Stability of a Jensen Type Cubic and Quartic Functional Equations over Non-Archimedean Normed Space

A. Ramachandran and S. Sangeetha*

***Abstract*—In this paper, we introduce the cubic and quartic Jensen type functional equations:**

$$\mathfrak{f}\left(\frac{3x+y}{2}\right)+\mathfrak{f}\left(\frac{x+3y}{2}\right)=12\mathfrak{f}\left(\frac{x+y}{2}\right)+2\left[\mathfrak{f}(x)+\mathfrak{f}(y)\right]$$

$$\mathfrak{f}\left(\frac{3x+y}{2}\right)+\mathfrak{f}\left(\frac{x+3y}{2}\right)=24\mathfrak{f}\left(\frac{x+y}{2}\right)-6\mathfrak{f}\left(\frac{x-y}{2}\right)+4\left[\mathfrak{f}(x)+\mathfrak{f}(y)\right]$$

and discussed the Hyers-Ulam stability over non-Archimedean normed space.

***Index Terms*—Hyers-Ulam Stability (HUS), Jensen functional equation, Cubic function, Quartic function, Non-Archimedean Normed (NAN) space.**

I. Introduction

THE stability problem of functional equations originated from a question of Ulam [15] in 1940, concerning the stability of group homomorphisms. The question was "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?".

Hyers [6] gave the positive response to the question of Ulam for Banach spaces. Aoki [1] generalized the Hyers theorem for additive mappings. Hyers theorem was generalized by Rassias [11] by allowing the Cauchy difference to be unbounded. In response to Rassias question regarding $p > 1$, Gajada replied for it in [5]. Moslehian and Rassias [9] proved generalized HUS of the Cauchy functional equation and the quadratic functional equation in NAN spaces.

In [8], Kenary and Cho proved the HUS of mixed additive-quadratic Jensen type functional equation in Non-Archimedean normed spaces and random normed spaces. Yang et.al.[17] proved the HUS of mixed additive-quadratic Jensen type functional equation in multi-Banach spaces. Also, many authors have been extensively studied the stability problem of functional equations and Non-Archimedean spaces (see [2], [4], [7], [10], [13], [18]). The Jensen type additive functional equation was solved by Trif and the HUR (Hyers-Ulam-Rassias) stability was investigated in [14].

In this paper we introduce a new cubic and quartic functional equation of Jensen type

$$\mathcal{D}_3\mathfrak{f}(x,y)=\mathfrak{f}\left(\frac{3x+y}{2}\right)+\mathfrak{f}\left(\frac{x+3y}{2}\right)-12\mathfrak{f}\left(\frac{x+y}{2}\right)-2\left[\mathfrak{f}(x)+\mathfrak{f}(y)\right] \quad (1)$$

$$\mathcal{D}_4\mathfrak{f}(x,y)=\mathfrak{f}\left(\frac{3x+y}{2}\right)+\mathfrak{f}\left(\frac{x+3y}{2}\right)-24\mathfrak{f}\left(\frac{x+y}{2}\right)+6\mathfrak{f}\left(\frac{x-y}{2}\right)-4\left[\mathfrak{f}(x)+\mathfrak{f}(y)\right] \quad (2)$$

in NAN space.

II. Preliminaries

Definition 2.1. [12] A functional equation is an equation in which both sides contain a finite number of functions, some are known and some are unknown.

Example 2.1. $\mathfrak{f}(x+y)=\mathfrak{f}(x)+\mathfrak{f}(y)$ is the Cauchy additive functional equation

Definition 2.2. [12] A solution of a functional equation is a function which satisfies the equation.

Example 2.2. (i) $\mathfrak{f}(x)=kx$ is a solution of the Cauchy functional equation $\mathfrak{f}(x+y)=\mathfrak{f}(x)+\mathfrak{f}(y)$
(ii) $\mathfrak{f}(x)=cx+a$ is the solution of the Jensen functional equation $\mathfrak{f}(\frac{x+y}{2})=\frac{\mathfrak{f}(x)+\mathfrak{f}(y)}{2}$

Definition 2.3. [12] A functional equation F is stable if any function f satisfying the equation F approximately is near to exact solution of F.

Definition 2.4. [3], [16]. If $\mathbb{F}$ is any field then a valuation (of rank 1) is a map $|.|:\mathbb{F}\to\mathbb{R}$, satisfying the following axioms:

$$(i)|x|\geq 0$$
$$(ii)|x|=0,\quad when\quad x=0$$
$$(iii)|xy|=|x||y|$$
$$(iv)|x+y|\leq|x|+|y|$$

for all $x,y\in\mathbb{F}$.
The valuation is said to be non-Archimedean, if the following stronger form of inequality (iv) holds, namely

$$|x+y|\leq max\{|x|,|y|\}.$$

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Definition 2.5. [16] A sequence $\{x_n\}$ in $\mathbb{K}$ is called a Cauchy sequence with respect to a non-Archimedean valuation $|.|$, if and only if

$$|x_{n+1} - x_n| \to 0, \quad as \quad n \to \infty.$$

Definition 2.6. [3] If every Cauchy sequence of $\mathbb{K}$ has a limit in $\mathbb{K}$, then $\mathbb{K}$ is said to be complete.

Example 2.3. [16] The field $\mathbb{Q}_p$ of p-adic number is the completion of $\mathbb{Q}$ with respect to $|.|_p$.

Definition 2.7. [16] A complete normed linear space is called a Banach space.

Definition 2.8. [3], [16] Let X be a vector space over a field $\mathbb{K}$ with a non-trivial non-Archimedean valuation $|.|$. Then, a function $\|.\| : X \to \mathbb{R}$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$ for all $x \in$X
(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in$ X and $\alpha \in \mathbb{K}$
(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in$ X

and the space (X,$\|.\|$) is called a non-Archimedean normed space.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for $x, y > 0$, there exists $\eta \in \mathcal{N}$ such that $x < \eta y$.

Example 2.4. [3] Let p be a positive prime number. For every non-zero rational number x there exists a unique integer α such that

$$x = p^{\alpha} \left(\frac{a}{b}\right)$$

with some integer a and b not divisible by p, define p-adic absolute value

$$|x|_p = p^{-\alpha}.$$

Example 2.5. [3] Take $x = \frac{162}{13}$. Suppose we want to find its 3-adic absolute value (hence $p = 3$). Expressed in the p-adic form, we obtain

$$x = 81.\frac{2}{13} = 3^4.\frac{2}{13}$$

which mean $|x|_3 = \frac{1}{3^4}$.
13-adic absolute value for x. It will simply be $|x|_{13} = 13$ because

$$x = 13^{-1}.162$$
$$|x|_{13} = \frac{1}{13^{-1}} = 13.$$

III. Main Results

Throughout this paper, it is assumed that $\mathcal{G}$ is an additive group, $\mathcal{X}$ is a complete NAN space and $\mathcal{X}_1$, $\mathcal{X}_2$ are vector spaces. We start this section with the following lemmas.

Lemma 3.1. If a mapping $\mathfrak{f}$ from $\mathcal{X}_1$ to $\mathcal{X}_2$ satisfies (1) and $\mathfrak{f}(0) = 0$ then $\mathfrak{f}$ is a cubic mapping.

Proof: Putting $y = 0$ in (1), we get

$$\mathfrak{f}\left(\frac{3x}{2}\right) - 11\mathfrak{f}\left(\frac{x}{2}\right) - 2\mathfrak{f}(x) = 0 \qquad \text{for all } x \in \mathcal{G}. \quad (3)$$

$$\frac{1}{8}\mathfrak{f}(3x) - \frac{11}{8}\mathfrak{f}(x) - 2\mathfrak{f}(x) = 0 \qquad \text{for all } x \in \mathcal{G}. \quad (4)$$

$$\mathfrak{f}(3x) - 27\mathfrak{f}(x) = 0 \qquad \text{for all } x \in \mathcal{G}. \quad (5)$$

This means that $\mathfrak{f}$ is a cubic mapping.

■

Lemma 3.2. If a function $\mathfrak{f}$ from $\mathcal{X}_1$ to $\mathcal{X}_2$ satisfies (2) and $\mathfrak{f}(0) = 0$ then $\mathfrak{f}$ is a quartic mapping.

Proof: Putting $y = 0$ in (2), we get

$$\mathfrak{f}\left(\frac{3x}{2}\right) - 17\mathfrak{f}\left(\frac{x}{2}\right) - 4\mathfrak{f}(x) = 0 \qquad \text{for all } x \in \mathcal{G}. \quad (6)$$

$$\frac{1}{16}\mathfrak{f}(3x) - \frac{17}{16}\mathfrak{f}(x) - 4\mathfrak{f}(x) = 0 \qquad \text{for all } x \in \mathcal{G}. \quad (7)$$

$$\mathfrak{f}(3x) - 81\mathfrak{f}(x) = 0 \qquad \text{for all } x \in \mathcal{G}. \quad (8)$$

This means that $\mathfrak{f}$ is a quartic mapping.

■

Theorem 3.1. Fix $\ell = \pm 1$. Suppose that ξ from $\mathcal{G}^2 \to [0, \infty)$ is a mapping such that

$$\lim_{\eta \to \infty} \frac{1}{|27|^{\eta\ell}} \xi\left(3^{\eta\ell}x, 3^{\eta\ell}y\right) = 0 \qquad \text{for all } x, y \in \mathcal{G}. \quad (9)$$

Also, the limit

$$\Phi(x) = \lim_{\eta \to \infty} \max\left\{\frac{|8|}{|27|} \frac{1}{|27|^{\kappa\ell - \left(\frac{1-\ell}{2}\right)}} \xi\left(3^{\kappa\ell - \left(\frac{1-\ell}{2}\right)}x, 0\right) : 0 \leq \kappa < \eta\right\} \text{ for all } x \in \mathcal{G}, \quad (10)$$

exists and $\mathfrak{f} : \mathcal{G} \to \mathcal{X}$ is a cubic function satisfying

$$\|\mathcal{D}_3\mathfrak{f}(x, y))\| \leq \xi(x, y) \qquad \text{for all } x, y \in \mathcal{G}. \quad (11)$$

Then for all $x \in \mathcal{G}$,

$$\mathcal{C}_3(x) = \lim_{\eta \to \infty} \frac{1}{27^{\eta}} \mathfrak{f}(3^{\eta}x)$$

exists such that

$$\left\|\mathfrak{f}(x) - \mathcal{C}_3(x)\right\| \leq \Phi(x) \qquad \text{for all } x \in \mathcal{G}. \quad (12)$$

Moreover, if

$$\lim_{\jmath \to \infty} \lim_{\eta \to \infty} \max\left\{\frac{1}{|27|^{\kappa\ell}} \xi\left(3^{\kappa\ell}x, 0\right) : \jmath \leq \kappa < \eta + \jmath\right\} = 0, \quad (13)$$

then $\mathcal{C}_3$ is a unique cubic mapping satisfying (12).

Proof: **Case(i)**. Let us prove the theorem for $\ell = 1$. It follows by replacing $y = 0$ in (11), we obtain

$$\left\| \mathfrak{f}(3x) - 27\mathfrak{f}(x) \right\| \leq |8|\xi(x, 0) \qquad \text{for all } x \in \mathcal{G}. \quad (14)$$

Replacing x by $3^\eta x$ in (14), we get

$$\left\| \mathfrak{f}\frac{(3^{\eta+1}x)}{27^{\eta+1}} - \mathfrak{f}\frac{(3^\eta x)}{27^\eta} \right\| \leq \frac{|8|}{|27|^{\eta+1}}\xi(3^\eta x, 0) \text{ for all } x \in \mathcal{G}. \quad (15)$$

Thus, it follows from (9) and (15) that the sequence $\left\{ \frac{\mathfrak{f}(3^\eta x)}{27^\eta} \right\}$ is Cauchy sequence. Since $\mathcal{X}$ is complete. Therefore $\left\{ \frac{\mathfrak{f}(3^\eta x)}{27^\eta} \right\}$ is convergent.

$$\text{Let } \mathcal{C}_3(x) = \lim_{\eta\to\infty} \mathfrak{f}\left(\frac{3^\eta x}{27^\eta}\right) \qquad \text{for all } x \in \mathcal{G}. \quad (16)$$

By induction, one can show that

$$\left\| \mathfrak{f}\frac{(3^\eta x)}{27^\eta} - \mathfrak{f}(x) \right\| \leq \max\left\{ \frac{|8|}{|27|^{\kappa+1}}\xi(3^\kappa x, 0) : 0 \leq \kappa < \eta \right\}, \quad (17)$$

by taking the limit $\eta \to \infty$ in (17) and using (10) one obtain (12).

By (9) and (11), we get

$$\begin{aligned} \|\mathcal{D}_3\mathfrak{f}(x,y)\| &= \lim_{\eta\to\infty} \left\| \mathcal{D}_3\mathfrak{f}\left(\frac{3^\eta x}{27^\eta}, \frac{3^\eta y}{27^\eta}\right) \right\| \\ &= \lim_{\eta\to\infty} \frac{1}{|27|^\eta} \left\| \mathcal{D}_3\mathfrak{f}(3^\eta x, 3^\eta y) \right\| \\ &\leq \lim_{\eta\to\infty} \frac{1}{|27|^\eta}\, \xi(3^\eta x, 3^\eta y) = 0 \quad \text{for all } x, y \in \mathcal{G}. \end{aligned}$$

Therefore $\mathcal{C}_3(x)$ is a cubic mapping.
To prove uniqueness, let $\mathcal{C}'_3$ be another mapping satisfying (12) we obtain

$$\begin{aligned} &\left\| \mathcal{C}_3(x) - \mathcal{C}'_3(x) \right\| \\ &= \lim_{\eta\to\infty} \frac{1}{|27|^\eta} \|\mathcal{C}_3(3^\eta x) - \mathcal{C}'_3(3^\eta x)\| \\ &\leq \lim_{\eta\to\infty} \frac{1}{|27|^\eta} \max\Big\{ \|\mathcal{C}_3(3^\eta x) - \mathfrak{f}(3^\eta x)\|, \\ &\qquad \|\mathfrak{f}(3^\eta x) - \mathcal{C}'_3(3^\eta x)\| \Big\} \\ &\leq \lim_{\jmath\to\infty} \lim_{\eta\to\infty} \max\left\{ \frac{1}{|27|^\kappa}\xi(3^\kappa x, 0) : \jmath \leq \kappa < \eta + \jmath \right\} \\ &= 0 \quad \text{for all } x \in \mathcal{G}. \end{aligned}$$

Therefore $\mathcal{C}_3(x) = \mathcal{C}'_3(x)$. This completes the proof.
Case (ii). Let us prove the theorem for $\ell = -1$. It follows by replacing $y = 0$ in (11), we obtain

$$\left\| \mathfrak{f}(3x) - 27\mathfrak{f}(x) \right\| \leq |8|\xi(x, 0) \qquad \text{for all } x \in \mathcal{G}. \quad (18)$$

Replacing x by $\frac{x}{3^{\eta+1}}$ in (18), we get

$$\left\| 27^{\eta+1}\mathfrak{f}\left(\frac{x}{3^{\eta+1}}\right) - 27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right) \right\| \leq |8||27|^\eta\xi\left(\frac{x}{3^{\eta+1}}, 0\right) \quad \text{for all } x \in \mathcal{G}. \quad (19)$$

Thus, it follows from (9) and (19) that the sequence $\left\{27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right)\right\}$ is Cauchy sequence. Since $\mathcal{X}$ is complete. Therefore $\left\{27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right)\right\}$ is convergent.

$$\text{Let } \mathcal{C}_3(x) = \lim_{\eta\to\infty} 27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right) \qquad \text{for all } x \in \mathcal{G}. \quad (20)$$

By induction, one can show that

$$\left\| 27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right) - \mathfrak{f}(x) \right\| \leq \max\left\{ |8||27|^\eta\xi\left(\frac{x}{3^{\eta+1}}, 0\right) : 0 \leq \kappa < \eta \right\}, \quad (21)$$

by taking the limit $\eta \to \infty$ in (21) and using (10) one obtain (12).

By (9) and (11), we get

$$\begin{aligned} \|\mathcal{D}_3\mathfrak{f}(x,y)\| &= \lim_{\eta\to\infty} \left\| \mathcal{D}_3\mathfrak{f}\left(27^\eta\frac{x}{3^\eta}, 27^\eta\frac{y}{3^\eta}\right) \right\| \\ &= \lim_{\eta\to\infty} |27|^\eta \left\| \mathcal{D}_3\mathfrak{f}\left(\frac{x}{3^\eta}, \frac{y}{3^\eta}\right) \right\| \\ &\leq \lim_{\eta\to\infty} |27|^\eta\, \xi\left(\frac{x}{3^\eta}, \frac{y}{3^\eta}\right) = 0 \quad \text{for all } x, y \in \mathcal{G}. \end{aligned}$$

Therefore $\mathcal{C}_3(x)$ is a cubic mapping. To prove uniqueness, let $\mathcal{C}'_3$ be another mapping satisfying (12) we obtain

$$\begin{aligned} &\left\| \mathcal{C}_3(x) - \mathcal{C}'_3(x) \right\| \\ &= \lim_{\eta\to\infty} |27|^\eta \left\| \mathcal{C}_3\left(\frac{x}{3^\eta}\right) - \mathcal{C}'_3\left(\frac{x}{3^\eta}\right) \right\| \\ &\leq \lim_{\eta\to\infty} |27|^\eta \max\Big\{ \left\| \mathcal{C}_3\left(\frac{x}{3^\eta}\right) - \mathfrak{f}\left(\frac{x}{3^\eta}\right) \right\|, \\ &\qquad \left\| \mathfrak{f}\left(\frac{x}{3^\eta}\right) - \mathcal{C}'_3\left(\frac{x}{3^\eta}\right) \right\| \Big\} \\ &\leq \lim_{\jmath\to\infty} \lim_{\eta\to\infty} \max\left\{ |27|^\kappa\xi\left(\frac{x}{3^\kappa}, 0\right) : \jmath \leq \kappa < \eta + \jmath \right\} \\ &= 0 \qquad \text{for all } x \in \mathcal{G}. \end{aligned}$$

Therefore $\mathcal{C}_3(x) = \mathcal{C}'_3(x)$. This completes the proof. ■

Corollary 3.1. Let $\delta \geq 0$ and prime $p > 3$. Define a function $\mathfrak{f}$ from $\mathcal{G}$ to $\mathcal{X}$ and if $\mathfrak{f}$ is a cubic mapping that fulfills the inequality

$$\left\| \mathcal{D}_3\mathfrak{f}(x,y) \right\| \leq \delta \qquad \text{for all } x, y \in \mathcal{G}.$$

Then, there exists a unique cubic function $\mathcal{C}_3(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\|\mathfrak{f}(x) - \mathcal{C}_3(x)\| \leq \frac{|8|}{|27|}\, \delta$$

Proof:

By Theorem 3.1, if $\xi(x,y)=\delta$ then

$$\|\mathfrak{f}(x)-\mathcal{C}_3(x)\| \le \Phi(x),$$

$$\text{where } \Phi(x)=\lim_{\eta\to\infty}\max\left\{\frac{|8|}{|27|^{\kappa+1}}\ \xi(3^{\kappa}x,0): 0\le\kappa<\eta\right\}.$$

Therefore,

$$\|\mathfrak{f}(x)-\mathcal{C}_3(x)\| \le \lim_{\eta\to\infty}\max\left\{\frac{|8|}{|27|^{\kappa+1}}\delta : 0\le\kappa<\eta\right\}$$

$$\le \frac{|8|}{|27|}\delta.$$

■

Corollary 3.2. Let $r,s,\delta>0$ and $r+s>3$. Define a function $\mathfrak{f}$ from $\mathcal{G}$ to $\mathcal{X}$ and if $\mathfrak{f}$ is a cubic mapping satisfying the inequality

$$\left\|\mathcal{D}_3\mathfrak{f}(x,y)\right\| \le \delta\ (\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s) \quad \text{for all } x,y\in\mathcal{G}.$$

Then, there is a unique cubic function $\mathcal{C}_3(x):\mathcal{G}\to\mathcal{X}$ such that

$$\|\mathfrak{f}(x)-\mathcal{C}_3(x)\| \le \frac{\delta\ |8|\ \|x\|^{r+s}}{|27|}$$

Proof: Let $\xi(x,y)=\delta(\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s)$

From Theorem (3.1),

$$\left\|\mathfrak{f}(x)-\mathcal{C}_3(x)\right\| \le \Phi(x) \qquad \text{for all } x\in\mathcal{G}.$$

where,

$$\Phi(x)=\lim_{\eta\to\infty}\max\left\{\frac{|8|}{|27|}\frac{1}{|27|^{\kappa\ell-\left(\frac{1-\ell}{2}\right)}}\xi\left(3^{\kappa\ell-\left(\frac{1-\ell}{2}\right)}x,0\right) : 0\le\kappa<\eta\right\} \text{ for all } x\in\mathcal{G}.$$

Taking $\ell=1$, we obtatin

$$\Phi(x)=\lim_{\eta\to\infty}\max\left\{\frac{|8|}{|27|^{\kappa+1}}\xi\left(3^{\kappa}x,0\right) : 0\le\kappa<\eta\right\} \text{ for all } x\in\mathcal{G}.$$

$$=\lim_{\eta\to\infty}\max\left\{\frac{|8|}{|27|^{\kappa+1}}\delta|3|^{\kappa(r+s)}\|x\|^{r+s} : 0\le\kappa<\eta\right\}$$

$$=\frac{\delta\ |8|\ \|x\|^{r+s}}{|27|}$$

Therefore,

$$\|\mathfrak{f}(x)-\mathcal{C}_3(x)\| \le \frac{\delta\ |8|\ \|x\|^{r+s}}{|27|}.$$

■

For the case $r+s=3$, we have the following counter example.

Example 3.1. Let $p>3$ be a prime number and $\mathbb{Q}_p\to\mathbb{Q}_p$ be defined by $\mathfrak{f}(x)=x^3+1$. Since $|3^\eta|_p=1$ for all $\eta\in\mathcal{N}$. Then for $\delta>0$,

$$\|\mathcal{D}_3\mathfrak{f}(x,y)\| \le 1 \le \delta(\|x\|^{r+s}+\|y\|^{r+s}+\|x\|^r\|y\|^s) \quad \text{for all } x,y\in\mathcal{G},$$

and

$$\left\|\mathfrak{f}\frac{(3^{\eta+1}x)}{27^{\eta+1}}-\mathfrak{f}\frac{(3^{\eta}x)}{27^{\eta}}\right\| \nrightarrow 0 \quad \text{as} \quad \eta\to\infty.$$

Hence $\left\{\frac{\mathfrak{f}(3^\eta x)}{27^\eta}\right\}$ is not a Cauchy sequence.

Theorem 3.2. Fix $\ell=\pm1$. Suppose that ξ from $\mathcal{G}^2\to[0,\infty)$ is a mapping such that

$$\lim_{\eta\to\infty}\frac{1}{|81|^{\eta\ell}}\xi\left(3^{\eta\ell}x,3^{\eta\ell}y\right)=0 \qquad \text{for all } x,y\in\mathcal{G}. \quad (22)$$

Also, the limit

$$\Phi(x)=\lim_{\eta\to\infty}\max\left\{\frac{|16|}{|81|}\frac{1}{|81|^{\kappa\ell-\left(\frac{1-\ell}{2}\right)}}\xi\left(3^{\kappa\ell-\left(\frac{1-\ell}{2}\right)}x,0\right) : 0\le\kappa<\eta\right\} \quad \text{for all } x\in\mathcal{G}, \quad (23)$$

exists and $\mathfrak{f}:\mathcal{G}\to\mathcal{X}$ is an even mapping satisfying

$$\|\mathcal{D}_4\mathfrak{f}(x,y)\| \le \xi(x,y) \qquad \text{for all } x,y\in\mathcal{G}. \quad (24)$$

Then for all $x\in\mathcal{G}$,

$$\mathcal{Q}_4(x)=\lim_{\eta\to\infty}\mathfrak{f}\left(\frac{3^\eta x}{81^\eta}\right)$$

exists such that

$$\left\|\mathfrak{f}(x)-\mathcal{Q}_4(x)\right\| \le \Phi(x) \quad \text{for all } x\in\mathcal{G}. \quad (25)$$

Moreover, if

$$\lim_{\jmath\to\infty}\lim_{\eta\to\infty}\max\left\{\frac{1}{|81|^{\kappa\ell}}\xi\left(3^{\kappa\ell}x,0\right) : \jmath\le\kappa<\eta+\jmath\right\}=0, \quad (26)$$

then $\mathcal{Q}_4$ is unique quartic mapping Satisfying (25).

Proof: **Case (i)**. Let us prove the theorem for $\ell=1$. It follows by replacing $y=0$ in (24), we obtain

$$\left\|\mathfrak{f}(3x)-81\mathfrak{f}(x)\right\| \le |16|\xi(x,0) \quad \text{for all } x\in\mathcal{G}. \quad (27)$$

Replacing x by $3^\eta x$ in (27), we get

$$\left\|\mathfrak{f}\frac{(3^{\eta+1}x)}{81^{\eta+1}}-\mathfrak{f}\frac{(3^{\eta}x)}{81^{\eta}}\right\| \le \frac{|16|}{|81|^{\eta+1}}\xi(3^\eta x,0) \text{ for all } x\in\mathcal{G}. \quad (28)$$

Thus, it follows from (22) and (28) that the sequence $\left\{\frac{\mathfrak{f}(3^\eta x)}{81^\eta}\right\}$ is Cauchy sequence. Since $\mathcal{X}$ is complete. Therefore $\left\{\frac{\mathfrak{f}(3^\eta x)}{81^\eta}\right\}$ is convergent.

$$\text{Let } \mathcal{Q}_4(x) = \lim_{\eta\to\infty} \mathfrak{f}\left(\frac{3^\eta x}{81^\eta}\right) \quad \text{for all } x \in \mathcal{G}. \tag{29}$$

By induction, one can show that

$$\left\|\mathfrak{f}\frac{(3^\eta x)}{81^\eta} - \mathfrak{f}(x)\right\| \le \max\left\{\frac{|16|}{|81|^{\kappa+1}}\xi(3^\kappa x, 0) : 0 \le \kappa < \eta\right\} \tag{30}$$

by taking the limit $\eta \to \infty$ in (30) and using (23) one obtain (25).

By (22) and (24) we get

$$\begin{aligned}\|\mathcal{D}_4\mathfrak{f}(x,y)\| &= \lim_{\eta\to\infty} \left\|\mathcal{D}_4\mathfrak{f}\left(\frac{3^\eta x}{81^\eta}, \frac{3^\eta y}{81^\eta}\right)\right\| \\ &= \lim_{\eta\to\infty} \frac{1}{|81|^\eta} \left\|\mathcal{D}_4\mathfrak{f}(3^\eta x, 3^\eta y)\right\| \\ &\le \lim_{\eta\to\infty} \frac{1}{|81|^\eta}\ \xi(3^\eta x, 3^\eta y) \\ &= 0 \qquad\qquad \text{for all } x, y \in \mathcal{G}.\end{aligned}$$

Therefore $\mathcal{Q}_4(x)$ is a quartic mapping.
To prove uniqueness, let $\mathcal{Q}_4'$ be another mapping satisfying (25) we obtain

$$\begin{aligned}&\left\|\mathcal{Q}_4(x) - \mathcal{Q}_4'(x)\right\| \\ &= \lim_{\eta\to\infty} \frac{1}{|81|^\eta} \left\|\mathcal{Q}_4(3^\eta x) - \mathcal{Q}_4'(3^\eta x)\right\| \\ &\le \lim_{\eta\to\infty} \frac{1}{|81|^\eta} \max\Big\{ \left\|\mathcal{Q}_4(x) - \mathfrak{f}(3^\eta x)\right\|, \\ &\qquad\qquad \left\|\mathfrak{f}(3^\eta x) - \mathcal{Q}_4'(x)\right\| \Big\} \\ &\le \lim_{\jmath\to\infty}\lim_{\eta\to\infty} \max\left\{\frac{1}{|81|^\kappa}\xi(3^\kappa x, 0) : \jmath \le \kappa < \eta + \jmath\right\} \\ &= 0 \quad \text{for all } x \in \mathcal{G}.\end{aligned}$$

Therefore $\mathcal{Q}_4(x) = \mathcal{Q}_4'(x)$. This completes the proof.

Case (ii). Let us prove the theorem for $\ell = -1$.
It follows by replacing $y = 0$ in (24), we obtain

$$\left\|\mathfrak{f}(3x) - 81\mathfrak{f}(x)\right\| \le |16|\xi(x, 0) \qquad \text{for all } x \in \mathcal{G} \tag{31}$$

Replacing x by $\frac{x}{3^{\eta+1}}$ in (31), we get

$$\left\|81^{\eta+1}\mathfrak{f}\left(\frac{x}{3^{\eta+1}}\right) - 81^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right)\right\| \le |16||81|^\eta\xi\left(\frac{x}{3^{\eta+1}}, 0\right) \quad \text{for all } x \in \mathcal{G}. \tag{32}$$

Thus, it follows from (22) and (32) that the sequence $\left\{27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right)\right\}$ is Cauchy sequence. Since $\mathcal{X}$ is complete. Therefore $\left\{27^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right)\right\}$ is convergent.

$$\text{Let } \mathcal{Q}_4(x) = \lim_{\eta\to\infty} 81^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right) \quad \text{for all } x \in \mathcal{G}. \tag{33}$$

By induction, one can show that

$$\left\|81^\eta\mathfrak{f}\left(\frac{x}{3^\eta}\right) - \mathfrak{f}(x)\right\| \le \max\left\{|16||81|^\eta\xi\left(\frac{x}{3^{\eta+1}}, 0\right) : 0 \le \kappa < \eta\right\} \tag{34}$$

by taking the limit $\eta \to \infty$ in (30) and using (23) one obtain (25).

By (22) and (24), we get

$$\begin{aligned}\|\mathcal{D}_4\mathfrak{f}(x,y)\| &= \lim_{\eta\to\infty} \left\|\mathcal{D}_4\mathfrak{f}\left(81^\eta\frac{x}{3^\eta}, 81^\eta\frac{y}{3^\eta}\right)\right\| \\ &= \lim_{\eta\to\infty} |81|^\eta \left\|\mathcal{D}_4\mathfrak{f}\left(\frac{x}{3^\eta}, \frac{y}{3^\eta}\right)\right\| \\ &\le \lim_{\eta\to\infty} |81|^\eta\ \xi\left(\frac{x}{3^\eta}, \frac{y}{3^\eta}\right) = 0 \quad \text{for all } x, y \in \mathcal{G}.\end{aligned}$$

Therefore $\mathcal{Q}_4(x)$ is a quartic mapping.
To prove uniqueness, let $\mathcal{Q}_4'$ be another mapping satisfying (25) we obtain

$$\begin{aligned}&\left\|\mathcal{Q}_4(x) - \mathcal{Q}_4'(x)\right\| \\ &= \lim_{\eta\to\infty} |81|^\eta \left\|\mathcal{Q}_4\left(\frac{x}{3^\eta}\right) - \mathcal{Q}_4'\left(\frac{x}{3^\eta}\right)\right\| \\ &\le \lim_{\eta\to\infty} |81|^\eta \max\Big\{ \left\|\mathcal{Q}_4\left(\frac{x}{3^\eta}\right) - \mathfrak{f}\left(\frac{x}{3^\eta}\right)\right\|, \\ &\qquad\qquad \left\|\mathfrak{f}\left(\frac{x}{3^\eta}\right) - \mathcal{Q}_4'\left(\frac{x}{3^\eta}\right)\right\| \Big\} \\ &\le \lim_{\jmath\to\infty}\lim_{\eta\to\infty} \max\left\{|81|^\kappa\xi\left(\frac{x}{3^\kappa}, 0\right) : \jmath \le \kappa < \eta + \jmath\right\} \\ &= 0 \quad \text{for all } x \in \mathcal{G}.\end{aligned}$$

Therefore $\mathcal{Q}_4(x) = \mathcal{Q}_4'(x)$. This completes the proof. ■

Corollary 3.3. Let $\delta \ge 0$ and prime $p > 3$. Define a function $\mathfrak{f}$ from $\mathcal{G}$ to $\mathcal{X}$ and if $\mathfrak{f}$ is a quartic mapping that fulfills the inequality

$$\left\|\mathcal{D}_4\mathfrak{f}(x,y)\right\| \le \delta \qquad \text{for all } x, y \in \mathcal{G}.$$

Then, there exists a unique quartic function $\mathcal{Q}_4(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\left\|\mathfrak{f}(x) - \mathcal{Q}_4(x)\right\| \le \frac{|16|}{|81|}\delta$$

Proof: By Theorem 3.2, if $\xi(x,y) = \delta$ then

$$\|\mathfrak{f}(x) - \mathcal{Q}_4(x)\| \le \Phi(x),$$

$$\text{where } \Phi(x) = \lim_{\eta\to\infty} \max\left\{\frac{|16|}{|81|^{\kappa+1}}\ \xi(3^\kappa x, 0) : 0 \le \kappa < \eta\right\}.$$

Therefore,

$$\begin{aligned}\left\|\mathfrak{f}(x) - \mathcal{Q}_4(x)\right\| &\le \lim_{\eta\to\infty} \max\left\{\frac{|16|}{|81|^{\kappa+1}}\delta : 0 \le \kappa < \eta\right\}. \\ &\le \frac{|16|}{|81|}\delta.\end{aligned}$$

■

Corollary 3.4. Let $r, s, \delta > 0$ and $r + s > 4$. Define a function $\mathfrak{f}$ from $\mathcal{G}$ to $\mathcal{X}$ and if $\mathfrak{f}$ is a quartic mapping satisfying the inequality

$$\left\|\mathcal{D}_4\mathfrak{f}(x, y)\right\| \leq \delta\ (\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s)$$

for all $x, y \in \mathcal{G}$. Then, there is a unique quartic function $\mathcal{Q}_4(x) : \mathcal{G} \to \mathcal{X}$ such that

$$\|\mathfrak{f}(x) - \mathcal{Q}_4(x)\| \leq \frac{\delta\ |16|\ \|x\|^{r+s}}{|81|}.$$

Proof: Let $\xi(x, y) = \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s)$.

From Theorem (3.2),

$$\left\|\mathfrak{f}(x) - \mathcal{Q}_4(x)\right\| \leq \Phi(x) \quad \text{for all } x \in \mathcal{G}.$$

where,

$$\Phi(x) = \lim_{\eta\to\infty} \max\left\{\frac{|16|}{|81|}\frac{1}{|81|^{\kappa\ell-\left(\frac{1-\ell}{2}\right)}}\xi\left(3^{\kappa\ell-\left(\frac{1-\ell}{2}\right)}x, 0\right) : 0 \leq \kappa < \eta\right\} \text{ for all } x \in \mathcal{G}.$$

$$\Phi(x) = \lim_{\eta\to\infty} \max\left\{\frac{|16|}{|81|^{\kappa+1}}\xi\left(3^{\kappa}x, 0\right) : 0 \leq \kappa < \eta\right\} \text{ for all } x \in \mathcal{G}.$$

Taking $\ell = 1$, we obtain

$$= \lim_{\eta\to\infty} \max\left\{\frac{|16|}{|81|^{\kappa+1}}\delta|3|^{\kappa(r+s)}\|x\|^{r+s} : 0 \leq \kappa < \eta\right\}.$$

$$= \frac{\delta\ |16|\ \|x\|^{r+s}}{|81|}.$$

Therefore,

$$\|\mathfrak{f}(x) - \mathcal{Q}_4(x)\| \leq \frac{\delta\ |16|\ \|x\|^{r+s}}{|81|}.$$

■

For the case $r + s = 4$, we have the following counter example.

Example 3.2. Let $p > 3$ be a prime number and $\mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $\mathfrak{f}(x) = x^4 + 1$. Since $|3^\eta|_p = 1$ for all $\eta \in \mathcal{N}$. Then for $\delta > 0$,

$$\|\mathcal{D}_3\mathfrak{f}(x, y)\| \leq 1 \leq \delta(\|x\|^{r+s} + \|y\|^{r+s} + \|x\|^r\|y\|^s) \quad \text{for all } x, y \in \mathcal{G},$$

and

$$\left\|\mathfrak{f}\frac{(3^{\eta+1}x)}{81^{\eta+1}} - \mathfrak{f}\frac{(3^\eta x)}{81^\eta}\right\| \nrightarrow 0 \quad \text{as} \quad \eta \to \infty.$$

Hence $\left\{\frac{\mathfrak{f}(3^\eta x)}{81^\eta}\right\}$ is not a Cauchy sequence.

IV. Conclusion

Many authors discussed the HUS of Jensen type functional equation in NAN space in recent years. In this current article, we have proved a new cubic and quartic Jensen type Cauchy functional equations (1) and (2) in NAN space.

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