# Further Examination for Resolving Inventory Models through Algebraic Procedures

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*Abstract*—**We propose to study a solution approach for an inventory model regarding the use of the algebraic method to solve inventory problems. His process is more straightforward than other analytic methods based on calculus. However, there are questionable results contained in his approach. The purpose of this paper is threefold. First, we point out the problems included in his process. Second, we prepare our corrected solution approach by the algebraic method. Third, for completeness, we list the questionable results in his paper.**

*Index Terms*—**The complete squares method, Analytic process, Non-convex, Lost sales, Backorders, Inventory models**

## I. Introduction

Recently, there has been a tendency to solve inventory system problems by adopting only the algebraic method without referring to calculus so that inventory models may be present to high school students. Grubbström [1] is the first author to derive the optimal solution for the EOQ model without using calculus. Following this trend, Leung [2], Minner [3], Sphicas [4], Chang *et al.* [5], Ronald et al. [6], Cardenas-Barron [7], and Grubbström and Erdem [8] studied inventory models by the algebraic approach. The purpose of this paper is threefold. First, we review the algebraic method of Leung [2] for a deterministic inventory model with a mixture of backorders and lost sales of Montgomery et al. [9]. We provide three simple examples to illustrate that Leung [2] neglected the sign of two terms to make his approach efficient in obtaining the minimum value. Second, according to the sign of two terms, we divide the problem into three cases, and then we present our revised patchwork to solve the inventory model of Montgomery et al. [9]. Third, we afford some further discussions with Leung [2] to demonstrate that (a) he was not aware of the non-convex property of the objective function mentioned in his paper; (b) he had quoted a questionable result of Chu and Chung [10] that was already improved by Yang [11]; (c) for the perticular case where the shortage cost per backorder per year is assumed to be zero, then his replenishment policy is false.

## II. Assumptions and Notation

We use the same notation and assumptions as Montgomery et al. [9] and Leung [2] for the EOQ model with partial backorders.

$D =$ Demand per year.

$Q =$ Order quantity (units per cycle, a decision variable).

$K(U,S) =$ Average annual cost as a function of $U$ and $S$ (dollar per year, the objective function).

$V =$ Maximum inventory level, $V = Q - bS$ (units per cycle, a decision variable).

$S =$ Total demand per cycle during the stockout period (units per cycle, a decision variable).

$U =$ Total demand per cycle, $U = Q + (1-b)S$ (units per cycle, a decision variable).

$b =$ A fraction of the demand is backordered during the stockout period, while the remaining fraction $(1-b) = \bar{b}$ is lost.

$\pi_0 =$ Profit per unit (dollar per order).

$\pi =$ Fixed penalty cost per unit short (dollar per unit).

$\bar{\pi} =$ Shortage cost per backorder per year (dollar per unit per year).

$I =$ Inventory carrying cost per year, as a percentage of $C$.

$C =$ Unit cost of each item (dollar per unit).

$A =$ Fixed ordering cost per inventory cycle (dollar per order).

## III. Review of Previous Works

Motivated by Montgomery et al. [9], Leung [2] expressed the average annual cost as

$$K(U,S) = \frac{AD}{U} + \frac{IC(U-S)^2}{2U} + \frac{\bar{\pi}bS^2}{2U} + \frac{D\pi S}{U} + \frac{D\pi_0\bar{b}S}{U}, \tag{3.1}$$

Using the complete square method, Leung [2]rewrites Equation (3.1) as

$$K(U,S) = a_1U + \frac{a_2}{U} + a_3 + \frac{IC+\bar{\pi}b}{2U}\left[S - \frac{ICU - D(\pi+\pi_0\bar{b})}{IC+\bar{\pi}b}\right]^2, \tag{3.2}$$

where

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$$a_1 = \frac{IC\pi b}{2(IC + \pi b)}, \tag{3.3}$$

$$a_2 = \frac{2AD(IC + \pi b) - \left[D\left(\pi + \pi_0 \bar{b}\right)\right]^2}{2(IC + \pi b)}, \tag{3.4}$$

and

$$a_3 = \frac{ICD\left(\pi + \pi_0 \bar{b}\right)}{IC + \pi b}, \tag{3.5}$$

are three auxiliary expressions to simplify the expression, and then Leung [2] derived that

$$K(U,S) = \frac{IC + \pi b}{2U}\left[S - \frac{ICU - D\left(\pi + \pi_0 \bar{b}\right)}{IC + \pi b}\right]^2$$

$$+\left(\sqrt{a_1 U} - \sqrt{\frac{a_2}{U}}\right)^2 + 2\sqrt{a_1 a_2} + a_3 \cdot \tag{3.6}$$

Leung [2] mentioned that $K(U,S)$ attains its minimum when the two quadratic non-negative terms, depending on $U$ and $S$, are made equal to zero so that the optimal values of the two decision variables are determined by

$$U^* = \sqrt{a_2 / a_1} =$$

$$\sqrt{\frac{2AD(IC + \pi b) - \left[D\left(\pi + \pi_0 \bar{b}\right)\right]^2}{IC\pi b}}, \tag{3.7}$$

and

$$S^* = \frac{ICU^* - D\left(\pi + \pi_0 \bar{b}\right)}{IC + \pi b}$$

$$= \left(\frac{IC}{IC + \pi b}\right)U^* - \frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC + \pi b}. \tag{3.8}$$

After $U^*$ and $S^*$, Leung [2] also derived some other results for some exceptional cases that we will discuss in Section 5.

## IV. The Hidden Problems in His Solution Approach

Let us provide several familiar problems in two variables, x, and y, to highlight Leung's approach's hidden problem. First, we assume that for $x > 0$ and $y \geq 0$ to find the minimum of

$$f(x,y) = \frac{5}{x}(y-4)^2 + 3x + \frac{12}{x} + 6, \tag{4.1}$$

Following the solution procedure of Leung [2], we rewrite Equation (4.1) as

$$f(x,y) = \frac{5}{x}(y-4)^2$$

$$+\left(\sqrt{3x} - \sqrt{12/x}\right)^2 + 18, \tag{4.2}$$

Based on the expression of Equation (4.2), Leung's approach is workable to imply that

$$y^* = 4, \tag{4.3}$$

and

$$x^* = 2\,. \tag{4.4}$$

Second, we assume another example that for $x > 0$ and $y \geq 0$ to find the minimum of

$$g(x,y) = \frac{5}{x}(y-4)^2 + 3x + \frac{-7}{x} + 6, \tag{4.5}$$

then his approach will imply that

$$g(x,y) = \frac{5}{x}(y-4)^2 + 2\sqrt{3(-7)}$$

$$+\left(\sqrt{3x} - \sqrt{-7/x}\right)^2 + 6, \tag{4.6}$$

so that the optimal solutions proposed by Leung [2] will be

$$y^* = 4, \tag{4.7}$$

and

$$x^* = \sqrt{-7/3}\,. \tag{4.8}$$

Third, we assume the last example that for $x > 0$ and $y \geq 0$ to find the minimum of

$$h(x,y) = \frac{5}{x}(y+4)^2 + 3x + \frac{-7}{x} + 6, \tag{4.9}$$

then his approach will imply that

$$h(x,y) = \frac{5}{x}(y+4)^2 + 6$$

$$+\left(\sqrt{3x} - \sqrt{-7/x}\right)^2 + 2\sqrt{3(-7)}, \tag{4.10}$$

so that the optimal solutions proposed by Leung [2] will be obtained as

$$y^* = -4, \tag{4.11}$$

and

$$x^* = \sqrt{-7/3}\,. \tag{4.12}$$

The above three examples show that

$$a_2 \geq 0, \tag{4.13}$$

and

$$ICU - D\left(\pi + \pi_0 \bar{b}\right) \geq 0, \tag{4.14}$$

was implicitly assumed by Leung [2].

Moreover, inequalities in Equations (4.13) and (4.14) generally do not always hold. Therefore, Leung [2] only considered some partial conditions, ignoring more general cases. Hence, we reveal that the algebraic method proposed by Leung [2] contains questionable results.

## V. Our Improvement

We reconsider Equation (3.2) to divide the domain of $U$, from $U > 0$, into two cases: (a) $\frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC} \geq U > 0$, and (b) $U \geq \frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC}$.

For case (a), motivated by Equation (4.10), we know that $S^* = 0$ and then plug $S^* = 0$ into Equation (3.1) to obtain that

$$K\left(U, S^* = 0\right) = \frac{AD}{U} + \frac{IC}{2}U$$
$$= \left(\sqrt{AD/U} - \sqrt{IC/2 \cdot U}\right)^2 + 2\sqrt{ADIC/2}, \tag{5.1}$$

Based on Equation (5.1), we assume that

$$U^{\#} = \sqrt{2AD/IC}, \tag{5.2}$$

We know that $U^{\#}$ is the minimum solution for $K\left(U, S^* = 0\right)$ without considering the condition of (a) $\frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC} \geq U > 0$.

We compare $\frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC}$ and $U^{\#}$, then divide case (a) into two sub-cases: (a1) $\frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC} \geq U^{\#}$ and (a2) $U^{\#} > \frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC}$.

For case (a1) $\frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC} \geq U^{\#}$, it yields that $U^{\#}$ satisfying the condition of (a), so that $U^{\#}$ is indeed the minimum solution under the condition (a). We summarize our findings abstractly for later application in the following Lemma 1 and Corollary 1.

**Lemma 1.** We assume that $f(x) = ax + \frac{b}{x}$ for $c \geq x > 0$ with $a > 0$ and $b > 0$, if $\sqrt{b/a} \leq c$, then $\sqrt{b/a}$ is the minimum solution.

**Corollary 1.** We assume that $f(x) = ax + \frac{b}{x}$ for $x \geq d > 0$ with $a > 0$ and $b > 0$, if $\sqrt{b/a} \geq d$, then $\sqrt{b/a}$ is the minimum solution.

For case (a2) $U^{\#} > \frac{D\left(\pi + \pi_0 \bar{b}\right)}{IC}$, we will show that $K\left(U, S^* = 0\right)$ is a decreasing function of $U$.

For the later application, we abstractly handle this problem to assume that

$$f(x) = ax + \frac{b}{x}, \tag{5.3}$$

for $c \geq x > 0$, with $a > 0$ and $b > 0$, under the condition $\sqrt{b/a} > c$.

First, for the domain $c \geq x > 0$, we assume two points $x$ and $x + \delta$, with $c \geq x + \delta > x > 0$ where $\delta > 0$, and then our goal will prove that $f(x + \delta) > f(x)$. It shows that

$$f(x) - f(x + \delta)$$
$$= \frac{a\delta}{2x(x + \delta)}\left[\frac{b}{a} - x(x + \delta)\right], \tag{5.4}$$

Under the condition $\sqrt{b/a} > c$, and $c \geq x + \delta > x > 0$, we obtain $0 < \frac{b}{a} - x(x + \delta)$ to verify that $f(x)$ is a decreasing function. Therefore, $f(x)$ attains its minimum at $c$. We derive the next lemma.

**Lemma 2.** We assume that $f(x) = ax + \frac{b}{x}$ for

$c \geq x > 0$ with $a > 0$ and $b > 0$, if $\sqrt{b/a} > c$, then $c$ is the minimum solution.

Based on a similar argument, we know the next corollary to show the objective function is an increasing function.

**Corollary 2.** We assume that $f(x) = ax + \frac{b}{x}$ for $x \geq d > 0$ with $a > 0$ and $b > 0$, if $\sqrt{b/a} < d$, then $d$ is the minimum solution.

We summarize the results for Lemmas 1 and 2 in the next theorem.

**Theorem 1.** If $f(x) = ax + \frac{b}{x}$ for $c \geq x > 0$ with $a > 0$ and $b > 0$, then the minimum solution is $\min\left\{\sqrt{b/a}, c\right\}$.

We are applying Theorem 1 to yield the following.

**Corollary 3.** For case (a) $\frac{D(\pi + \pi_0 \bar{b})}{IC} \geq U > 0$, the objective function $K(U,S)$ attains its minimum at $S^* = 0$, and $U^* = \min\left\{\sqrt{\frac{2AD}{IC}}, \frac{D(\pi + \pi_0 \bar{b})}{IC}\right\}$.

Next, we consider the case (b) $U \geq \frac{D(\pi + \pi_0 \bar{b})}{IC}$. It yields that $\frac{ICU - D(\pi + \pi_0 \bar{b})}{IC + \bar{\pi} b}$ is a non-negative number such that we take

$$S^*(U) = \frac{ICU - D(\pi + \pi_0 \bar{b})}{IC + \bar{\pi} b}, \tag{5.5}$$

to clearly indicate that $S^*(U)$ is a function of $U$. We simplify the objective function

$$K(U, S^*(U)) = a_1 U + \frac{a_2}{U} + a_3, \tag{5.6}$$

with

$$a_1 = \frac{IC\bar{\pi} b}{2(IC + \bar{\pi} b)}, \tag{5.7}$$

$$a_2 = \frac{2AD(IC + \bar{\pi} b) - \left[D(\pi + \pi_0 \bar{b})\right]^2}{2(IC + \bar{\pi} b)}, \tag{5.8}$$

and

$$a_3 = \frac{ICD(\pi + \pi_0 \bar{b})}{IC + \bar{\pi} b}, \tag{5.9}$$

are three auxiliary expressions to simplify the results. It follows that $a_1 > 0$, and $a_3 > 0$. Owing to the sign of $a_2$, we divide case (b) into two sub-cases: (b1) $a_2 > 0$, and (b2) $a_2 \leq 0$.

We summarize the results for Corollary 1 and Corollary 2 in the next theorem.

**Theorem 2.** If $f(x) = ax + \frac{b}{x}$ for $x \geq d > 0$ with $a > 0$ and $b > 0$, then the minimum solution is $\max\left\{\sqrt{b/a}, d\right\}$.

We are applying Theorem 2 to yield the following.

**Corollary 4.** For case (b1), $a_2 > 0$, the objective function $K(U, S^*(U))$ attains its minimum at $S = S^*(U^*)$, and $U^* = \max\left\{\sqrt{\frac{a_2}{a_1}}, \frac{D(\pi + \pi_0 \bar{b})}{IC}\right\}$.

Next, we consider the case (b2) with $a_2 \leq 0$. We assume two points, $U$, and $U + \delta$, with $\delta > 0$ and a relationship $U + \delta > U \geq \frac{D(\pi + \pi_0 \bar{b})}{IC}$ and then we compute

$$\begin{aligned} &K(U + \delta, S^*(U)) - K(U, S^*(U)) \\ &= a_1 \delta + \frac{-a_2 \delta}{U(U + \delta)} > 0. \end{aligned} \tag{5.10}$$

Based on Equation (5.10), $K(U, S^*(U))$ is an increasing function to attain its minimum at the boundary point $\frac{D(\pi + \pi_0 \bar{b})}{IC}$ for case (b2).

For convenience, we summarize our findings in the above table 1.

Table 1. Summary of our results

| cases | Domain of $U$ | $a_2$ | $U^*$ | $S^*$ |
|---|---|---|---|---|
| (a) | $\frac{D(\pi+\pi_0\bar{b})}{IC}\ge U>0$ | | $\min\left\{\sqrt{\frac{2AD}{IC}},\frac{D(\pi+\pi_0\bar{b})}{IC}\right\}$ | 0 |
| (b1) | $U\ge\frac{D(\pi+\pi_0\bar{b})}{IC}$ | $a_2>0$ | $\max\left\{\sqrt{\frac{a_2}{a_1}},\frac{D(\pi+\pi_0\bar{b})}{IC}\right\}$ | $S^*(U^*)=\frac{ICU^*-D(\pi+\pi_0\bar{b})}{IC+\bar{\pi}b}$ |
| (b2) | | $a_2\le 0$ | $\frac{D(\pi+\pi_0\bar{b})}{IC}$ | |

## VI. A List of Questionable Results in the Previous Paper

Leung [2] only derived one pair minimum solution for $U^*$ and $S^*$ in Equations (3.7) and (3.8), respectively. In his paper, he mentioned "The non-convex cost function." He realized that $K(U,S)$ is not a convex function, but still derived only one minimum solution. Let us recall Table 1: when $\sqrt{\frac{2AD}{IC}}<\frac{D(\pi+\pi_0\bar{b})}{IC}$ and $\sqrt{\frac{a_2}{a_1}}>\frac{D(\pi+\pi_0\bar{b})}{IC}$ both happen, there is a local minimum at $\sqrt{2AD/IC}$ for the domain $\frac{D(\pi+\pi_0\bar{b})}{IC}\ge U>0$, and there is another local minimum at $\sqrt{a_2/a_1}$ for the domain $U\ge\frac{D(\pi+\pi_0\bar{b})}{IC}$. It points out that the non-convex property of $K(U,S)$.

Leung [2] also discussed the comparison between $\sqrt{2ADIC}$ and $D(\pi+\pi_0)$ when $\sqrt{2ADIC}>D(\pi+\pi_0)$. Leung [2] claimed that based on Chu and Chung [10], pages 290-291, "no inventory system exists because this condition implies that the incurring the lost sales all the time, $D(\pi+\pi_0)$ is less than operating an EOQ system with cost $\sqrt{2ADIC}$." However, Yang ([11], pages 867-868) already pointed out that the statement in Chu and Chung [10] is false and then revised them into three cases. Consequently, his discussions for the condition $\sqrt{2ADIC}>D(\pi+\pi_0)$ cannot be supported by Chu and Chung [10].

We want to mention another questionable result in Leung [2]. For a particular case of $\sqrt{2ADIC}>D(\pi+\pi_0)$ and $\bar{\pi}=0$, he derived that $U^*=\infty$ and $S^*=\infty$ such that his optimum strategy is to partially backorder everything. However, $U^*=\infty$ and $S^*=\infty$ are mathematically acceptable solutions but $U^*=\infty$ and $S^*=\infty$ are not feasible solutions in the real world. If $U^*=\infty$, then the cost to buy the product will be beyond the limit of the business budget. We may advise researchers referring to Yang ([11], page 867) to buy the total demand, $U$, as large as capital allowed, and then to extend the shortage period as long as possible, with $S=U-\frac{D(\pi+\pi_0\bar{b})}{IC}$. It points out that the inventory carrying period always exists with time length $(\pi+\pi_0\bar{b})/IC$. Therefore, his procedure of partial backorder everything is false.

## VII. A Related Problem

We study the paper of Hua et al. [12] to examine their arrangement of locations for car sensors under a network consideration. In the past, many researchers aimed to construct such plans, setting the traffic information of connected transit models on all paths within a network by personal on-site surveys or sophisticated transit sensor approaches. However, in the natural environment, this assumption of having sensors installed on all links is often not applicable owing to budgetary considerations. This limitation highlights the motivation to predict transit currents on all paths of a transit network according to the estimation of path currents relying on only a subfamily of paths equipped with proper detective sensors. Our paper pays attention to discussing the arrangement of locations for the least amount of sensors to predict all transit information within the network. To achieve this goal, Hua et al. [12] applied an algebraic method to find the minimum base for the transit matrices that contained the maximum amount of transit information.

It is supposed that there are $C_1$, $C_2,\ldots,C_n$ column vectors, where the weight of $C_i$ is denoted as $w_i$ for $i=1,\ldots,n$. We predict that the assumption should be added as non-zero column vectors. In Hua et al. [12], they assumed that let $\Omega$ be the collection of all independent subsets of $\{C_i:i=1,2,\ldots,n\}$, and then their goal is to find

$$\max\left\{\sum_{w_i\in A}w_i : A\in\Omega\right\}. \tag{7.1}$$

The method in Hua et al. [12] is to rearrange the order of $\{C_i:i=1,2,\ldots,n\}$ relying on their weights such that

$$w_{f(1)}\ge w_{f(2)}\ge\cdots\ge w_{f(n)}. \tag{7.2}$$

Hua et al. [12] considered that (i) $\{C_{f(j)}:j=1\}$, (ii) $\{C_{f(j)}:j=1,2\}$, ..., $\{C_{f(j)}:j=1,2,\ldots n\}$ in this order to select an independent subset to derive a subset as $\{C_{g(k)}:k=1,\ldots,m\}$ then your total weight for this independent sub-family is derived as

$$\sum_{k=1}^{m} w_{g(k)}. \tag{7.3}$$

The goal of Hua et al. [12] is to prove that

$$\sum_{k=1}^{m} w_{g(k)} = \max\left\{ \sum_{w_i \in A} w_i : A \in \Omega \right\}. \tag{7.4}$$

In Hua et al. [12], they rearranged the column vectors according to their weight from the largest to the smallest. If their weight values are all different, then this procedure can solve the problem. However, we assume that

$$\mathrm{w}_1 = \mathrm{w}_2 > \mathrm{w}_3 = \mathrm{w}_4, \tag{7.5}$$

and then there are four possible sequences constructed by the method proposed by Hua et al. [12]:

$$\mathrm{w}_1 \geq \mathrm{w}_2 > \mathrm{w}_3 \geq \mathrm{w}_4, \tag{7.6}$$

$$\mathrm{w}_1 \geq \mathrm{w}_2 > \mathrm{w}_4 \geq \mathrm{w}_3, \tag{7.7}$$

$$\mathrm{w}_2 \geq \mathrm{w}_1 > \mathrm{w}_3 \geq \mathrm{w}_4, \tag{7.8}$$

and

$$\mathrm{w}_2 \geq \mathrm{w}_1 > \mathrm{w}_4 \geq \mathrm{w}_3. \tag{7.9}$$

Based on the abovementioned four sequences, they may imply different subfamilies of the maximum independent set. Hua et al. [12] did not prove that different sequences will result in the same optimal value. We will provide a new solution approach to provide a patchwork for the incomplete solution procedure proposed by Hua et al. [12].

## VIII. Our Improvement

We can rearrange $\{w_i : i = 1,...,n\}$ in a finite community such that in each community, $w_i$ has the same value that is, if there are $h$ communities to partition $\{w_i : i = 1,...,n\}$ as

$$\{w_i : i = 1,...,n\} = \bigcup_{i=1}^{h} \Delta_i, \tag{8.1}$$

such that if $w_\alpha$ and $w_\beta$ in the same community, then $w_\alpha = w_\beta$, if $w_a \in \Delta_i$ and $w_b \in \Delta_j$ then $w_a > w_b$ if and only if $i < j$ such that we express $\{w_i : i = 1,...,n\}$ as

$$w_a = w_b = \cdots (\in \Delta_1) > w_c = w_d = \cdots (\in \Delta_2)$$

$$> \cdots > w_y = w_z = \cdots (\in \Delta_h). \tag{8.2}$$

Hence, we define the corresponding $C_i$ as follows,

$$\{C_i : i = 1,...,n\} = \bigcup_{i=1}^{h} \Phi_i. \tag{8.3}$$

We will prove by the Principle of Finite Induction on the rank of $\{C_i : i = 1,2,...,n\}$.

We assume the rank of $\{C_i : i = 1,2,...,n\}$ is $m$.

We assume that $m = 1$ then any $C_j$ is a base for $\{C_i : i = 1,...,n\}$. By the approach of Hua et al. [12], $C_{f(1)} \in \Phi_1$ and then

$$w_{f(1)} = \max\{w_i : i = 1,...,n\} \geq w_j, \tag{8.4}$$

for any $C_j$ being a base for $\{C_i : i = 1,...,n\}$ to prove that based on our proposed approach we can attain the maximum. We assume that our approach is valid for $m = 1,2,...,k$, and then we assume that the rank of $\{w_i : i = 1,...,n\}$ is $k+1$.

For the subspace generated by $\{C_a, C_b, \cdots\} = \Phi_1$, we assume that the rank of $\Phi_1$ is $k_0$ with $k_0 \leq k+1$.

By our approach, there is a subset consisting of $k_0$ independent vectors that is a base for the subspace generated by $\Phi_1$.

We assume that there is a base, denoted by Π that is selected from $\bigcup_{i=1}^{h} \Phi_i$. For any other selection of base, denoted as Y, for $\{C_i : i = 1,...,n\}$, we may denote the base as $\overline{\Phi_1}$ union other vectors in $\bigcup_{i=2}^{h} \Phi_i$, with $\overline{\Phi_1} \subseteq \Phi_1$.

We denote the cardinal of $\overline{\Phi_1}$ as $\#(\overline{\Phi_1})$. Owing to $\#(\overline{\Phi_1})$ must less than the rank of $\Phi_1$ to imply the $\#(\overline{\Phi_1}) \leq k_0$. We will divide into two cases: Case (a) $\#(\overline{\Phi_1}) = k_0$ and Case (b) $\#(\overline{\Phi_1}) < k_0$.

For case (a), we derive that

$$\sum_{w_i \in \mathrm{Y}} w_i = k_0\ w_a + \sum_{w_i \in \mathrm{Y} - \Phi_1} w_i. \tag{8.5}$$

The cardinal of "$\mathrm{Y} - \Phi_1$" must be less than or equal to k so we can apply the result that our approach is valid for $m = 1,2,...,k$ to derive that

$$\sum_{w_i \in \Pi - \Phi_1} w_i = \max\left\{ \sum_{w_i \in B} w_i : B \in \Omega_1 \right\} \geq \sum_{w_i \in Y - \Phi_1} w_i, \tag{8.6}$$

where $\Omega_1$ be the collection of all independent subsets of $\{C_i : i = 1,2,...,n\} - \Phi_1$.

Hence, we show that

$$\sum_{w_i \in \Pi} w_i = \sum_{w_i \in \Pi - \Phi_1} w_i + k_0\ w_a$$

$$\geq \sum_{w_i \in Y} w_i = \sum_{w_i \in Y-\Phi_1} w_i + k_0 \; w_a . \tag{8.7}$$

For case (b), we derive that

$$\sum_{w_i \in Y} w_i = \#\left(\overline{\Delta}_1\right) w_a + \sum_{w_i \in Y-\Phi_1} w_i , \tag{8.8}$$

with $\#\left(\overline{\Delta}_1\right) < k_0$.

Owing to those changing base theorem, we know that there are $k_0 - \#\left(\overline{\Phi}_1\right)$ independent vectors in $\Phi_1$ that those independent vectors can be selected to join Y to form a new base, denoted as H, when there are $k_0 - \#\left(\overline{\Delta}_1\right)$ independent vectors in $\bigcup_{i=2}^{h} \Phi_i$ which are kicked off from the base, Y.

$$\sum_{w_i \in H} w_i = k_0 \; w_a + \sum_{w_i \in H-\Phi_1} w_i \geq$$
$$\sum_{w_i \in Y} w_i = \#\left(\overline{\Delta}_1\right) w_a + \sum_{w_i \in Y-\Phi_1} w_i , \tag{8.9}$$

Owing to those newly joined vectors with weights $w_a \in \Delta_1$ that are greater than those weights in $\bigcup_{j=2}^{h} \Delta_j$ .

Our method presented in this study proves to be valuable for path-based applications in transit networks and long-term planning. Strategically selecting sensor locations on a limited number of links allows for cost-effective yet accurate flow estimation on the entire network, under budget constraints.

## IX. A Second Approach

Based on the values of $\{w_k: k = 1,2, \ldots, n\}$, we may rename the index system such that

$$\{w_k: k = 1,2, \ldots, n\} = \{t_k: k = 1,2, \ldots, n\}, \tag{9.1}$$

under our new index system that $t_i \geq t_j$ if and only if $i < j$.

Our new approach is to slightly modification of $t_k$, for $k = 1,2, \ldots, n$, from $t_k$ to $t_k + \varepsilon_k$ such that

$$t_k + \varepsilon_k > t_i + \varepsilon_i, \tag{9.2}$$

if and only if $i < k$.

We assume the family of all non-empty subsets of $\{t_k: k = 1,2, \ldots, n\}$ as $\nabla$. If the cardinal number of $\{t_k: k = 1,2, \ldots, n\}$ is n, then the cardinal number of $\nabla$ is $2^n - 1$, because we only consider non-empty subsets. We express the total weight, denoted as $f(K)$, of an element, denote as K, in $\nabla$ as follows,

$$f(K) = \textstyle\sum_{t_j \in K} t_j. \tag{9.3}$$

We define a new number as follows,

$$\delta = \min\{|f(K) - f(J)|: K, J \in \nabla, f(K) \neq f(J)\}. \tag{9.4}$$

Owing to $\nabla$ is a finite, we imply that $\{f(K): K \in \nabla\}$ is also a finite set, such that

$$\{|f(K) - f(J)|: K, J \in \nabla, f(K) \neq f(J)\}, \tag{9.5}$$

is also finite. The minimum value of a finite family of positive numbers is a positive number. Hence, we show that

$$\delta > 0. \tag{9.6}$$

For convenience, we assume a new expression, denoted as $\varepsilon$, where

$$\varepsilon = \delta / 2n^2. \tag{9.7}$$

Based on $\{t_k: k = 1,2, \ldots, n\}$, we construct a new sequence

$$\{t_k + (n - k + 1)\varepsilon: k = 1,2, \ldots, n\}. \tag{9.8}$$

We know that

$$t_j \geq t_{j+1}, \tag{9.9}$$

for $j = 1,2, \ldots, n - 1$. Consequently, we derive that

$$t_j + (n - j + 1)\varepsilon > t_{j+1} + (n - j)\varepsilon. \tag{9.10}$$

Based on Equation (9.10), we construct a new sequence of weights that are all distinct.

Using our new sequence of Equation (9.8), we can apply the solution procedure proposed by Hua et al. [12] to find a base, denoted as B, that attains the maximum value.

In the following, we will prove that

$$\left\{\textstyle\sum t_j: t_j + (n - j + 1)\varepsilon \text{ in } B\right\}, \tag{9.11}$$

attains the maximum of Equation (7.4).

By way of contradiction, we assume that there is another base, denoted as C, such that

$$\left\{\textstyle\sum t_i: t_i \text{ in } C\right\} > \left\{\textstyle\sum t_j: t_j + (n - j + 1)\varepsilon \text{ in } B\right\}. \tag{9.12}$$

Owing to B attains the maximum value of Equation (7.4), we derive that

$$\left\{\textstyle\sum t_j + (n - j + 1)\varepsilon: t_j + (n - j + 1)\varepsilon \text{ in } B\right\} \geq,$$
$$\left\{\textstyle\sum t_i + (n - i + 1)\varepsilon: t_i \text{ in } C\right\}. \tag{9.13}$$

We show that

$$\left\{\textstyle\sum t_j + n\varepsilon: t_j + (n - j + 1)\varepsilon \text{ in } B\right\} > \left\{\textstyle\sum t_i: t_i \text{ in } C\right\}. \tag{9.14}$$

Based on Equation (9.14), we derive that

$$n^2\varepsilon \geq \left\{n\varepsilon: t_j + (n - j + 1)\varepsilon \text{ in } B\right\}$$
$$\geq \left\{\textstyle\sum t_i: t_i \text{ in } C\right\} - \left\{\textstyle\sum t_j: t_j + (n - j + 1)\varepsilon \text{ in } B\right\}, \tag{9.15}$$

where from Equation (9.12), the right-hand side of Equation (9.15) is a positive number.

Recalling Equation (9.15), we obtain that

$$n^2\varepsilon \geq \delta. \tag{9.16}$$

The finding of Equation (9.16) is violated to Equation (9.7), and then we prove that the assertion of Equation (9.12) is false to finish our proof.

Based on our above discussion, we provide a patchwork for Hua et al. [12].

## X. Application of Our Discussion

In this section, we will show that our previous discussion can be apply to other inventory models to indicate its effectiveness. In the following, we begin to review Li et al. [13] that developed an inventory system with a retailer and a manufacturer such that the retailer decided the retail price to customers and the manufacturer decided the quality of the product item and the wholesale price.

## XI. Application of Our Discussion

In the following, we provide notation for the inventory system proposed by Li et al. [13].

$r$ is the cost coefficient with respect to quality.

$w$ is the wholesale price of one unit for the produced item, a decision variable.
$x$ is the quality level of the produced item, a decision variable.
$p$ is the retail price for the produced item.
$\alpha$ is sensitivity of the demand for the produced item with the quality.
$b$ is sensitivity of the demand to price of the retailer.
$c$ is the production cost of one unit item for the manufacturer.
$a$ is the expected value of stochastic market base for the retailer, with $0 < c \le a$.
$\sigma^2$ is the variance of the stochastic market base for the retailer.
$\lambda_r$ is the risk aversion coefficient of the retailer.
$\lambda_m$ is the risk aversion coefficient of the manufacturer.
$U_r(\Pi_r)(p)$ is the profit of a retailer.
$U_m(\Pi_m)(w,x,p)$ is the profit of a manufacturer.

## XII. Our Algebraic Approach

In this section, we demonstrate that we can apply algebraic method to solve the maximum problem studied by Li et al. [13] that tried to find the maximum solution for utility function of a retailer and a manufacturer,

$$U_r(\Pi_r)(p) = (p-w)(a-bp+\alpha x) - \frac{\lambda_r (p-w)^2 \sigma^2}{2}, \qquad (12.1)$$

and

$$U_m(\Pi_m)(w,x,p) = (w-c)(a-bp+\alpha x) - \frac{r}{2}x^2 - \frac{\lambda_m}{2}(w-c)^2\sigma^2. \qquad (12.2)$$

We will use algebraic method to find the optimal solution. We rewrite Equation (12.1) in the descending order of $p$ to imply that

$$U_r(\Pi_r)(p) = \frac{2b+\lambda_r\sigma^2}{-2}p^2 + \left(a+bw+\alpha x+\lambda_r\sigma^2 w\right)p - \left(aw+\alpha wx+\frac{\lambda_r}{2}\sigma^2 w^2\right), \qquad (12.3)$$

and then we complete the square for Equation (12.3) to derive that

$$U_r(\Pi_r)(p) = \frac{2b+\lambda_r\sigma^2}{-2}\left[p-\frac{a+bw+\alpha x+\lambda_r\sigma^2 w}{2b+\lambda_r\sigma^2}\right]^2 + \frac{\left(a+bw+\alpha x+\lambda_r\sigma^2 w\right)^2}{2\left(2b+\lambda_r\sigma^2\right)} - \left(aw+\alpha wx+\frac{\lambda_r}{2}\sigma^2 w^2\right). \qquad (12.4)$$

Owing to the coefficient of $\left[p-\frac{a+bw+\alpha x+\lambda_r\sigma^2 w}{2b+\lambda_r\sigma^2}\right]^2$ is denoted as $\frac{2b+\lambda_r\sigma^2}{-2}$ that is a negative number, to obtain the maximum value for the utility function of the retailer, we should take that

$$p = \frac{a+bw+\alpha x+\lambda_r\sigma^2 w}{2b+\lambda_r\sigma^2}. \qquad (12.5)$$

We plug Equation (12.5) into Equation (12.2) and to simplify the expression with a expression,

$$y = w-c, \qquad (12.6)$$

in the descending order of $x$ to yield that

$$U_m(\Pi_m)(y,x) = -\frac{r}{2}x^2 + \frac{\left(b+\lambda_r\sigma^2\right)\alpha y}{2b+\lambda_r\sigma^2}x - \frac{by}{2b+\lambda_r\sigma^2}\left[a+\left(b+\lambda_r\sigma^2\right)(y+c)\right] + ay - \frac{\lambda_m}{2}\sigma^2 y^2. \qquad (12.7)$$

We complete the square for $x$ in Equation (12.7) to find that

$$U_m(\Pi_m)(y,x) = -\frac{r}{2}\left[x-\frac{\left(b+\lambda_r\sigma^2\right)\alpha y}{r\left(2b+\lambda_r\sigma^2\right)}\right]^2 + \frac{\left[\left(b+\lambda_r\sigma^2\right)\alpha y\right]^2}{2r\left(2b+\lambda_r\sigma^2\right)^2} + ay - \frac{\lambda_m}{2}\sigma^2 y^2 - \frac{by}{2b+\lambda_r\sigma^2}\left[a+\left(b+\lambda_r\sigma^2\right)(y+c)\right]. \qquad (12.8)$$

Because the coefficient of $\left[x-\frac{\left(b+\lambda_r\sigma^2\right)\alpha y}{r\left(2b+\lambda_r\sigma^2\right)}\right]^2$ is denoted as $\frac{-r}{2}<0$, we know that the maximum value will occur when

$$x = \frac{\left(b+\lambda_r\sigma^2\right)\alpha}{r\left(2b+\lambda_r\sigma^2\right)}y. \qquad (12.9)$$

We plug the findings of Equation (12.9) into Equation (12.8) to obtain that

$$U_m(\Pi_m)(y) = \frac{\left(b+\lambda_r\sigma^2\right)^2\alpha^2 y^2}{2r\left(2b+\lambda_r\sigma^2\right)^2} + ay - \frac{\lambda_m}{2}\sigma^2 y^2 - \frac{by\left[a+\left(b+\lambda_r\sigma^2\right)(y+c)\right]}{2b+\lambda_r\sigma^2}. \qquad (12.10)$$

We rewrite Equation (12.10) in the descending order of $y$, and to simplify the expression, we assume that

$$A_3 = r\lambda_m\sigma^2\left(2b+\lambda_r\sigma^2\right)^2 - \alpha^2\left(b+\lambda_r\sigma^2\right)^2 + 2br\left(b+\lambda_r\sigma^2\right)\left(2b+\lambda_r\sigma^2\right), \qquad (12.11)$$

$$A_4 = 2r\left(2b+\lambda_r\sigma^2\right)^2, \qquad (12.12)$$

$$A_2 = A_3/A_4, \qquad (12.13)$$

and

$$A_1 = (a - bc)(b + \lambda_r \sigma^2)/(2b + \lambda_r \sigma^2), \quad (12.14)$$

to yield

$$U_m(\Pi_m)(y) = -A_2 y^2 + A_1 y. \quad (12.15)$$

Based on Equation (12.15), we rewrite Equation (12.15) as

$$U_m(\Pi_m)(y) = -A_2\left(y - \frac{A_1}{2A_2}\right)^2 + \frac{A_1^2}{4A_4} \quad (12.16)$$

to derive that

$$w - c = y = \frac{A_1}{2A_2}. \quad (12.17)$$

Li et al. [13] did not check whether or not $A_3 > 0$ and $a > bc$ to guarantee the well-defined of the solutions of (12.5), (12.9), and (12.17).

In the numerical example of Li et al. [13], we recall that $a = 500$, $b = 10$ and $c = 3$ such that

$$a - bc > 0, \quad (12.18)$$

is supported by the numerical example mentioned in Li et al. [13].

In the numerical example of Li et al. [13], we recall that $r = 100$, $\sigma = 60$, $b = 10$ to imply that

$$A_3 = 360000\lambda_m(20 + 3600\lambda_r)^2 - 64(10 + 3600\lambda_r)^2 + 2000(10 + 3600\lambda_r)(20 + 3600\lambda_r). \quad (12.19)$$

When $\lambda_m = 0.02$, to run a sensitivity analysis on $\lambda_r$, Li et al. [13] assumed the value of $\lambda_r$ is $0 \le \lambda_r \le 1$.

Owing to

$$360000\lambda_m > 64, \quad (12.20)$$

and

$$(20 + 3600\lambda_r)^2 > (10 + 3600\lambda_r)^2, \quad (12.21)$$

we derive that

$$A_3 > 0, \quad (12.22)$$

is supported by the numerical example.

On the other hand, $\lambda_r = 0.02$, to run a sensitivity analysis on $\lambda_m$, Li et al. [13] assumed the value of $\lambda_m$ is $0 \le \lambda_m \le 1$.

We observe that

$$2000 > 64, \quad (12.23)$$

and

$$20 + 3600\lambda_r > 10 + 3600\lambda_r, \quad (12.24)$$

we still obtain that

$$A_3 > 0, \quad (12.25)$$

is supported by the numerical example mentioned in Li et al. [13].

## XIII. Direction of Future Research

In this section, we begin to review some important papers that were recently published. Sulistiawanti et al. [14] used multivariate exponentially weighted moving variance charts and multivariate exponentially weighted moving averages to deal with monitoring water quality under residual XGBoost regression. Using naive Bayes classifier and Bayesian logistic regression, Yanuar et al. [15] classified death risk for those COVID-19 patients. Cheng and Chen [16] studied contradictory pairwise comparison matrices in the analytic hierarchy process to show that those improvements contained questionable findings. According to an improved Salp swarm algorithm, Long et al. [17] considered the optimal allocation of DGs in radial distribution networks. Wu [18] constructed a new inventory model under fuzzy restriction and fuzzy demand to derive a formulated optimal solution. Wang et al. [19] considered the consistency test in the analytic hierarchy process to point out several questionable results in previously published papers. For the flooded passenger vehicles, Al-Qadami et al. [20] examined a 3-dimensional numerical study on the critical orientation. Aripin et al. [21] amend compound emotional text classification performance in the multichannel convolutional neural network model. Chen and Cheng [22] developed a simple algorithm to evaluate the order time duration for inventory models with a linear demand. Referring to COVID-19 cases, Novianti et al. [23] developed geographically weighted logistic regression models with spatial binomial data to examine the weather non-stationarity. Prasetyo et al. [24] examined the fuel subsidies in Indonesia using clustering large applications, partitioning around medoids, and K-Means. Wang and Chiang [25] examined the ordered weighted averaging operator to provide a further study. According to our above referring, we can assume that practitioners will locate several hot research topics to help them obtain their research directions.

## XIV. Conclusion

Leung [2] considered the inventory model of Montgomery et al. [9] using algebraic methods to obtain the minimum solution, so that inventory systems may introduced to high school students without the knowledge of calculus. However, his approach overlooked the sign of two terms, so it obtained questionable results. Our improvement provides a sound patchwork for his shortcomings.

On the other hand, we also provide a new solution procedure to solve the optimal problem of Hua et al. [12].

At last, not least, we apply our algebraic method to deal with a supply chain problem with one manufacturer and one retailer. We point out that there are two necessary condition that must be hold to guarantee the well-defined of the optimal solution. Referring to numerical examples in Li et al. [13], we show that both conditions are valid to provide a sound foundation for our algebraic process.

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