Resonance Analysis for Weakly Nonlinear Duffing-van der Pol Oscillation

Songlin Chen, Nannan Wang

***Abstract*—The resonance phenomena of a weakly nonlinear, damped, Duffing-van der Pol oscillation is studied analytically and numerically. The methods of multiple scales is used to obtain uniformly valid asymptotic approximate solutions of the governing equation for various cases of primary harmonic resonance, super-harmonic resonance and sub-harmonic resonance respectively. The study shows that the steady amplitudes in the solutions of the nonlinear equation demonstrate the nonlinear phenomena involving jump and bistability at some bifurcation points. The quantitative relations of Frequency-Amplitude involving the parameters of damping, nonlinear, external force in the oscillator are obtained. The asymptotic approximation and numerical solutions are in vertically perfect agreement for all the cases considered. The results enrich previous researches just for Duffing or van der Pol oscillation respectively.**

***Index Terms*—resonance analysis, multiple scales, amplitude-frequency relations, bi-stability**

I. Introduction

THE Duffing oscillations or van der Pol oscillations are studied in many literatures [Holmes, 2013; Benney & Newell, 1967; Yang et al, 1998; Han & Bi, 2011, Macouo & Woafo, 2017; Pan & Chen, 2020] . The problem to be considered here is a damped weakly nonlinear Duffing-van der Pol oscillation that is forced at frequencies near resonance. The concrete problem that will be investigated is

$$u''+u+\varepsilon\mu(u^2-1)u'+\varepsilon\alpha u^3=\varepsilon F\cos\omega t \quad (1)$$

$$u(0)=0, u'(0)=0 \quad (2)$$

Wherein $0<\varepsilon\ll 1$ is the small parameter that describes the weak non-linearity and the small external forcing. $\mu,\alpha>0$ describe the damping term and non-linearity respectively, F contributes to the amplitude of external forcing, ω denotes the frequency of the external forcing.

Because of small external forcing and zero initial conditions, it is natural to expect that the solution to be small as usually, i.e. the amplitude of the solution is proportional to the amplitude of the external forcing, but this supposition is not true when the driving frequency ω takes some particular values. It will be seen that the amplitude of the solution may be the order of magnitude larger than the external forcing. To investigate these phenomena, we'll apply the method of multiple-scale [Holmes, 2013; Nayfeh, 2004; Johnson, 2005; Shen et al, 2008; Wang & Zhang, 2019 ; Ramos, 2007; Ghaleb et al, 2021, Ginoux, et al, 2022]to find the uniformly valid asymptotic approximation of the solution near the resonance frequencies and do some nonlinear analysis to it. The results enriches existing researches just for Duffing or van der Pol oscillation respectively.

The response of a weakly nonlinear, single-degree-of-freedom with near resonance external forces (1), (2) is studied analytically and numerically, the research shows that the peak amplitude in the solutions of the nonlinear equations can be several times those in the solutions of the reduced linearized equations. The obtained asymptotic approximations and numerical solutions of the cases considered here are in virtually perfect agreement, but differ markedly to the exact solution of the reduced linearized problem.

The rest of the paper is organized as follows. First, the multiple time scale version of the original problem is obtained. In section 2, the asymptotic analysis for the case of primary-harmonic resonances is underway, next, in section 3, the asymptotic behavior for the case of Super-harmonic and sub-harmonic resonances is investigated, finally, section 4 concludes the paper and gives some remarks.

Take the two scales

$$t_1=t, t_2=\varepsilon t \quad (3)$$

And consider the case of primary resonance, we assume the solution $u(t,\varepsilon)$ possess the asymptotic expansion

$$u(t,\varepsilon)\sim u_0(t_1,t_2)+\varepsilon u_1(t_1,t_2)+\cdots \quad (4)$$

then equation (1) becomes

$$\begin{aligned}&(\partial_{t_1}{}^2+2\varepsilon\partial_{t_1}\partial_{t_2}+\varepsilon^2\partial_{t_2}{}^2)u+u+\varepsilon\alpha u^3\\&\quad+\varepsilon\mu(u^2-1)(\partial_{t_1}+\varepsilon\partial_{t_2})u=\varepsilon F\cos\omega t_1\end{aligned} \quad (5)$$

Substituting (4) into (5) and making an effort to include only the terms that might contribute to the first two terms of the expansion, we have that

$$\begin{aligned}&\partial_{t_1}{}^2u_0+u_0+\varepsilon(\partial_{t_1}{}^2u_1+u_1+2\varepsilon\partial_{t_1}\partial_{t_2}u_0+\alpha u_0{}^3\\&\quad+\varepsilon\mu(u_0{}^2-1)\partial_{t_1}u_0)+\cdots=\varepsilon F\cos\omega t_1\end{aligned}$$

With this and the initial values (2), the following problem is resulted in

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$$O(1):\begin{cases}\partial_{t_1}{}^2 u_0 + u_0 = 0\\ u_0(0,0) = \partial_{t_1} u_0(0,0) = 0\end{cases}$$

The general solution of this problem is

$$u_0(t_1,t_2) = A(t_2)\cos(t_1+\theta(t_2)) \quad (6)$$

With

$$A(0) = 0, \theta(0) = -\frac{\pi}{2} \quad (7)$$

Similar to the order of $O(\varepsilon)$, we have

$$\begin{aligned}\partial_{t_1}{}^2 u_1 + u_1 &= F\cos\omega t_1 - 2\partial_{t_1}\partial_{t_2}u_0\\ &\quad -\alpha u_0{}^3 - \mu(u_0{}^2-1)\partial_{t_1}u_0\\ &= F\cos\omega t_1\\ &\quad +2(A'\sin(t_1+\theta)+A\theta'\cos(t_1+\theta))\\ &\quad +\mu(A^2\cos^2(t_1+\theta)-1)A\sin(t_1+\theta)\\ &\quad -\alpha A^3\cos^3(t_1+\theta)\end{aligned}$$

II. ASYMPTOTIC ANALYSIS FOR THE CASE OF PRIMARY-HARMONIC RESONANCES

Substituting $u_0(t_1,t_2)$ in (6) into the equation involving $u_1(t_1,t_2)$, we find that the choice of $A(t_2)$ and $\theta(t_2)$ can be made to demand the obtained equation possessing bounded solution, essentially nearby the resonance frequencies, as follows.

Considering the case of primary resonance for the external force, we set $\omega = 1+\varepsilon\sigma$ to describe quantitatively the nearness of ω to 1, the frequency of the linearized problem, by introducing the detuning parameter σ. Thus we have

$$\begin{aligned}\partial_{t_1}{}^2 u_1 + u_1 &= (2A'-\mu A+\frac{1}{4}\mu A^3)\sin(t_1+\theta)\\ &\quad +(2A\theta'-\frac{3}{4}\alpha A^3)\cos(t_1+\theta)\\ &\quad +\frac{1}{4}\mu A^3\sin 3(t_1+\theta)\\ &\quad -\frac{1}{4}\alpha A^3\cos 3(t_1+\theta)+F\cos(t_1+\sigma t_2)\end{aligned} \quad (8)$$

In order to eliminate secular terms in u1, the $\sin(t_1+\theta)$ terms and $\cos(t_1+\theta)$ terms in (8) must be removed, it is required that

$$2A'-\mu A+\frac{1}{4}\mu A^3 - F\sin(\sigma t_2-\theta) = 0 \quad (9a)$$

$$2A\theta'-\frac{3}{4}\alpha A^3 + F\cos(\sigma t_2-\theta) = 0 \quad (9b)$$

Solving differential equations (9) with initial values (7) will give the first term asymptotic approximation $u_0(t_1,t_2)$ to, $u(t,\varepsilon)$, the solution of (1),(2).

It is worth mentioned that initial value problem (9),(7) don't contain small parameter, it can be solved expediently by routine numerical methods such as Runge-Kutta method.

Unfortunately, the equations (9),(7) can't be solved explicitly in general, thus the nature of the solution of equations (9),(7) is not apparent. In order to obtain some qualitative behavior of the solution of equations (9),(7),We make a transformation to (9), $\theta_1 = \sigma t_2 - \theta$, such that (9) becomes a nonlinear autonomous one (10)

$$2A'-\mu A+\frac{1}{4}\mu A^3 = F\sin\theta_1 \quad (10a)$$

$$2A(\sigma-\theta_1') - \frac{3}{4}\alpha A^3 = -F\cos\theta_1 \quad (10b)$$

The equations (10) can't be solved explicitly still. In particular, we can investigate what value, denoted by A_∞, the amplitude A approaches if it goes to a steady state as $t\to+\infty$. Thus it brings about the following algebraic equation

$$(-\mu A_\infty + \frac{1}{4}\mu A_\infty{}^3)^2 + (2A_\infty\sigma - \frac{3}{4}\alpha A_\infty{}^3)^2 = F^2 \quad (11)$$

It shows that the positive solution, A_∞, is a function of the frequency parameter σ. Particularly, we can see $A_\infty(0)$ satisfies

$$(-\mu A_\infty + \frac{1}{4}\mu A_\infty{}^3)^2 + (\frac{3}{4}\alpha A_\infty{}^3)^2 = F^2$$

which shows obviously that $\varepsilon \ll A_\infty(0)$, it means that the resonance occurs.

For given parameters μ,α,F in the equation (1), equation (11) provides quick insight of the oscillation nature into how the steady amplitude A_∞ depends on the detuning parameter σ as in Fig.1.

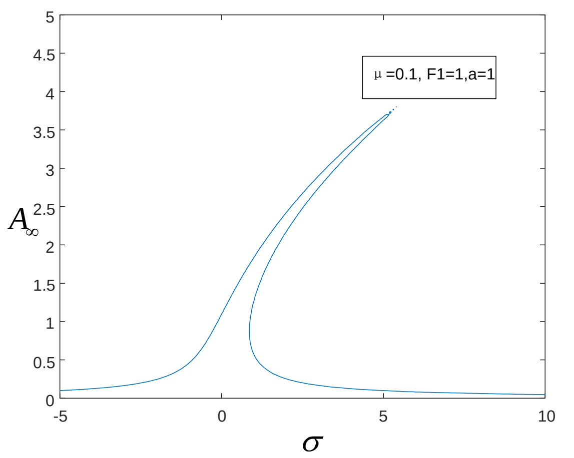

Fig.1 The relation between A_∞ and σ in case of Primary-harmonic resonance, wherein $\mu = 0.1, \alpha = 1, F = 1$

The Fig.1 shows that the phenomena of bi-stability and jump occur [Holmes, 2013; Nayfeh, 2004; Yang et al, 1998] at some bifurcation points.

Also, the equation (11) provides quantitative insight of the oscillation nature into how the steady amplitude A_∞ depends on the damping, nonlinear, external force parameters

μ, α and F in the equation (1).

For the case of $\omega = -1 + \varepsilon\sigma$, the discussions are analogous.

III. Asymptotic Analysis for the Case of Super-Harmonic and Sub-Harmonic Resonances

We can continue the above procedure involving searching for the asymptotic representation of u_1 to detect the super-harmonic and/or sub-harmonic resonance frequencies, and to obtain their amplitude-frequency relations. Alternatively, we adopt the Holmes strategy of setting the formal asymptotic expansion of the solution of (1), (2) as

$$u(t,\varepsilon) \sim \varepsilon(u_0(t_1,t_2) + \varepsilon u_1(t_1,t_2) + \cdots) \qquad (12)$$

Operating similar to segment 2, we have

$$O(\varepsilon): \begin{cases} \partial_{t_1}{}^2 u_0 + u_0 = F\cos\omega t_1 \\ u_0(0,0) = \partial_{t_1} u_0(0,0) = 0 \end{cases}$$

The general solution of this problem is

$$u_0(t_1,t_2) = A(t_2)\cos(t_1 + \theta(t_2)) + \frac{F}{1-\omega^2}\cos\omega t_1 \qquad (13)$$

$A(t_2)$ and $\theta(t_2)$ are to be determined later and with which the $A(0)$ and $\theta(0)$ can be assigned by (2) .

Denote $F_1 \equiv \frac{F}{1-\omega^2}$, Similar to the order of $O(\varepsilon)$, we have for $O(\varepsilon^2)$

$$\partial_{t_1}{}^2 u_1 + u_1 = -2\partial_{t_1}\partial_{t_2} u_0 - \mu(u_0{}^2 - 1)\partial_{t_1} u_0 - \alpha u_0{}^3$$
$$= 2(A'\sin(t_1+\theta) + A\theta'\cos(t_1+\theta))$$
$$+\mu(\frac{1}{2}(F_1^2 + F_1^2\cos 2\omega t_1 + A^2 + A^2\cos(2t_1 + 2\theta)$$
$$+2F_1 A\cos((1+\omega)t_1+\theta) + 2F_1 A\cos((1-\omega)t_1+\theta))$$
$$-1)*(A\sin(t_1+\theta) + F_1\omega\sin\omega t_1)$$
$$-\alpha(A\cos(t_1+\theta) + F_1\cos\omega t_1)^3$$
$$= -3\alpha F_1^3\cos\omega t_1 - \alpha F_1^3\cos 3\omega t_1 - 4F_1\mu\omega\sin\omega t_1$$
$$+F_1^3\mu\omega\sin\omega t_1 + F_1^3\mu\omega\sin 3\omega t_1$$
$$-4A^3\cos^2(t_1+\theta)(\alpha\cos(t_1+\theta) - \mu\sin(t_1+\theta))$$
$$-F_1 A^2(6\alpha\cos\omega t_1 + 3\alpha\cos((-2+\omega)t_1 - 2\theta)$$
$$+3\alpha\cos((2+\omega)t_1+2\theta) - 2\mu\omega\sin\omega t_1$$
$$-2\mu\sin((2-\omega)t_1+2\theta) + \mu\omega\sin((2-\omega)t_1+2\theta)$$
$$-2\mu\sin((2+\omega)t_1+2\theta) - \mu\omega\sin((2+\omega)t_1+2\theta))$$
$$+8A'\sin(t_1+\theta) + A(-6\alpha F_1^2\cos(t_1+\theta)$$
$$-3\alpha F_1^2\cos((1-2\omega)t_1+\theta) - 3\alpha F_1^2\cos((1+2\omega)t_1+\theta)$$
$$-4\mu\sin(t_1+\theta) + 2F_1^2\mu\sin(t_1+\theta) + 8\theta'\cos(t_1+\theta)$$
$$+F_1^2\mu\sin((1-2\omega)t_1+\theta) - 2F_1^2\mu\omega\sin((1-2\omega)t_1+\theta)$$
$$+F_1^2\mu\sin((1+2\omega)t_1+\theta) + 2F_1^2\mu\omega\sin((1+2\omega)t_1+\theta))$$

(14)

For the problem (1),(2) possessing bounded solution, and from the right hand side of equation (14), it is founded that the second resonances that contain the so-called superharmonic and subharmonic resonance frequencies are $\pm\frac{1}{3}$ and ± 3 ,0 respectively.

We just discuss the cases of $\omega \approx \frac{1}{3}$ and $\omega \approx 3, 0$ respectively, the case of second resonance frequency $\omega \approx -\frac{1}{3}$ and $\omega \approx -3$ can be discussed similarly.

A. *Super-harmonic resonances* ($\omega \approx \frac{1}{3}$)

For the case of super-harmonic resonance of the external force, we set $\omega = \frac{1}{3} + \varepsilon\sigma$ to describe quantitatively the nearness of ω to $\frac{1}{3}$ by introducing the detuning parameter σ. Eliminating secular term of the solution of equation (14) gives

$$-6\alpha F_1^2 A - 3\alpha A^3 + \mu\omega F_1^3\sin(3\sigma t_2 - \theta) + 8A\theta' = 0 \quad (15a)$$
$$\mu\omega F_1^3\cos(3\sigma t_2 - \theta) - 4\mu A + 2\mu A F_1^2 + \mu A^3 + 8A' = 0 \quad (15b)$$

The transformation $\theta_1 = 3\sigma t_2 - \theta$ to (15) will result in

$$-6\alpha F_1^2 A - 3\alpha A^3 + \mu\omega F_1^3\sin\theta_1 + 8A(3\sigma - \theta_1{}') = 0 \quad (16a)$$
$$\mu\omega F_1^3\cos\theta_1 - 4\mu A + 2\mu A F_1^2 + \mu A^3 + 8A' = 0 \quad (16b)$$

When we consider the steady state, we thus have

$$(\mu A_\infty{}^3 - 4\mu A_\infty + 2\mu A_\infty F_1{}^2)^2 + (24A_\infty\sigma - 6\alpha F_1^2 A_\infty - 3\alpha A_\infty{}^3)^2 = (\mu\omega F_1{}^3)^2 \quad (17)$$

Wherein A_∞ represents the steady state of *A* in (15).

The positive solution, A_∞, obtained from (17), is shown in Fig.2.

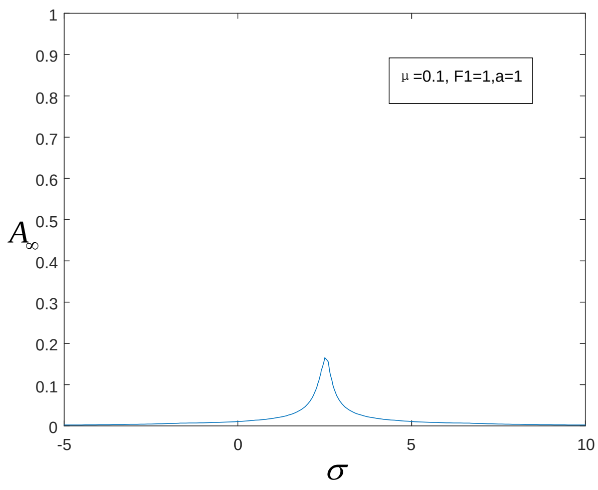

Fig.2 The relationship between A_∞ and σ in case of Super-harmonic

resonance

The Fig.2 shows that the Super-harmonic resonance is small compared with the Primary-harmonic resonance that is studied in Fig. 1.

B. *Sub-harmonic resonance* ($\omega \approx 3$)

For the case of sub-harmonic resonance of the external force, we set $\omega = 3 + \varepsilon\sigma$ to describe quantitatively the nearness of ω to 3 by introducing the detuning parameter σ. Eliminating secular term of the solution of equation (14) gives

$$-3\alpha F_1 A^2 \cos(\sigma t_2 - 3\theta) - 6\alpha F_1^2 A - 3\alpha A^3 + (\mu\omega F_1 A^2 - 2\mu F_1 A^2)\sin(\sigma t_2 - 3\theta) + 8A\theta' = 0$$

$$3\alpha F_1 A^2 \sin(\sigma t_2 - 3\theta) - 4\mu A + 2\mu A F_1^2 + \mu A^3 + (\mu\omega F_1 A^2 - 2\mu F_1 A^2)\cos(\sigma t_2 - 3\theta) + 8A' = 0$$

Set $3\theta_1 = \sigma t_2 - 3\theta$, we have

$$-3\alpha F_1 A^2 \cos 3\theta_1 - 6\alpha F_1^2 A - 3\alpha A^3 + (\mu\omega F_1 A^2 - 2\mu F_1 A^2)\sin 3\theta_1 + 8A(\frac{\sigma}{3} - \theta_1') = 0$$

$$3\alpha F_1 A^2 \sin 3\theta_1 - 4\mu A + 2\mu A F_1^2 + \mu A^3 + (\mu\omega F_1 A^2 - 2\mu F_1 A^2)\cos 3\theta_1 + 8A' = 0$$

Similarly to (11), we obtain

$$(\mu(A_\infty^{\ 2} + 2F_1^2 - 4))^2 + (6\alpha F_1^{\ 2} + 3\alpha A_\infty^{\ 2} - \frac{8\sigma}{3})^2 = ((\mu\omega - 2\mu)^2 + 9\alpha^2) F_1^{\ 2} A_\infty^{\ 2} \quad (18)$$

The positive solution, A_∞, obtained from (18), is shown in Fig.3.

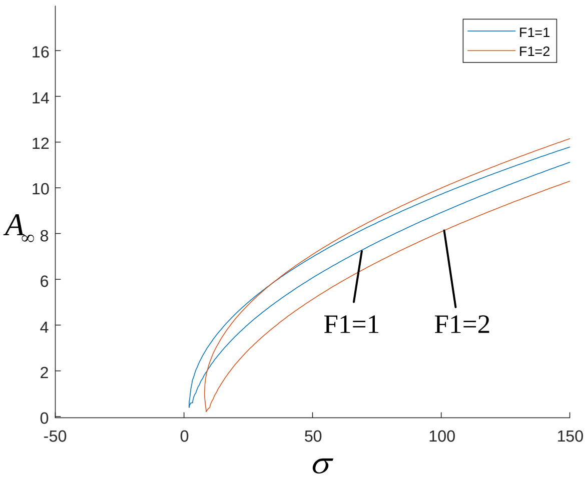

Fig.3 The relationship between A_∞ and σ in case of Sub-harmonic resonance

Fig.3 shows that the behavior of Sub-harmonic resonance of the Duffing-van der Pol oscillator studied here is almost the same to the one of the Duffing oscillator expounded by AH Nayfeh[Nayfeh,2004].

C. *Sub-harmonic resonance*($\omega \approx 0$)

For this case of the second resonance of the external force, we set $\omega = \varepsilon\sigma$ to describe quantitatively the nearness of ω to 0 by introducing the detuning parameter σ. This case is similarly to slow variable external force oscillator discussed by Holmes in [Holmes 2013]. Eliminating secular term of the solution of equation (14) gives

$$(\frac{1}{2}F_1^2\mu - \mu)A + \frac{1}{4}\mu F_1^2 A\cos(2\sigma t_2) + \frac{1}{4}\mu A^3 + 2A' = 0 \quad (19)$$

$$-3\alpha A^3 - 6\alpha A F_1^2 - 6\alpha A F_1^2 \cos(2\sigma t_2) + 8A\theta' = 0 \quad (20)$$

It is noticeable here that the nonlinear equations (19) and (20) involving A and θ is in fact decoupled. Fortunately, the nonlinear equation (19) involving A is Bernoulli type, thus we can solve A explicitly to get

$$A(t_2) = (e^{\frac{\mu}{4}((2-F_1^2)t_2 - \frac{F_1^2}{2\sigma}\sin 2\sigma t_2)} * (C - \frac{\mu}{4}\int_0^{t_2} e^{\frac{\mu}{4}((2-F_1^2)\tau - \frac{F_1^2}{2\sigma}\sin 2\sigma\tau)} d\tau))^{-\frac{1}{2}} \quad (21)$$

The elementary discussion shows that nearby the Sub-harmonic resonance ($\omega \approx 0$), the amplitude A is bounded when $F_1 \le \sqrt{2}$, the threshold value, $\sqrt{2}$, is thus obtained.

IV. Conclusions and Remarks

The resonance phenomena of a weakly nonlinear, damped, Duffing-van der Pol oscillation is studied analytically and numerically by using the methods of multiple scales. the cases of primary resonance frequency, super-harmonic resonance frequencies and sub-harmonic resonance frequencies are obtained respectively. The study shows the nonlinear behaviors such as jump and bistability at some bifurcation points for nonlinear Duffing-van der Pol oscillator. The results enriches existing researches just for Duffing or van der Pol oscillation respectively.

The problems involving multi-frequency external excitations can be considered by the method of multiple scales to observe combination resonance phenomena similarly.

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