Edge-Magic Total Labeling of Some Composite Graphs

Cong Huang, Jingwen Li, Liangjing Sun, Jiang Wang

Abstract - This composition presents a judgment method for Edge-Magic Total Labeling. The algorithm determines provided that the labels of all the edges and points connected to a given network G(p,q) can be related to a set {1,2,3,…,p+q}. Additionally, if the amount of the tags on some side of the graph and associated points remains constant, subsequently the labeling method conforms to the Edge-Magic Total Labeling. The resulting network that adheres to this labeling method is referred to as the edge illusion graph, as shown in Figure G(p,q). By applying the algorithm, we are able to obtain all atlases within 17 points. By using as well as studying the obtained conclusions, the tagging laws of some union network about the Edge-Magic Total Labeling is found, and the theorems are outlined and demonstrated.

***Index Terms*—connected graph, connected graph, edge-magic total labeling, iterative optimization.**

I. Introduction

In 1736, Euler solved the issue of whether the seven bridges in the city of Konigsberg could be crossed at once without repeating them[1].This groundbreaking achievement laid the foundation for the basic concepts of graph theory. Graph theory, an interdisciplinary field in computer science and mathematics, delves into the analysis of graphs. These graphs consist of nodes (otherwise vertices) and edges, which portrays relationships between objects. Currently, the graph theory has found substantial applications in diverse areas such as computer science, biology, sociology, and the network design.

To enhance comprehension and facilitate the depiction of relationships among various elements within a graph's structure, researchers have explored and refined the notion of edge-magic total labeling since its introduction by Kotzig and Rosa in 1970 [2]. Consequently, extensive investigation into graph labeling has yielded significant findings, establishing a strong basis for the ongoing advancement of Edge-Magic Total Labeling.

Enomoto[3] proved that all graphs C_n are super edge-magic when n is queer. Additionally, when m=1 or n=1, graph $K_{m,n}$ is also a super edge-magic network, in 1998. Moreover, provided that n≠3 (mod 4), graph W_n is an edge-magic graph. In 2001, Berkman[4] provided a proof that all C_n is edge-magic when n≥3. In 2002, Lin[5] and others contributed edge-magic labels for certain F_n and friendship graphs. Literature[6] proved that when n and m meet specific situations, the figure $C_m \cup C_n$ is a super edge-magic graph. Gallian[7] summarized and organized all label types and research findings, along with their corresponding proofs. Literature[8-10] demonstrated that all fan graphs are edge-magic when n≤6. Furthermore, literature[11-16] has proven that C_n, P_n+K_1, $K_{m,n}$, fan graphs, as well as binary trees, is also edge-magic graphs.

This paper focuses on utilizing the point generation algorithm[17] to acquire edge-magic and total labels for all non-isomorphic graphs in a finite set of points. Afterwards, it examines the labeling patterns from the outcome set, summarizes various union graph theorems, and presents proofs for them.

II. Preparation

This paper explores special graphs, including road, star, and wheel graphs, among others, and examines the edge-magic total tagging of the associated networks.

Definition 1: Let f be a one-to-one correspondence within the collection $V(G) \cup E(G)$ of a simple undirected network $G(p,q)$ to the set $\{1,2,3,\cdots,p+q\}$ satisfies the condition that pertains to some edge $uv \in E(G)$ with vertices $u,v \in V(G)$, the equation $f(u)+f(v)+f(uv)=k$ holds, where k represents a fixed number known as the edge-magic constant, then f is an edge-magic tagging of the figure G, G is an edge-magic figure. Provided that the figure G does not have a vertex-magic total tagging, then it is an NVMTL figure.

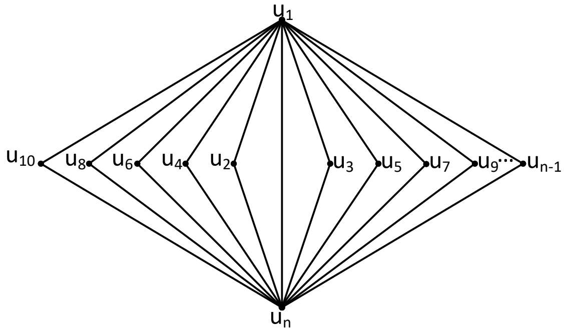

Fig. 1. side phantom and full mark example diagram

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Definition 2: Let G_1 and G_2 be graphs belonging to the path figure p_n, cycle figure C_n, star figure S_n, the fan figure F_n, and the wheel figure W_n. The union figure $G_1 \uparrow_{ab} G_2$ represents the connection of node a from G_1 to node b from G_2, where node a represents the central node of star, fan, or the wheel graph; the 1-degree node of path graph; or any node of cycle graph. Node b represents the non-central node of star or wheel graph, the 2-degree node of fan graph, or the 2-degree node of path graph.

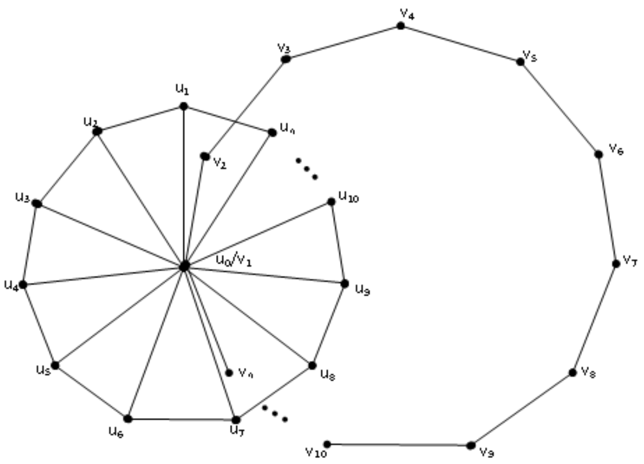


Fig. 2. Labeling example for $W_n \uparrow_{aa} C_n$

Definition 3: Let G_1 and G_2 be two simple figures. The graph formed by connecting each node a from G_1 to a copy of the node a from G_2 (where a represents the central node of a star, fan, or wheel graph; the 1-degree node of a path graph; or any node of a cycle figure) is known as a generalized corona figure. It's denoted as $G_1^a \circ G_2^a$. For example, provided that $G_1 = C_{11}$ and $G_2 = S_3$, the generalized corona graph of G_1 and G_2 is denoted as $C_{11}^a \circ S_3^a$, as shown in Fig 3.

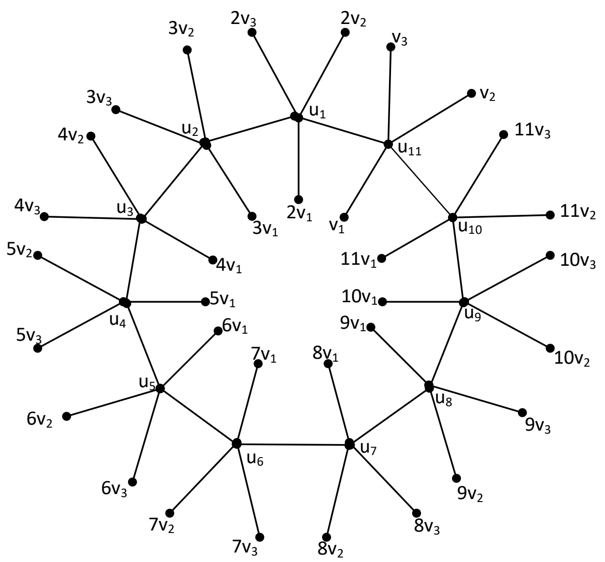


Fig. 3. Generalized crown graph $C_{11}^a \circ S_3^a$

III. EMTL ALGORITHM

A. BASIC COMPUTATIONS OF VMTL ALGORITHM

As per the definition of EMTL, the edge labels and vertex labels of an EMTL figure $G(p,q)$ are related one-to-one onto}. The total sum of labels in graph G equals the total of vertex labels $S(p)$ and edge tags $S(q)$:

$$C = S(p) + S(q) = \sum_{i=1}^{p+q} i = \frac{(p+q)(p+q+1)}{2} \quad (1)$$

Based on the definition, it can be inferred that the product of the vertex tags as well as edge tags of an EMTL graph must satisfy a condition where $f(u)+f(v)= K- f(uv)$. Hence, the following formula can be derived:

$$q*k - c = \sum_{i=1}^{p} (\deg(v_i) - 1) f(v_i) \quad (2)$$

To simplify the formula, let $\mathrm{adc}(v_i)=\deg(v_i)-1$, yielding:

$$q*k - c = \sum_{i=1}^{p} adc(v_i) f(v_i) \quad (3)$$

Additionally, as equation (3) exclusively incorporates vertex labels and does not encompass the complete solution space, the coefficients of all edge labels are assigned a value of 0 and subsequently summed, yielding Equation (4):

$$q*k = c + \sum_{i=1}^{p} adc(v_i) f(v_i) + \sum_{j=1}^{q} 0 * f(e_j) \quad (4)$$

$$q*k = C + \sum_{i=1}^{p} \mathrm{adc}(v_i) f(v_i) + \sum_{j=1}^{q} 0 * f(e_j)$$

Let $S = \sum_{i=1}^{p} \mathrm{adc}(v_i) f(v_i) + \sum_{j=1}^{q} 0 * f(e_j)$, then:

$$q = (C+S)/k$$

B. EMTL ALGORITHM CONCEPT:

(1) To calculate the values of the degree sequence coefficients $adc(v_i)$, C, and S for the graph G, use the given formulas. Once you have obtained these values, substitute them into Equation (4) to compute the value of K.

(2) Initialize $f(v_i)$ and label the vertices in descending order based on their degrees.

(Ⅰ) Check if the range of satisfies $p+q+3 \leq 2(p+q)$ the equation. If there exists a non-negative whole number k which satisfies the formula, proceed to the labelling. Otherwise, perform a permutation of the combination of $adc(v_i)$ and 0.

(Ⅱ) If the allocation is successful, indicating the presence of an EMTL in the graph, terminate the loop and conclude the algorithm. Otherwise, conduct a permutation of the combination of " $adc(v_i)$" and "0" and repeat this process in a loop.

(Ⅲ) Continue this process until a successful allocation is achieved otherwise all permutations are exhausted. The algorithm ends in either case .

(3) If the graph remains unlabeled after all permutations have been exhausted, it indicates that the graph is a NON-EMTL graph.

C. EMTL ALGORITHM FLOW

1)PSEUDOCODE FOR EMTL ALGORITHM

Table 1 presents the pseudocode for the algorithm utilized in edge-magic total labeling.

TABLE 1. EMTL PSEUDOCODE

Input	The adjacency matrix (store in A) of figure G (p, q)
Output	Non-EMTL diagrams or matrices satisfying EMTL
1	Calculate deg(vi)-1, C, f(vi);
2	is Conflict = true;
3	do
4	Sum = CalculateSum(deg(vi)-1, f(vi));
5	if (C + S) / q == 0
6	Calculate k;
7	if (Labelling(deg(vi)-1, k))
8	is Conflict = false;
9	is Success = true;
10	return;
11	end if
12	end if
13	Find the right(deg(vi)-1);
14	while (is Conflict)

D. ANALYSIS OF EMTL ALGORITHM RESULTS

According to the EMTL algorithm, experiments were conducted on special graphs with 3 to 17 vertices, resulting in the edge magic total labeling (EMTL) sets for these vertices, as shown in Table 2(Table 2 is on the appendix page).

E. EXAMPLES

Figure 4 shows examples of edge-magic total labeling graphs with 46 vertices (Figure 4 is on the appendix page).

IV. THEOREM && PROOF

Theorem 1: The composite figure $W_n\uparrow_{ab}P_n$, is an EMTL graph when $n \geq 3$ and n≡1(mod2).

Proof:

$V(W_n \uparrow_{ab} P_n) = \{u_0,u_1,u_2,...,u_n\} \cup \{v_1,v_2,v_3,...,v_n\}$ is the vertex set of the composite graph $W_n\uparrow_{ab}P_n$. The edge set $E(W_n\uparrow_{ab}P_n)$ is referred to as: $E(W_n\uparrow_{ab}P_n)=\{u_iu_{i+1}|1\leq i\leq n-1\}\cup\{u_nu_1\}\cup\{u_0u_i|1\leq i\leq n\}\cup\{v_iv_{i+1}|1\leq i\leq n-1\}$, Here, the central vertex of the wheel graph W_n is connected to any degree-1 vertex in the path graph P_n.

The composite graph $W_n\uparrow_{ab}P_n$ ($n=1(mod2)$) as follows:
Let $n = 2m+1$, where m=1,2,3…

$$f(u_0) = 3m+2$$

$$f(u_i) = \begin{cases} \dfrac{i+1}{2}, i \equiv 1(\mathrm{mod}\ 2), 1 \leq i \leq 2m+1 \\ \dfrac{2m+2+i}{2}, i \equiv 0(\mathrm{mod}\ 2), 2 \leq i \leq 2m \end{cases}$$

$$f(v_i) = \begin{cases} \dfrac{6m+3+i}{2}, i \equiv 1(\mathrm{mod}\ 2), 1 \leq i \leq 2m+1 \\ \dfrac{4m+2+i}{2}, i \equiv 0(\mathrm{mod}\ 2), 2 \leq i \leq 2m \end{cases}$$

$$f(u_nu_1) = 10m+4$$

$$f(u_iu_{i+1}) = 10m+4-i, \quad 1 \leq i \leq 2m$$

$$f(u_0u_i) = \begin{cases} \dfrac{16m+7-i}{2}, i \equiv 1(\mathrm{mod}\ 2), 1 \leq i \leq 2m+1 \\ \dfrac{14m+6-i}{2}, i \equiv 0(\mathrm{mod}\ 2), 2 \leq i \leq 2m \end{cases}$$

$$f(v_iv_{i+1}) = 6m+3-i, \quad 1 \leq i \leq 2m$$

In this case, the labels for each edge and associated vertices in the composite graph $W_n\uparrow_{ab}P_n$ are as follows:

(1) $f(u_i)+f(u_{i+1})+f(u_iu_{i+1})$
$= [\frac{i+1}{2}]+[(i+1)+\frac{2m+1-i}{2}]+[10m+4-i]$
$= 11m+6$

(2) $f(u_n)+f(u_1)+f(u_nu_1)$
$= [2m+1]+[m-1]+[10m+4]$
$= 11m+6$

(3) $f(u_0)+f(u_i)+f(u_0u_i)$
$= [3m+2]+[\frac{i+1}{2}]+[4(2m+1)-\frac{i+1}{2}]$
$= 11m+6$

(4) $f(v_i)+f(v_{i+1})+f(v_iv_{i+1})$
$= [10m+5]+[\frac{2m+2i}{2}]+[1-i]$
$= 11m+6$

According to the above proof process, it can be established that f is a bijective function from the set of vertices $V(G_n)$ as well as the set of edge $E(G_n)$ to the set $\{1,2,3,\cdots,5n-1\}$, as stated in definitions of edge-magic total tagging, when n≥ 3 as well as $n=1(mod2)$, the composite graph $W_n\uparrow_{ab}P_n$ is an EMTL graph. This concludes the verification of Theorem 1.

The labeling result for the composite graph $W_{11}\uparrow_{aa}P_{11}$ is shown in Fig 5.

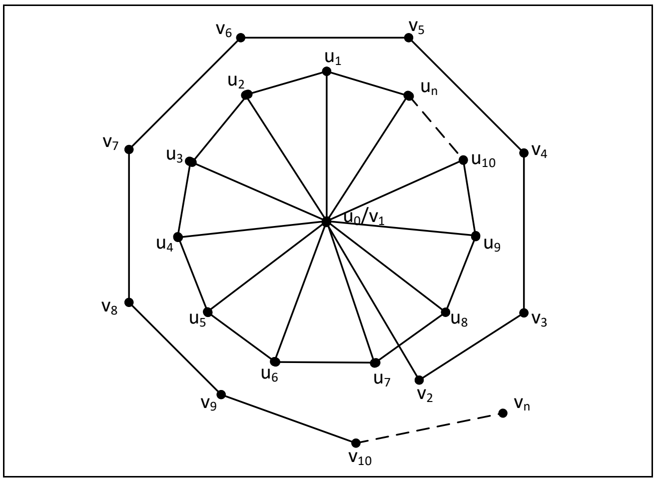


Fig. 5. Labeling example for $W_n\uparrow_{ab}P_n$

Theorem 2: For the composite graph $W_n\uparrow_{aa}S_m$, when $1 \leq m \leq \frac{n-1}{2}$, $n \geq 3$, as well as $n \equiv 1(\mathrm{mod}\ 2)$, it is an EMTL figure.

Proof:

Let $V(W_n\uparrow_{aa}S_m)=\{u_0,u_1,\cdots,u_n\}\cup V(W_n\uparrow_{aa}S_m)$ is the vertex assembly of the composite figure $W_n\uparrow_{aa}S_m$. The edge set $E(W_n\uparrow_{aa}S_m)$ is defined as: $E(W_n\uparrow_{aa}S_m)=\{u_iu_{i+1}|\ 1\le i \le n-1\}\cup\{u_nu_1\}\cup\{u_0u_i\,|1\le i\le n\}\cup\{v_0v_I\,|\ 1\le i\le m\}$,here, the central vertex of the wheel graph W_n is linked to the central vertex of the star figure S_m.

The composite graph $W_n\uparrow_{aa}S_m$ ($n=1(mod2)$)as follows:
Let $n=2m+1$, where m=1,2,3…

$$f(u_0)=3m+2$$

$$f(u_i)=\begin{cases}\frac{i+1}{2}, i\equiv 1(\mathrm{mod}\ 2), 1\le i\le 2m+1\\ \frac{2m+2+i}{2}, i\equiv 0(\mathrm{mod}\ 2), 2\le i\le 2m\end{cases}$$

$$f(u_nu_1)=8m+4$$

$$f(u_iu_{i+1})=8m+4-i,, \quad 1\le i\le 2m$$

$$f(u_0u_i)=\begin{cases}\frac{12m+7-i}{2}, i\equiv 1(\mathrm{mod}\ 2), 1\le i\le 2m+1\\ \frac{10m+6-i}{2}, i\equiv 0(\mathrm{mod}\ 2), 2\le i\le 2m\end{cases}$$

$f(v_0)=3m+2$

$f(v_i)=2m+1+i, \quad 1\le i\le m$

$f(v_0v_i)=4m+3-i, \quad 1\le i\le m$

In this case, the labels for each edge and associated vertices in the composite graph $W_n\uparrow_{aa}S_m$ are as follows:

(1) $f(u_i)+f(u_{i+1})+f(u_iu_{i+1})$
$=[\frac{i+1}{2}]+[(i+1)+\frac{2m+1-i}{2}]+[8m+4-i]$
$=9m+6$

(2) $f(u_n)+f(u_1)+f(u_nu_1)$
$=[2m+1]+[m-1]+[8m+4]$
$=9m+6$

(3) $f(u_0)+f(u_i)+f(u_0u_i)$
$=[3m+2]+[\frac{i+1}{2}]+[6m+4-\frac{i+1}{2}]$
$=9m+6$

(4) $f(v_0)+f(v_i)+f(v_0v_i)$
$=[3m+2]+[2m+1+i]+[4m+3-i]$
$=9m+6$

According to the above proof process, it can be established that f is a bijective function from the set of vertices $V(G_n)$ as well as the set of edge $E(G_n)$ to the set $\{1,2,3,\cdots,4n\}$. As stated in definitions of edge-magic total tagging, when $1\le m\le \frac{n-1}{2}$, $n\ge 3$,and $n\equiv 1(\mathrm{mod}\ 2)$, the composite graph $W_n\uparrow_{aa}S_m$ is an EMTL graph. This concludes the verification of Theorem 2.

The tagging result for the composite graph $W_{11}\uparrow_{aa}S_5$ is shown in Fig 6.

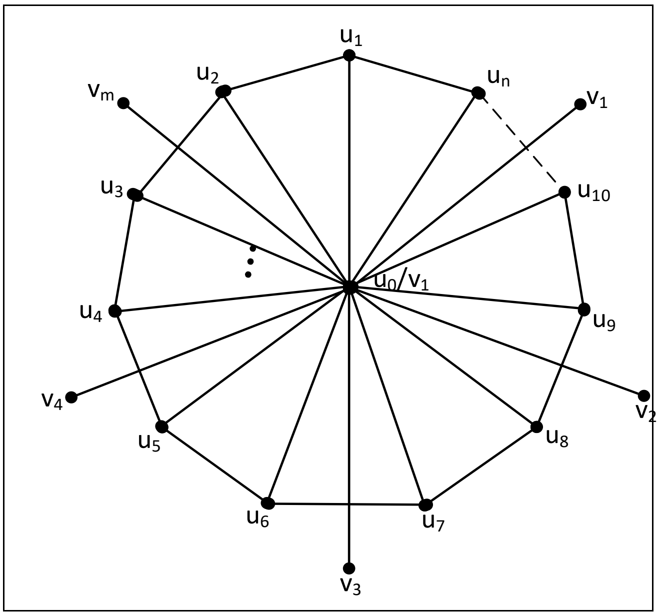


Fig. 6. Labeling example for $W_n\uparrow_{aa}S_m$

Theorem 3: For the n Prism($n-side$ Prism) , when $n\ge 3$ as well as $n\equiv 1(\mathrm{mod}\ 2)$, it is an EMTL graph.

Proof:

Let the vertex set of the n Prism($n-$ $side$ Prism) is $V(n\ \mathrm{Prism})=\{u_1,u_2,\cdots,u_n, 2u_1, 2u_2,\cdots,2u_n\}$, as well as the set of edge be $E(n$ Prism$)$ as follows: $E(n\ Prism)=\{u_iu_{i+1}|1\le i\le n\text{-}1\}\cup\{u_nu_1\}\cup\{u_i2u_{i+1}|1\le i\le n\text{-}1\}\cup\{u_n2u_1\}\cup\{u_nu_1\}\cup\{2u_i2u_{i+1}|1\le i\le n\text{-}1\}\cup\{2u_n2u_1\}$.

The labeling for the n Prism is as follows:

Let $n=2m+1$, where m=1,2,3…

When $h=1$:

$$f(hu_i)=\begin{cases}\frac{i+1}{2}, i\equiv 1(\mathrm{mod}\ 2), 1\le i\le 2m+1\\ \frac{2m+2+i}{2}, i\equiv 0(\mathrm{mod}\ 2), 2\le i\le 2m\end{cases}$$

$$f(hu_nhu_1)=10m+5$$

$$f(hu_ihu_{i+1})=10m-i+5, \quad 1\le \mathrm{i}\le 2\mathrm{m}$$

When $h=2$:

$$f(hu_i)=\begin{cases}(2m+1)+\frac{i+1}{2}, i\equiv 1(\mathrm{mod}\ 2), 1\le i\le 2m+1\\ (2m+1)+\frac{2m+2+i}{2}, i\equiv 0(\mathrm{mod}\ 2), 2\le i\le 2m\end{cases}$$

$$f(hu_nhu_1)=6m+3$$

$$f(hu_ihu_{i+1})=6m-i+3, \quad 1\le \mathrm{i}\le 2\mathrm{m}$$

$$f(u_n2u_1)=8m+4$$

$$f(u_i2u_{i+1})=8m-i+4, \quad 1\le \mathrm{i}\le 2\mathrm{m}$$

In this situation, the labeling for each edge and associated vertices of the n Prism($n-side$ Prism) is as follows:

(1) $f(hu_i)+f(hu_{i+1})+f(hu_i hu_{i+1})$

$=\left[(h-1)(2m+1)+\frac{i+1}{2}\right]+\left[(h-1)(2m+1)+m+\frac{i+3}{2}\right]$

$+\left[14m-4hm-2h-i+7\right]$

$=11m+7$

(2) $f(hu_n)+f(hu_1)+f(hu_n hu_1)$

$=\left[(h-1)(2m+1)+m+1\right]+[(h-1)(2m+1)+1]$

$+[6(2m+1)-(2h-1)(2m+1)]$

$=11m+7$

(3) $f(u_n)+f(2u_1)+f(u_n 2u_1)$

$=[m+1]+[2m+2]+[8m+4]$

$=11m+7$

(4) $f(u_i)+f(2u_{i+1})+f(u_i 2u_{i+1})$

$=[\frac{i+1}{2}]+\left[3m+2+\frac{i+1}{2}\right]+[8m+4-i]$

$=11m+7$

According to the above proof process, it can be determined that f is a bijection from the set of vertices $V(G_n)$ and the set of edge $E(G_n)$ towards the set $\{1,2,3,\cdots,5n\}$.

As stated in the definitions of edge-magic total tagging, when $n\ge 3$ as well as $n\equiv 1(\mathrm{mod}\,2)$, the n Pr*ism*($n-$*side* Pr*ism*) is an EMTL graph. This concludes the verification of Theorem 3.

The tagging result for the 5 Pr*ism*(5$-$*side* Pr*ism*) is shown in Fig 7:

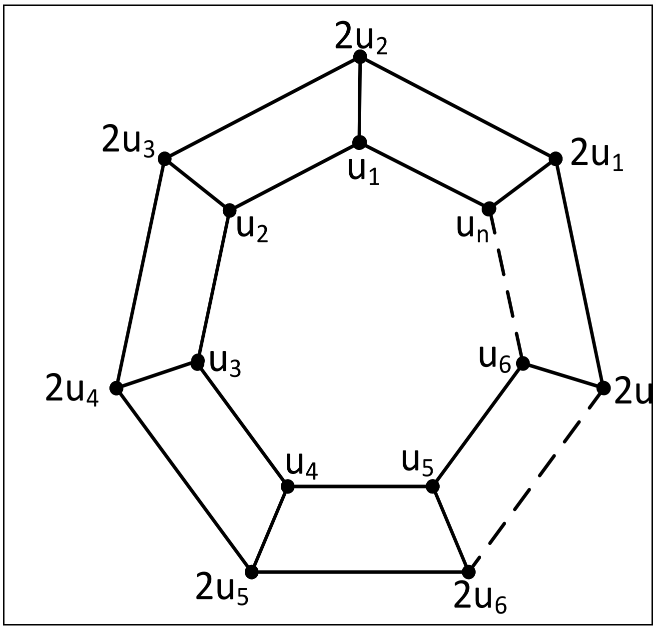


Fig. 7. Labeling example for 5 Pr*ism*

Theorem 4: For the $W_n\uparrow_{aa}p_l\uparrow_{aa}W_n$, when $1\le l\le n-1$, $n\ge 3$ as well as $n\equiv 1(\mathrm{mod}\,2)$, it is an EMTL graph.

Proof :

The graph $W_n\uparrow_{aa}p_l\uparrow_{aa}W_n$ is defined with the set of vertices $V(W_n\uparrow_{aa}p_l\uparrow_{aa}W_n)=\{u_0,u_1,u_2,\cdots,u_n\}\cup\{\{v_1,v_2,\cdots,v_n\}\cup\{w_0,w_1,w_2,\cdots,w_n\}$ and the set of edge $E(W_n\uparrow_{aa}p_l\uparrow_{aa}W_n)$ given by: $E(W_n\uparrow_{aa}p_l\uparrow_{aa}W_n)=\{u_iu_{i+1}\,|\,1\le i\le n-1\}\cup\{u_nu_1\}\cup\{u_0u_i\,|\,1\le i\le n\}\cup\{v_iv_{i+1}\,|\,1\le i\le l-1\}\cup\{w_iw_{i+1}\,|\,1\le i\le n-1\}\cup\{w_0w_i\,|\,1\le i\le n\}$, here, the central point of two wheel graphs w_n are respectively connected to the two degree-1 points in the path graph P_n.

The composite graph $W_n\uparrow_{aa}p_l\uparrow_{aa}W_n$ $(n\equiv 1(\mathrm{mod}\,2))$ as follows:

Let $n=2m+1, l=2m$, where m=1,2,3…

$$f(u_0)=3m+2$$

$$f(u_i)=\begin{cases}\frac{i+1}{2}, i\equiv 1(\mathrm{mod}\,2), 1\le i\le 2m+1\\ \frac{2m+2+i}{2}, i\equiv 0(\mathrm{mod}\,2), 2\le i\le 2m\end{cases}$$

$$f(u_nu_1)=16m+5$$

$$f(u_iu_{i+1})=16m-i+5 \quad 1\le i\le 2m$$

$$f(u_nu_i)=\begin{cases}\frac{28m+9-i}{2}, i\equiv 1(\mathrm{mod}\,2), 1\le i\le 2m+1\\ \frac{26m+8-i}{2}, i\equiv 0(\mathrm{mod}\,2), 2\le i\le 2m\end{cases}$$

$$f(v_i)=\begin{cases}\frac{6m+i+3}{2}, i\equiv 1(\mathrm{mod}\,2), 1\le i\le 2m-1\\ \frac{4m+2+i}{2}, i\equiv 0(\mathrm{mod}\,2), 2\le i\le 2m\end{cases}$$

$$f(v_iv_{i+1})=12m+4-i, \quad 1\le i\le 2m-1$$

$$f(w_0)=f(v_l)=3m+1$$

$$f(w_i)=\begin{cases}\frac{8m+i+3}{2}, i\equiv 1(\mathrm{mod}\,2), 1\le i\le 2m+1\\ \frac{10m+i+4}{2}, i\equiv 0(\mathrm{mod}\,2), 2\le i\le 2m\end{cases}$$

$$f(w_0w_i)=\begin{cases}\frac{10m+9-i}{2}, i\equiv 1(\mathrm{mod}\,2), 1\le i\le 2m+1\\ \frac{18m+8-i}{2}, i\equiv 0(\mathrm{mod}\,2), 2\le i\le 2m\end{cases}$$

$$f(w_nw_1)=8m+3$$

$$f(w_iw_{i+1})=8m+3-i, \quad 1\le i\le 2m$$

In this case, the labeling for each edge and associated vertices of the $W_n\uparrow_{aa}p_l\uparrow_{aa}W_n$ is as follows:

(1) $f(u_i)+f(u_{i+1})+f(u_iu_{i+1})$

$=[\frac{i+1}{2}]+[(i+1)+\frac{2m+1-i}{2}]+[16m+5-i]$

$=17m+7$

(2) $f(u_n)+f(u_1)+f(u_nu_1)$

$=[2m+1-m]+1+[16m+5]$

$=17m+7$

(3) $f(u_0)+f(u_i)+f(u_0u_i)$

$=[3m+2]+\left[i-\frac{i-1}{2}\right]+[14m+5-\frac{i+1}{2}]$

$=17m+7$

(4) $f(v_i)+f(v_{i+1})+f(v_i v_{i+1})$

$$=\left[2m+1+i+\frac{2m+1-i}{2}\right]+\left[2m+2+i-\frac{(i+1)}{2}\right]+[12m+4-i]$$

$=17m+7$

(5) $f(w_0)+f(w_i)+f(w_0 w_i)$

$$=[3m+1]+\left[4m+2+\frac{i-1}{2}\right]+[10m+5-\frac{i+1}{2}]$$

$=17m+7$

(6) $f(w_n)+f(w_1)+f(w_n w_1)$

$=[5m+2]+[4m+2]+[8m+3]$

$=17m+7$

(7) $f(w_i)+f(w_{i+1})+f(w_i w_{i+1})$

$$=\left[4m+2+\frac{i-1}{2}\right]+\left[5m+2+\frac{i}{2}\right]+[8m+3-i]$$

$=17m+7$

According to the above proof process, it can be determined that f is a bijective function from the set of vertex $V(G_n)$ as well as the set of edge $E(G_n)$ towards the set $\{1,2,3,\cdots,8n-3\}$. As stated in the definitions of edge-magic total labeling, when $1\le l\le n-1$, $n\ge 3$ and $n\equiv 1(\bmod 2)$, the $W_n\uparrow_{aa}p_l\uparrow_{aa}W_n$ is an EMTL graph. This concludes the verification of Theorem 4.

The tagging result for the $W_{11}\uparrow_{aa}p_{10}\uparrow_{aa}W_{11}$ is shown in Fig 8:

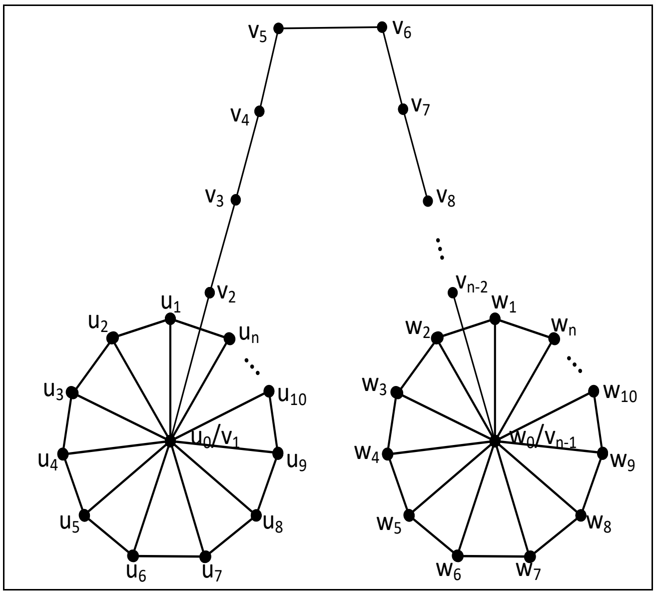


Fig. 8. Labeling example for $W_{11}\uparrow_{aa}p_{10}\uparrow_{aa}W_{11}$

Theorem 5: For the $W_n\uparrow_{aa}S_t\uparrow_{aa}C_n$, when $n\ge 3$ and $n\equiv 1(\bmod 2)$, it is an EMTL graph.

Proof :

The graph $W_n\uparrow_{aa}S_t\uparrow_{aa}C_n$ is defined with the vertex set $V(W_n\uparrow_{aa}S_t\uparrow_{aa}C_n)=\{u_0,u_1,u_2,...,u_n\}\cup\{v_0,v_1,v_2,\cdots,v_t\}\cup\{w_1,w_2,\cdots,w_n\}$ and the edge set $E(W_n\uparrow_{aa}S_t\uparrow_{aa}C_n)$ given by: $E(W_n\uparrow_{aa}S_t\uparrow_{aa}C_n)=\{u_i u_{i+1}\mid 1\le i\le n-1\}\cup\{u_n u_1\}\cup\{u_0 u_i\mid 1\le i\le n\}\cup\{v_0 v_i\mid 1\le i\le t\}\cup\{w_i w_{i+1}\mid 1\le i\le n-1\}\cup\{w_n w_1\}$, here, the central point of the wheel graph W_n and the star graph S_t are jointly connected to any point in the cycle graph C_n.

The composite graph $W_n\uparrow_{aa}S_t\uparrow_{aa}C_n(n=1(mod 2))$ as follows:

Let $n=2m+1, t=1,2,3,\cdots$ where m=1,2,3…

$$f(u_0)=3m+2$$

$$f(u_i)=\begin{cases}\dfrac{i+1}{2}, i\equiv 1(\bmod 2), 1\le i\le 2m+1\\ \dfrac{2m+2+i}{2}, i\equiv 0(\bmod 2), 2\le i\le 2m\end{cases}$$

$$f(u_n u_1)=10m+5+2t$$

$$f(u_i u_{i+1})=10m+5+2t-i,\quad 1\le i\le 2m$$

$$f(u_0 u_i)=\begin{cases}\dfrac{16m+4t-9+i}{2}, i\equiv 1(\bmod 2), 1\le i\le 2m+1\\ \dfrac{14m+4t-8+i}{2}, i\equiv 0(\bmod 2), 2\le i\le 2m\end{cases}$$

$$f(v_i)=\begin{cases}\dfrac{6m+3+i}{2}, i\equiv 1(\bmod 2), 1\le i\le 2m+1\\ \dfrac{4m+2+i}{2}, i\equiv 0(\bmod 2), 2\le i\le 2m\end{cases}$$

$$f(v_i v_{i+1})=6m+4+2t-i,\quad 1\le i\le 2m$$

$$f(v_n v_i)=4m+3+2t$$

$$f(w_0)=f(u_0)=3m+2$$

$$f(w_i)=4m+2+i,\quad 1\le i\le t$$

$$f(w_0 w_i)=4m+3+2t-i,\quad 1\le i\le t$$

In this case, the labeling for each edge and associated the points of the $W_n\uparrow_{aa}S_t\uparrow_{aa}C_n$ is as follows:

(1) $f(u_i)+f(u_{i+1})+f(u_i u_{i+1})$

$$=\left[\frac{i+1}{2}\right]+\left[\frac{2m+3+i}{2}\right]+[10m+5+2t-i]$$

$=11m+7+2t$

(2) $f(u_n)+f(u_1)+f(u_n u_1)$

$=[m+1]+1+[10m+5+2t-i]$

$=11m+7+2t$

(3) $f(u_0)+f(u_i)+f(u_0 u_i)$

$$=[3m+2]+\left[i-\frac{i-1}{2}\right]+[8m+4+2t-\frac{i-1}{2}]$$

$=11m+7+2t$

(4) $f(v_i)+f(v_{i+1})+f(v_i v_{i+1})$

$$=\left[\frac{6m+3+i}{2}\right]+\left[2m+\frac{(i+3)}{2}\right]+[6m+4+2t-i]$$

$=11m+7+2t$

(5) $f(w_0)+f(w_i)+f(w_0 w_i)$

$=[3m+2]+[4m+2+i]+[4m+3+2t-i]$

$=11m+7+2t$

According to the above proof process, it can be determined that f is a bijective function from the set of vertex $V(G_n)$ as well as the set of vertex $E(G_n)$ towards the set $\{1, 2, 3, \cdots, 5n+2t\}$. As stated in the definitions of edge-magic total tagging, when $n \geq 3$ and n ≡ 1(mod2) , the $W_n \uparrow_{aa} S_t \uparrow_{aa} C_n$ is an EMTL graph. This concludes the verification of Theorem 5.

The tagging result for the $W_n \uparrow_{aa} S_t \uparrow_{aa} C_n$ is shown in Fig 9:

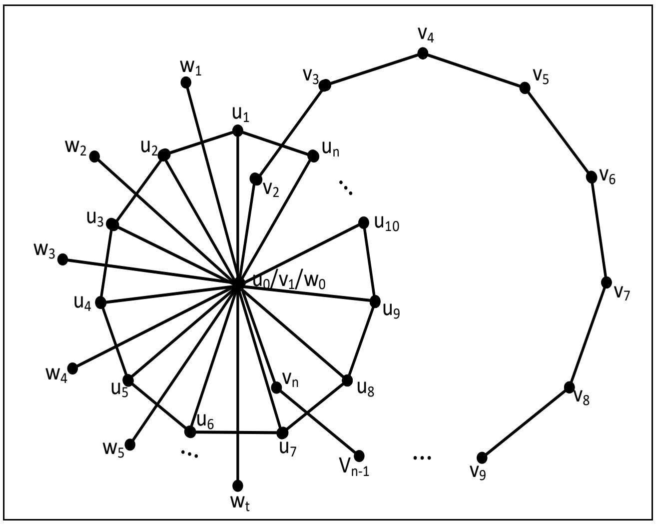


Fig. 9. Labeling example for $W_{11} \uparrow_{aa} S_6 \uparrow_{aa} C_{11}$

Theorem 6: For the $C_n \circ S_t^a$, when $t \geq 2, n \geq 3$ and $n \equiv 1 (\text{mod } 2)$, it is an EMTL graph.

Proof:

The graph $C_n \circ S_t^a$ is defined with the set of vertices $V(C_n \circ S_t^a) = \{u_1, u_2, ..., u_n\} \cup \{hv_1, hv_2, \cdots, hv_t\}$ as well as the set of edge $E(C_n \circ S_t^a)$ given by: $E(C_n \circ S_t^a) = \{u_i u_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{hv_0 hv_i \mid 1 \leq i \leq t\}$,here, h star graphs S_t are individually connected to each point in the cycle graph C_n.

The composite graph $C_n \circ S_t^a$ as follows:

Let $n = 2m+1, t = 1, 2, 3, \cdots$ where m=1,2,3…

$$f(u_i) = \begin{cases} \dfrac{i+1}{2}, i \equiv 1 (\text{mod } 2), 1 \leq i \leq 2m+1 \\ \dfrac{2m+2+i}{2}, i \equiv 0 (\text{mod } 2), 2 \leq i \leq 2m \end{cases}$$

$$f(u_n u_1) = (4m+2)(1+t)$$

$$f(u_i u_{i+1}) = (4m+2)(1+t) - i, \quad 1 \leq i \leq 2m$$

$$f(hv_0) = \begin{cases} \dfrac{h+2m+3}{2}, h \equiv 1 (\text{mod } 2), 1 \leq h \leq 2m+1 \\ \dfrac{h}{2}, h \equiv 0 (\text{mod } 2), 2 \leq h \leq 2m \end{cases}$$

$$f(hv_i) = \begin{cases} \dfrac{h+1}{2} + i(2m+1), h \equiv 1 (\text{mod } 2), 1 \leq i \leq t, 1 \leq h \leq 2m+1 \\ m+1+i(2m+1)+\dfrac{h}{2}, h \equiv 0 (\text{mod } 2), 1 \leq i \leq t, 2 \leq h \leq 2m \end{cases}$$

$$f(hv_0 hv_i) = (4+4t+2i)m + 2t + 3 - h - i, \quad 1 \leq i \leq t, 1 \leq h \leq 2m+1$$

In this case, the labeling for each edge and associated vertices of the $C_n \circ S_t^a$ is as follows:

$$(1)\ f(u_i) + f(u_{i+1}) + f(u_i u_{i+1})$$
$$= \left[\frac{i+1}{2}\right] + [i+1+\frac{2m+1-i}{2}] + [(4m+2)(1+t) - i]$$
$$= 5m + 4mt + 4 + 2t$$
$$(2)\ f(hv_0) + f(hv_i) + f(hv_0 hv_i)$$
$$= \left[\frac{h+2m+1}{2}\right] + \left[\frac{h+1}{2} + i(2m+1)\right] + \left[\begin{matrix}(1+2t)(2m+1) \\ +1-h-(i-1)(2m+1)\end{matrix}\right]$$
$$= 5m + 4mt + 4 + 2t$$

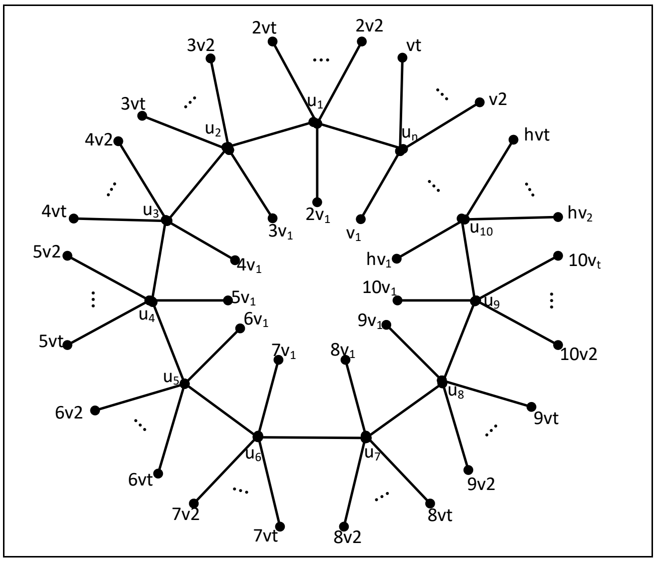


Fig. 10. Labeling example for $C_{11} \circ S_3^a$

According to the above proof process, it can be determined that f is a bijective function from the set of vertex $V(G_n)$ as well as the set of edge $E(G_n)$ to the set $\{1, 2, 3, \cdots, 2n(1+t)\}$, as stated by the definitions of edge-magic total tagging, when n⩾3, t⩾2 as well as $n \equiv 1 (\text{mod } 2)$, the $C_n \circ S_t^a$ is an EMTL graph. This concludes the verification of Theorem 6.

The tagging result for the $C_n \circ S_t^a$ is shown in Fig 10.

V. Conclusion

According to the E MTL algorithm, the special graphs within 3-17 vertices are tested to obtain the result set of edge-magic full marks of these vertices, as shown in Table 3(Table 3 is on the appendix page).

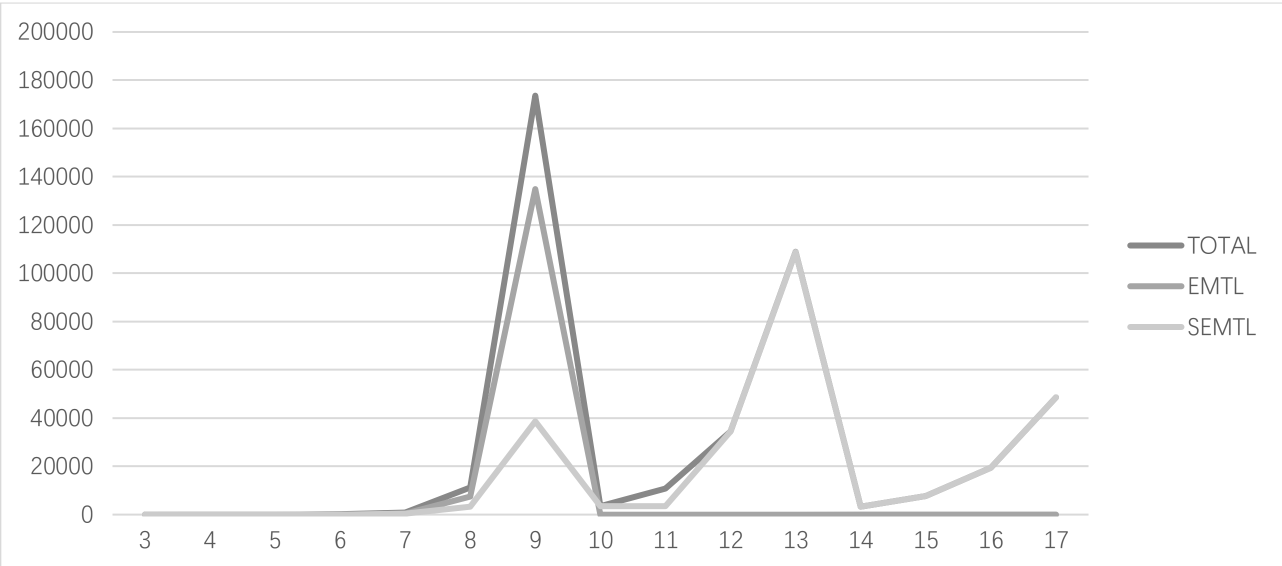
The text applies graph generation algorithms to acquire sets of all graphs with a limited number of vertices.Then devises an method utilizing Edge-Magic Total tagging to ascertain edge-magic tagging within a finite set of vertices. The obtained results are analyzed to identify labeling patterns for specific composite graphs. The text concludes by summarizing and presenting six theorems, along with their accompanying proofs.

APPENDIX

TABLE 2. LABELING RESULT STATISTICS TABLE

Number of vertices	Total number of graphs	Number of EMTL graphs	Number of NEMTL graphs	Total time consumed
3	2	0	0	2
4	10	1	1	5
5	21	8	0	22
6	112	54	5	219
7	853	490	4	1755
8	11117	7552	102	2084457
9	173609	134887	15	4050294
10	3441	18	0	3621
11	10874	46	0	225561
12	34485	24	0	3875
13	108869	63	0	72737
14	3159	0	0	23552
15	7741	0	0	334361
16	19320	0	0	41311
17	48629	0	0	305223

TABLE 3 CHART OF RESULTS



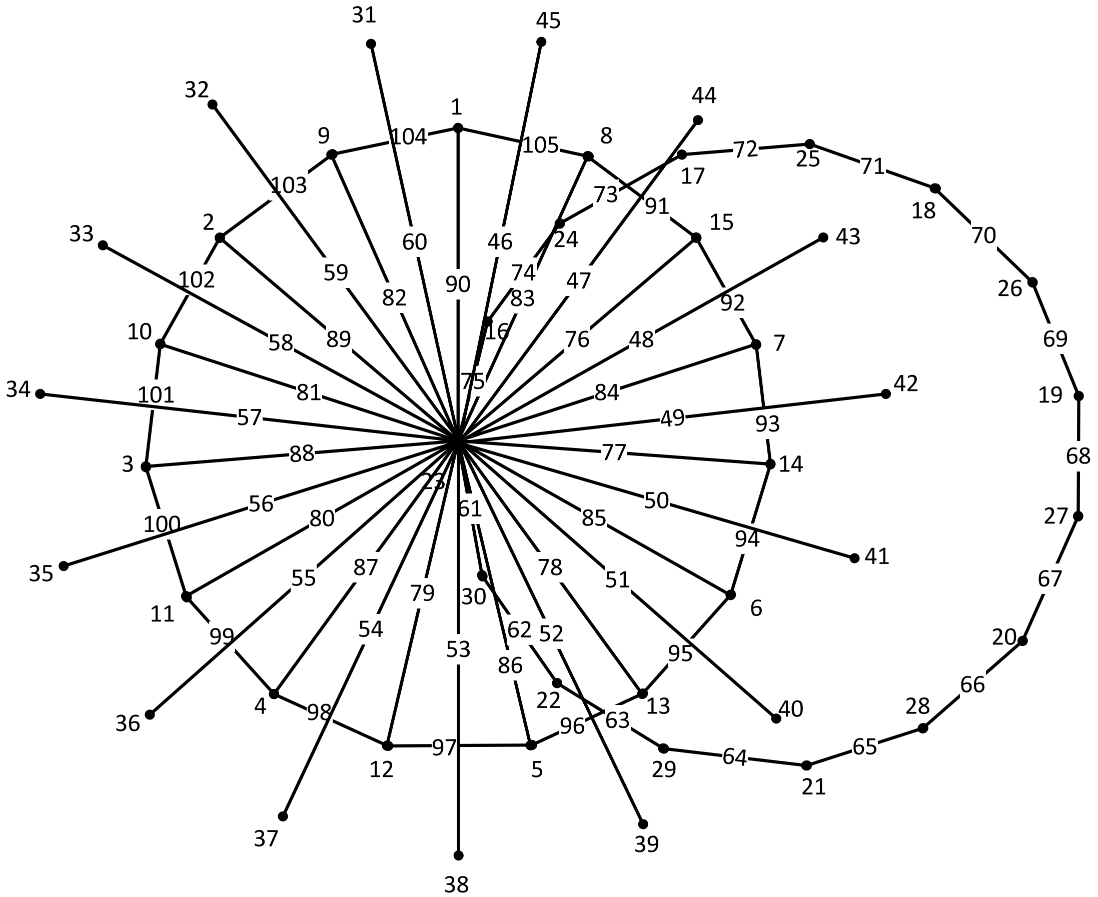


Fig. 4. edge-magic total labeling example graph G(46,60)

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