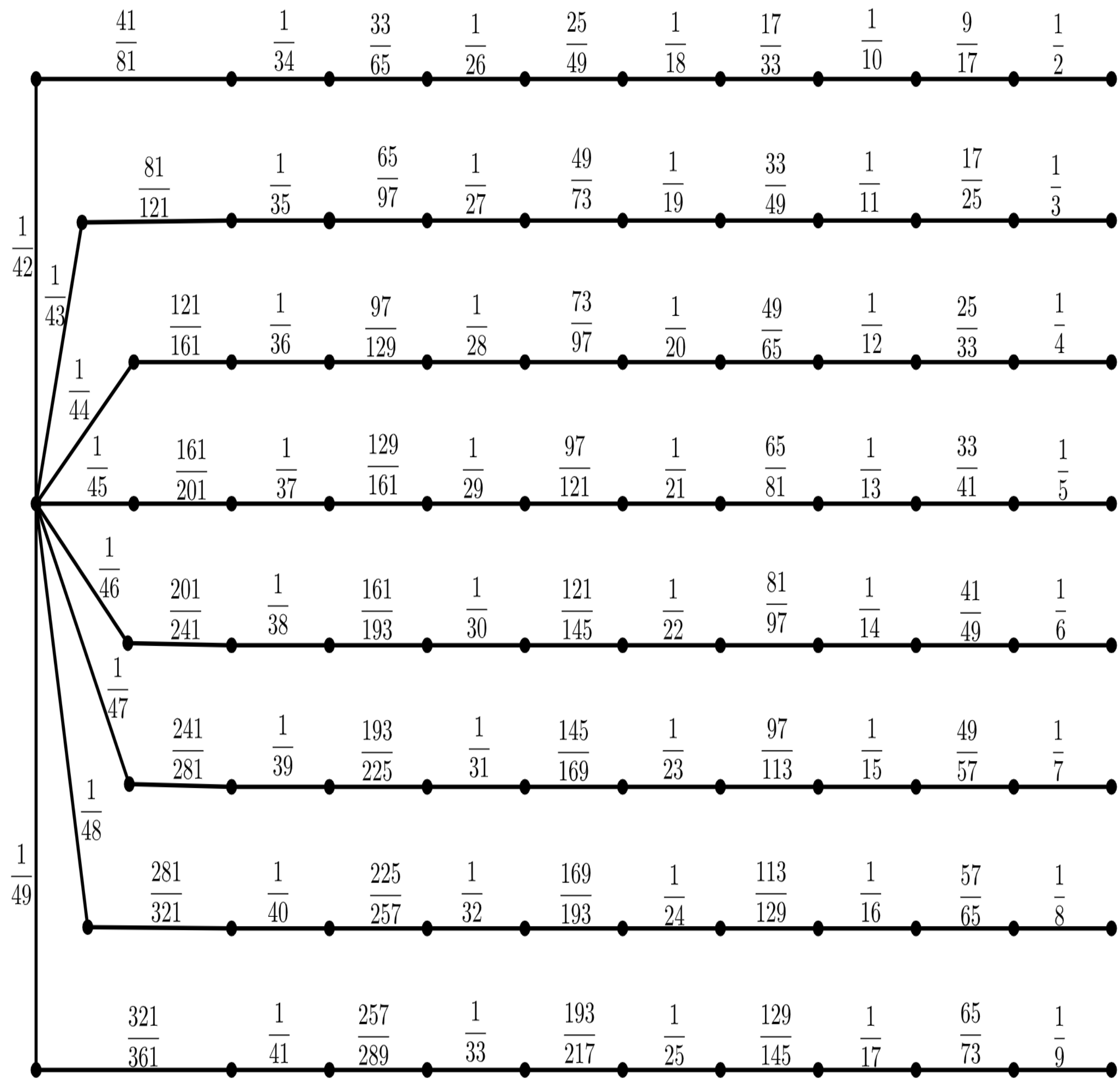

Fig. 6: Farey edge graceful labeling of S_8^{11}

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