An Approximation Method of Nonlinear Mapping for a Modified General Equilibrium and System of Variational Inequality Problems

Pimpawee Khumsup and Atid Kangtunyakarn

***Abstract*—In this article, we present a new iteration for approximating the solutions to generalized equilibrium problems, fixed point problems of κ-strictly pseudononspreading mapping, and modifications of the system of variational inequality. The variational inequality problem and the general split feasibility problem were then solved using our primary theorem. Additionally, we provide a numerical example to support our primary theorem.**

***Index Terms*—fixed points, variational inequalites, equilibrium problems, κ-strictly pseudononspreading mapping, inverse-strongly monotone.**

I. Introduction

LET K be a nonempty closed convex subset of H and let H be a real Hilbert space for the purposes of this article. Assume $O : K \to H$. Making a point $j \in K$ to the extent that

$$\langle Oj, h - j\rangle \geq 0 \tag{1}$$

for all $h \in K$ referred to be the variational inequality problem (VIP).

The set of solutions to (1) is indicated by $VI(K,O)$. With numerous applications in business, economics, and the pure and applied sciences, VIP has evolved as a captivating and intriguing subfield of mathematics and engineering [1].

In 2008, Ceng et al. [2] modified VIP to another way for finding $(l^*, e^*) \in K \times K$ such that

$$\begin{cases} \langle \lambda_1 D_1 e^* + l^* - e^*, l - l^*\rangle \geq 0, \forall e \in K, \\ \langle \lambda_2 D_2 l^* + e^* - l^*, l - e^*\rangle \geq 0, \forall l \in K, \end{cases} \tag{2}$$

which is called a system of variational inequalities problem (SVIP) where $D_1, D_2 : K \to H$ are mappings and parameters $\lambda_1, \lambda_2 > 0$. In the case of $\lambda_1 = \lambda_2, D_1 = D_2, l^* = e^*$, SVIP is reduced to VIP.

After that, Kangtunyakarn [3] modified (2) for finding $(l^*, e^*) \in K \times K$ such that

$$\begin{cases} \langle l^* - (I - \lambda_1 D_1)(bl^* + (1-b)e^*), l - l^*\rangle \geq 0, \\ \langle e^* - (I - \lambda_2 D_2)e^*, l - e^*\rangle \geq 0, \end{cases} \tag{3}$$

for all $l \in K$ which is called a modification of SVIP (MSVIP), for every $\lambda_1, \lambda_2 > 0$ and $b \in [0,1]$. If $b = 0$, (3) reduces to (2). He presented the following relationship between the fixed point of the mapping U and solutions of (3):

Lemma 1.1: Let $D_1, D_2 : K \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $b \in [0,1]$, the statements that follow are interchangeable:

1) $(t^*, z^*) \in K \times K$ is a solution of problem (3),
2) t^* is a fixed point of the mapping $U : K \to K$, i.e., $t^* \in F(U)$, defined by

$$U(t) = P_K(I - \lambda_1 D_1)(bt + (1-b)P_K(I - \lambda_2 D_2)t),$$

where $z^* = P_K(I - \lambda_2 D_2)t^*$.

Moreover, he proved the following strong convergence theorem for VIP and the fixed point problem for ν-strictly pseudononspreading mapping, which modified the Halpern iterative method [4] generated by (4).

Theorem 1.2: For every $r = 1,2,3,...,N$ let $B_r : K \to H$ be δ_r-ism mappings and let $J : K \to K$ be ν-strictly pseudononspreading mapping for some $\nu \in [0,1)$. Let $G_r : K \to K$ be defined by $G_r x = P_K(I - \iota B_r)x$ for every $x \in K$ and $\iota \in (0, 2\delta_r)$ for every $r = 1,2,3,...,N$, and let $\delta_l = (\vartheta_1^l, \vartheta_2^l, \vartheta_3^l) \in I \times I \times I, l = 1,2,3,...,N$, where $I = [0,1], \vartheta_1^l + \vartheta_2^l + \vartheta_3^l = 1, \vartheta_1^l \in (0,1)$ for all $l = 1,2,3,...,N-1, \vartheta_1^N \in (0,1], \vartheta_2^l, \vartheta_3^l \in (0,1]$ for all $l = 1,2,3,...,N$. Let $M : K \to K$ be the M-mappings generated by $G_1, G_2, ..., G_N$ and $\delta_1, \delta_2, ..., \delta_N$. Assume that $F = F(J) \cap \bigcap_{r=1}^N VI(K, B_r) \neq \phi$. For every $n \in N$, $r = 1,2,3,...,N$, let $x_1, y \in K$ and $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \vartheta_n y + \zeta_n P_K(I - \varrho_n(I - J))x_n + \xi_n M x_n \tag{4}$$

where $\{\vartheta_n\}, \{\zeta_n\}, \{\xi_n\}, \{\varrho_n\} \subset (0,1)$ such that $\vartheta_n + \zeta_n + \xi_n = 1, \zeta_n \in [o,p] \subset (0,1), \{\varrho_n\} \subset (0, 1-\nu)$ and suppose the following conditions hold:

(i) $\lim_{n\to\infty} \vartheta_n = 0$ and $\sum_{n=0}^\infty \vartheta_n = \infty$,
(ii) $\sum_{n=1}^\infty \varrho_n < \infty$,
(iii) $\sum_{n=1}^\infty |\varrho_{n+1} - \varrho_n| < \infty, \sum_{n=1}^\infty |\xi_{n+1} - \xi_n| < \infty$, $\sum_{n=1}^\infty |\vartheta_{n+1} - \vartheta_n| < \infty, \sum_{n=1}^\infty |\zeta_{n+1} - \zeta_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_F y$.

In 2012, Kangtunyakarn [5] modified (1). He introduced the combination of VIP (CVIP) by letting $P, B : K \to H$ such that

$$\langle r - m, (bP + (1-b)B)m\rangle \geq 0 \tag{5}$$

for all $m, r \in K$ and $b \in (0,1)$. The set of CVIP is denoted by $VI(K, bP + (1-b)B)$. Moreover, if $P \equiv B$, then CVIP can be reduced to VIP.

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Let $L : K \times K \to R$ be a bifunction. The equilibrium problem for L is to determine its equilibrium points and denote

$$EP(L) = \{x \in K : L(r, m) \geq 0, \forall m \in K\} \tag{6}$$

as the set of all solutions to the equilibrium problem.

From (1) and (6), we have the following generalized equilibrium problem, i.e., find $g \in K$ such that

$$L(g, m) + \langle Og, m - g\rangle \geq 0 \tag{7}$$

for all $m \in K$. The set of such $g \in K$ is denoted by $EP(L, O)$. When $O \equiv 0, EP(L, O)$ is represented by $EP(L)$. In the case of $L \equiv 0, EP(L, O)$ is also denoted by $VI(K, O)$.

Blum and Oettli [6] introduced equilibrium problems in 1994. Solving $EP(L)$ is a reduction of several optimization and economics problems (see [6]). The iterative approach for identifying a common element between the set of solutions to the equilibrium problems and the set of solutions to the fixed point problem has garnered attention from several writers recently [7].

In 2008, Takahashi and Takahashi [8] introduced a general iterative method for finding a common element between $EP(L, O)$ and $F(J)$. They defined $\{y_n\}$ in the following way:

$$\begin{cases} h, y_1 \in K, \quad arbitrarily; \\ L(t_n, m) + \langle Oy_n, m - t_n\rangle + \frac{1}{\rho_n}\langle m - t_n, t_n - y_n\rangle \geq 0, \\ y_{n+1} = \eta_n y_n + (1 - \eta_n)J\big(b_n h + (1 - b_n)t_n\big), \end{cases} \tag{8}$$

for all $m \in K$ and $n \in N$ with O being an ν-ism mapping of K into H with positive real number ν and $\{b_n\} \in [0, 1], \{\eta_n\} \subset [0, 1], \{\rho_n\} \subset [0, 2\alpha]$, and proved strong convergence of the scheme (8) to $t \in F(J) \cap EP(L, O)$, where $t = P_{F(J)\cap EP}h$ in H, given suitable constraints on $\{b_n\}, \{\eta_n\}, \{\rho_n\}$, and bifunction L.

The following theorem was shown by Inchan [9] by modification of the viscosity approximation method:

Theorem 1.3: Let $K \pm K \subset K$, and let $J : K \to H$ be a ν-strictly pseudo-contractive mapping with a fixed point for some $0 \leq \nu < 1$. Let V be a strongly positive bounded linear operator on K with coefficient $\bar{\rho}$ and $f : K \to K$ be a contraction with the contractive constant $(0 < \vartheta < 1)$ such that $0 < \eta < \frac{\bar{\eta}}{\vartheta}$. Let $\{o_n\}$ be the sequence generated by

$$o_{n+1} = \vartheta_n \eta f(o_n) + \varsigma_n o_n + \big((1 - \varsigma_n)I - \vartheta_n V\big)P_K S o_n, \tag{9}$$

where $o_1 \in K$ and $S : K \to H$ is a mapping defined by

$$So = \nu o + (1 - \nu)Jo \tag{10}$$

If the control sequence $\{\vartheta_n\}, \{\varsigma_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n\to\infty} \vartheta_n = 0$ and $\lim_{n\to\infty} \varsigma_n = 0$,
(ii) $\sum_{n=1}^{\infty} \vartheta_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |\vartheta_{n+1} - \vartheta_n| < \infty, \; \sum_{n=1}^{\infty} |\varsigma_{n+1} - \varsigma_n| < \infty$.

Then $\{o_n\}$ converges strongly to a fixed point l of J, which solves the following solution of VIP;

$$\langle (V - \eta f)l, l - o\rangle \leq 0$$

for all $o \in F(J)$.

Furthermore, from (5) and (7), we introduce a problem relative to CVIP and equilibrium problems, i.e., find $m \in K$ such that

$$L(m, g) + \langle (oA + (1 - o)B)m, g - m\rangle \geq 0 \tag{11}$$

for all $g \in K$ and $o \in (0, 1)$. The set of all solutions to such problems is denoted by $EP(L, (oA + (1 - o)B))$.

Remember that a mapping $O : K \to K$ is deemed nonexpansive if $\|Om - Os\| \leq \|m - s\|$ for all $m, s \in K$.

The nonspreading mapping in H was presented by Kohsaka and Takahashi [10] in 2008. It is defined as follows:

$$2\|Om - Os\|^2 \leq \|Om - s\|^2 + \|m - Os\|^2$$

for all $m, s \in K$.

In 2011, Osilike and Isiogugu [12] introduced, using terminology from Browder and Petryshyn [11], that a mapping $O : K \to K$ is a ν-strictly pseudononspreading mapping if there exists $\nu \in [0, 1)$ such that

$$\|Om - Os\|^2 \leq \|m - s\|^2 + \nu\|(I - O)m - (I - O)s\|^2 + 2\langle m - Jm, s - Js\rangle$$

for all $m, s \in K$. It is evident that each nonspreading mapping is ν-strictly pseudononspreading.

A point $h \in C$ is called a fixed point of J if $Jh = h$. The set of fixed points of J is denoted by

$$F(J) = \{h \in K : Jh = h\}.$$

A mapping $R : K \to H$ is called τ-inverse strongly monotone (ism), if there exists a positive real number τ such that

$$\langle m - s, Rm - Rs\rangle \geq \tau\|Rm - Rs\|^2$$

for all $m, s \in K$.

Inspired and motivated by Theorem 1.2, (11) and the similar trend of research , we prove a strong convergence theorem for the MSVIP, generalized equilibrium problems and fixed point problems of κ-strictly pseudononspreading mapping. In addition, we applied our main result to solving the VIP and the general split feasibility problem. In conclusion, we present a numerical example to validate our primary finding.

II. PRELIMINARIES

Let P_K be the metric projection of H onto K i.e., for $m \in H, \; P_K m$ satisfies the property

$$\|m - P_K m\| = \min_{s\in C} \|m - s\|.$$

The following characterizes the projection $P_K m$.

Lemma 2.1 ([13]): Given $m \in H$ and $s \in K$. Then $P_K m = s$ if and only if there holds the inequality

$$\langle m - s, s - r\rangle \geq 0$$

for all $r \in K$.

Lemma 2.2 ([13]): Let M be a mapping of K into H. Let $y \in K$. Then for $\nu > 0$,

$$y = P_K (I - \nu M) y \Leftrightarrow y \in VI(K, M),$$

where P_K is the metric projection of H onto K.

Lemma 2.3 ([14]): Let $\{r_n\}$ be a sequence of nonnegative real number satisfying

$$r_{n+1} \leq (1 - \xi_n)r_n + \eta_n, \; \forall n \geq 0,$$

where $\{\xi_n\}$ is a sequence in $(0,1)$ and $\{\eta_n\}$ is a sequence such that

1) $\sum_{n=1}^{\infty} \xi_n = \infty$,
2) $\limsup_{n\to\infty} \frac{\eta_n}{\xi} \le 0$ or $\sum_{n=1}^{\infty} |\eta_n| < \infty$.

Then $\lim_{n\to\infty} r_n = 0$.

Lemma 2.4 ([14]): Let $\{r_n\}$ be a sequence of nonnegative real number satisfying

$$r_{n+1} \le (1-\xi_n)s_n + \xi_n\eta_n,\ \forall n \ge 0,$$

where $\{\xi_n\}, \{\eta_n\}$ satisfy the conditions

1) $\{\xi_n\} \subset [0,1],\ \sum_{n=1}^{\infty} \xi_n = \infty$;
2) $\limsup_{n\to\infty}\ \eta_n \le 0$ or $\sum_{n=1}^{\infty} |\xi_n\eta_n| < \infty$.

Then $\lim_{n\to\infty} r_n = 0$.

Lemma 2.5 ([15]): Let W be a uniformly convex Banach space, K be a nonempty closed convex subset of W and $D : K \to K$ be a nonexpansive mapping. Then, $I - D$ is demiclosed at zero.

For solving the equilibrium problem for a bifunction $L : K\times\ K \to R$, let us assume that L satisfies the following conditions:

(G1) $L(m,m) = 0,\ \forall m \in K$;
(G2) L is monotone, i.e. $L(m,s)+L(s,m) \le 0, \forall m,s \in K$;
(G3) $\forall m,s,h \in K,\ \lim_{t\to 0^+} L(th + (1-t)m, s) \le L(m,s)$;
(G4) $\forall m \in K,\ s \mapsto L(m,s)$ is convex and lower semicontinuous.

Lemma 2.6 ([6]): Let L be a bifunction of $K\times\ K$ into R satisfying (G1) - (G4). Let $l > 0$ and $m \in H$. Then, there exists $s \in K$ such that

$$L(s,d) + \frac{1}{l}\langle o-s, s-m\rangle \ge 0,$$

for all $m \in K$.

Lemma 2.7 ([7]): Assume that $L : K\times\ K \to R$ satisfies (G1) - (G4). For $l > 0$ and $m \in H$, define a mapping $T_l : H \to K$ as follows:

$$T_l(m) = \left\{ s \in K : L(s,o) + \frac{1}{l}\langle o-s, s-m\rangle \ge 0, \forall o \in K \right\}$$

for all $m \in H$. Then, the following hold:

1) T_l is single-valued;
2) T_l is firmly nonexpansive i.e.,
$$\|T_l(m) - T_l(o)\|^2 \le \langle T_l(m) - T_l(o), m-o\rangle$$
for all $m,o \in H$;
3) $F(T_l) = EP(L)$;
4) $EP(L)$ is closed and convex.

Lemma 2.8 ([16]): Let $M : K \to K$ be a ν-strictly pseudononspreading mapping with $F(M) \ne \phi$. Then $F(M) = VI(K,(I-M))$.

Remark 2.9: From Lemmas 2.2 and 2.8, we have $F(M) = F\left(P_K\left(I - \nu\left(I-M\right)\right)\right)$ for all $\nu > 0$.

III. Main result

In this section, we prove strong convergence theorem for approximating the solution of the modification of system variational inequality, generalized equilibrium problems, and fixed point problems of κ-strictly pseudononspreading mapping by modifying Halpern iterative method.

Theorem 3.1: Let K be a closed convex subset of Hilbert space H and let $F : K\times\ K \to R$ be a function satisfying (G1) - (G4), let $\widetilde{A}, \widetilde{B}, A'', B'' : K \to H$ be $\widetilde{\alpha}, \widetilde{\beta}, \alpha'', \beta''$-ism, correspondingly. Define $G : K \to K$ by $Gg = P_K(I - \xi_1 A'')(wg + (1-w)P_K(I-\xi_2 B'')g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha'')$ and $\xi_2 \in (0, 2\beta'')$. Let $J : K \to K$ be κ-strictly pseudononspreading mapping with $\mathcal{F} = F(J) \cap F(G) \cap EP(F, b\widetilde{A} + (1-b)\widetilde{B}) \ne \phi$ for all $w \in (0,1)$. Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$\begin{cases} F\left(h_n, d\right) + \left\langle \left(w\widetilde{A} + (1-w)\,\widetilde{B}\right) g_n, d - h_n \right\rangle \\ \qquad + \frac{1}{p_n}\left\langle d - h_n, h_n - g_n\right\rangle \ge 0,\ \forall d \in K, \\ g_{n+1} = \eta_n h + \zeta_n G g_n \\ \qquad + \gamma_n P_K\left(I - \xi_n\left(I - J\right)\right) h_n, \forall n \in N, \end{cases} \tag{12}$$

where $\{\eta_n\}$, $\{\zeta_n\}$, $\{\gamma_n\} \subset [0,1], \xi_n \in (0, 1-\kappa), \eta_n + \zeta_n + \gamma_n = 1, \forall n \in N$ and $\{p_n\} \subset [0, 2\gamma], \gamma = \min\{\widetilde{\alpha}, \widetilde{\beta}\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \eta_n = \infty,\ \lim_{n\to\infty} \eta_n = 0,\ \sum_{n=1}^{\infty} \xi_n < \infty$;
(ii) $0 < o \le \zeta_n \le p < 1,\ 0 < q \le\ p_n \le m < 2\gamma$;
(iii) $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$;
(iv) $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty,\ \sum_{n=1}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$.

Then $\{g_n\}$ converges strongly to $e_0 = P_{\mathcal{F}} h$.

Proof: There are seven steps in our proof: Step 1. We'll demonstrate that $Y = w\widetilde{A} + (1-w)\widetilde{B}$ is γ-ism. Let $g, d \in K$, we have

$$\begin{aligned} &\langle Yg - Yd, g - d\rangle \\ &= \left\langle \left(w\widetilde{A} + (1-w)\,\widetilde{B}\right) g - \left(w\widetilde{A} + (1-w)\,\widetilde{B}\right) d, g - d \right\rangle \\ &= w\left\langle \widetilde{A}g - \widetilde{A}d, g - d\right\rangle + (1-w)\left\langle \widetilde{B}g - \widetilde{B}d, g - d\right\rangle \\ &\ge w\widetilde{\alpha}\left\|\widetilde{A}g - \widetilde{A}d\right\|^2 + (1-w)\,\widetilde{\beta}\left\|\widetilde{B}g - \widetilde{B}d\right\|^2 \\ &\ge \gamma\left(w\left\|\widetilde{A}g - \widetilde{A}d\right\|^2 + (1-w)\left\|\widetilde{B}g - \widetilde{B}d\right\|^2\right) \\ &\ge \gamma\left\|w\left(\widetilde{A}g - \widetilde{A}d\right) + (1-w)\left(\widetilde{B}g - \widetilde{B}d\right)\right\|^2 \\ &= \gamma\left\|Yg - Yd\right\|^2. \end{aligned} \tag{13}$$

For each $n \in N$, we obtain $I - p_n Y$ as a nonexpansive mapping by applying the same technique as [5].

Step 2. For every $b \in (0,1)$, we'll demonstrate that $\{g_n\}$ is bounded. Let $e \in \mathcal{F}$. Deriving from Lemma 2.7, we possess

$$h_n = T_{p_n}\left(I - p_n Y\right) g_n \tag{14}$$

and

$$e = T_{p_n}\left(I - p_n Y\right) e \tag{15}$$

for all $n \in N$. Based on Lemma 2.2 and Lemma 2.8, we may obtain

$$e = P_K(I - \xi_n(I-J))e \tag{16}$$

for all $n \in N$. By the nonexpansiveness of (16), we have

$$\begin{aligned} &\left\|P_K\left(I - \xi_n\left(I - J\right)\right) h_n - e\right\| \\ &= \left\|P_K\left(I - \xi_n\left(I - J\right)\right) h_n - P_K\left(I - \xi_n\left(I - J\right)\right) e\right\| \\ &\le \left\|\left(I - \xi_n\left(I - J\right)\right) h_n - \left(I - \xi_n\left(I - J\right)\right) e\right\|. \end{aligned} \tag{17}$$

Since J is κ-strictly pseudononspreading mapping and let $E = I - J$, we have

$$\begin{aligned} \|Jh_n - Je\|^2 &= \|(I-E)\,h_n - (I-E)\,e\|^2 \\ &= \|(h_n - e) - (Eh_n - Ee)\|^2 \\ &= \|h_n - e\|^2 - 2\langle h_n - e, Eh_n\rangle \\ &\quad + \|Eh_n\|^2 \\ &\le \|h_n - e\|^2 + \kappa\|Eh_n\|^2, \end{aligned}$$

it suggests that

$$(1-\kappa)\,\|Eh_n{}^2 \le 2\langle h_n - e, Eh_n\rangle. \tag{18}$$

From (18), we have

$$\begin{aligned} &\|(I - \xi_n E)\,h_n - (I - \xi_n E)\,e\|^2 \\ &= \|(h_n - e) - \xi_n(Eh_n - Ee)\|^2 \\ &= \|h_n - e\|^2 - 2\xi_n\langle h_n - e, Eh_n\rangle + \xi_n^2\|Eh_n\|^2 \\ &\le \|h_n - e\|^2 - \xi_n(1-\kappa)\|Eh_n\|^2 + \xi_n^2\|Eh_n\|^2 \\ &= \|h_n - e\|^2 - \xi_n((1-\kappa) - \xi_n)\|Eh_n\|^2 \\ &\le \|h_n - e\|^2. \end{aligned} \tag{19}$$

From (17) and (19), we can imply that

$$\|P_K(I - \xi_n(I - J))\,h_n - e\| \le \|h_n - e\|. \tag{20}$$

Since $e \in \mathcal{F}$, we have $e = G(e) = P_K(I - \xi_1 A'')(be + (1-b)P_K(I - \xi_2 B'')e)$. Put $M_n = bg_n + (1-b)P_K(I - \xi_2 B'')g_n$. Then, we have $G_n g_n = P_K(I - \xi_1 A'')M_n$. From definition of g_n, (20), and nonexpansiveness of G, we have

$$\begin{aligned} \|g_{n+1} - e\| &= \|\eta_n(h - e) + \zeta_n(Gg_n - e) \\ &\quad + \gamma_n(P_K(I - \xi_n(I-J))\,h_n - e)\| \\ &\le \eta_n\|h - e\| + \zeta_n\|g_n - e\| + \gamma_n\|h_n - e\| \\ &= \eta_n\|h - e\| + \zeta_n\|g_n - e\| \\ &\quad + \gamma_n\|T_{p_n}(I - p_n Y)\,g_n - e\| \\ &\le \eta_n\|h - e\| + \zeta_n\|g_n - e\| + \gamma_n\|g_n - e\| \\ &= \eta_n\|h - e\| + (1-\eta_n)\|g_n - e\| \\ &\le \max\{\|g_1 - e\|, \|h - e\|\}. \end{aligned}$$

We can demonstrate by induction that both $\{g_n\}$ and $\{h_n\}$ have bounded.

Step 3. We'll demonstrate that $\lim_{n\to\infty}\|g_{n+1} - g_n\| = 0$. Putting $l_n = g_n - p_n Y g_n$, we obtain $h_n = T_{p_n}(g_n - p_n Y g_n) = T_{p_n} l_n$. From definition of h_n, we obtain

$$F(h_n, d) + \frac{1}{p_n}\langle d - h_n, h_n - l_n\rangle \ge 0,\ \forall d \in K \tag{21}$$

and

$$F(h_{n+1}, d) + \frac{1}{p_{n+1}}\langle d - h_{n+1}h_{n+1} - l_{n+1}\rangle \ge 0,\ \forall d \in K. \tag{22}$$

Instead of d by h_{n+1} and h_n in (21) and (22), correspondingly, we have

$$F(h_n, h_{n+1}) + \frac{1}{p_n}\langle h_{n+1} - h_n, h_n - l_n\rangle \ge 0 \tag{23}$$

and

$$F(h_{n+1}, h_n) + \frac{1}{p_{n+1}}\langle h_n - h_{n+1}, h_{n+1} - l_{n+1}\rangle \ge 0. \tag{24}$$

Adding (23) and (24) and using (G2) , we obtain

$$\begin{aligned} 0 &\le \frac{1}{p_n}\langle h_{n+1} - h_n, h_n - l_n\rangle \\ &\quad + \frac{1}{p_{n+1}}\langle h_n - h_{n+1}, h_{n+1} - l_{n+1}\rangle \\ &= \left\langle h_{n+1} - h_n, \frac{h_n - l_n}{p_n}\right\rangle \\ &\quad + \left\langle h_n - h_{n+1}, \frac{h_{n+1} - l_{n+1}}{p_{n+1}}\right\rangle \\ &= \left\langle h_{n+1} - h_n, \frac{h_n - l_n}{p_n} - \frac{h_{n+1} - l_{n+1}}{p_{n+1}}\right\rangle. \end{aligned}$$

It implies that

$$\begin{aligned} 0 &\le \left\langle h_{n+1} - h_n, h_n - l_n - \frac{p_n}{p_{n+1}}(h_{n+1} - l_{n+1})\right\rangle \\ &= \langle h_{n+1} - h_n, h_n - h_{n+1} + h_{n+1} - l_n \\ &\quad - \frac{p_n}{p_{n+1}}(h_{n+1} - l_{n+1})\rangle. \end{aligned}$$

From (25), we obtain

$$\begin{aligned} &\|h_{n+1} - h_n\|^2 \\ &\le \left\langle h_{n+1} - h_n, h_{n+1} - l_n - \frac{p_n}{p_{n+1}}(h_{n+1} - l_{n+1})\right\rangle \\ &= \langle h_{n+1} - h_n, h_{n+1} - l_{n+1} + l_{n+1} - l_n \\ &\quad - \frac{p_n}{p_{n+1}}(h_{n+1} - l_{n+1})\rangle \\ &= \langle h_{n+1} - h_n, l_{n+1} - l_n \\ &\quad + (1 - \frac{p_n}{p_{n+1}})(h_{n+1} - l_{n+1})\rangle \\ &\le \|h_{n+1} - h_n\|(\|l_{n+1} - l_n\| \\ &\quad + \frac{1}{p_{n+1}}|p_{n+1} - p_n|\,\|h_{n+1} - l_{n+1}\|). \end{aligned}$$

Hence

$$\begin{aligned} \|h_{n+1} - h_n\| &\le \|l_{n+1} - l_n\| \\ &\quad + \frac{1}{q}|p_{n+1} - p_n|\,\|h_{n+1} - l_{n+1}\|. \end{aligned} \tag{25}$$

Since $l_n = g_n - p_n Y g_n$, we obtain

$$\begin{aligned} &\|l_{n+1} - l_n\| \\ &= \|(g_{n+1} - p_{n+1}Yg_{n+1}) - (g_n - p_n Y g_n)\| \\ &= \|(I - p_{n+1}Y)g_{n+1} - (I - p_{n+1}Y)g_n \\ &\quad + (I - p_{n+1}Y)g_n - (I - p_n Y)g_n\| \\ &\le \|(I - p_{n+1}Y)g_{n+1} - (I - p_{n+1}Y)g_n\| \\ &\quad + \|(p_n - p_{n+1})Yg_n\| \\ &\le \|g_{n+1} - g_n\| + |p_{n+1} - p_n|\,\|Yg_n\|. \end{aligned} \tag{26}$$

Substitute (26) into (25), we obtain

$$\begin{aligned} &\|h_{n+1} - h_n\| \\ &\le \|l_{n+1} - l_n\| + \frac{1}{q}|p_{n+1} - p_n|\,\|h_{n+1} - l_{n+1}\| \\ &\le \|g_{n+1} - g_n\| + |p_{n+1} - p_n|\,\|Yg_n\| \\ &\quad + \frac{1}{q}|p_{n+1} - p_n|\,\|h_{n+1} - l_{n+1}\| \\ &\le \|g_{n+1} - g_n\| + |p_{n+1} - p_n|L + \frac{1}{q}|p_{n+1} - p_n|L \end{aligned} \tag{27}$$

where $L = \max_{n\in N}\{\|Yg_n\|, \|h_n - l_n\|\}$.
From definition of g_n and let $E = I - J$, we obtain

$$\begin{aligned}
&\|g_{n+1} - g_n\| \\
=\ & \|\eta_n h + \zeta_n Gg_n + \gamma_n P_K(I-\xi_n E)h_n - \eta_{n-1}h \\
& -\zeta_{n-1}Gg_{n-1} - \gamma_{n-1}P_K(I-\xi_{n-1}E)h_{n-1}\| \\
=\ & \|\eta_n h + \zeta_n Gg_n - \zeta_n Gg_{n-1} + \zeta_n Gg_{n-1} \\
& +\gamma_n P_K(I-\xi_n E)h_n - \gamma_n P_K(I-\xi_{n-1}E)h_{n-1} \\
& +\gamma_n P_K(I-\xi_{n-1}E)h_{n-1} - \eta_{n-1}h \\
& -\zeta_{n-1}Gg_{n-1} - \gamma_{n-1}P_K(I-\xi_{n-1}E)h_{n-1}\| \\
\le\ & |\eta_n - \eta_{n-1}|\,\|h\| + \zeta_n\|Gg_n - Gg_{n-1}\| \\
& +|\zeta_n - \zeta_{n-1}|\,\|Gg_{n-1}\| \\
& +\gamma_n\|P_K(I-\xi_n E)h_n - P_K(I-\xi_{n-1}E)h_{n-1}\| \\
& +|\gamma_n - \gamma_{n-1}|\,\|P_K(I-\xi_{n-1}E)h_{n-1}\| \\
\le\ & |\eta_n - \eta_{n-1}|\,\|h\| + \zeta_n\|g_n - g_{n-1}\| \\
& +|\zeta_n - \zeta_{n-1}|\,\|Gg_{n-1}\|\,l + \gamma_n\big(\|h_n - h_{n-1}\| \\
& +\xi_n\|Eh_n - Eh_{n-1}\| + |\xi_n - \xi_{n-1}|\,\|Eh_{n-1}\|\big) \\
& +|\gamma_n - \gamma_{n-1}|\,\|P_K(I-\xi_{n-1}E)h_{n-1}\| \\
\le\ & |\eta_n - \eta_{n-1}|\,\|h\| + \zeta_n\|g_n - g_{n-1}\| \\
& +|\zeta_n - \zeta_{n-1}|\,\|Gg_{n-1}\| + \gamma_n\big(\|g_n - g_{n-1}\| \\
& +|p_{n-1} - p_n|L + \frac{1}{q}|p_{n-1} - p_n|L \\
& +\xi_n\|Eh_n - Eh_{n-1}\| + |\xi_n - \xi_{n-1}|\,\|Eh_{n-1}\|\big) \\
& +|\gamma_n - \gamma_{n-1}|\,\|P_K(I-\xi_{n-1}E)h_{n-1}\| \\
\le\ & |\eta_n - \eta_{n-1}|\,\|h\| + (1-\eta_n)\|g_n - g_{n-1}\| \\
& +|\zeta_n - \zeta_{n-1}|\,\|Gg_{n-1}\| + |p_{n-1} - p_n|O \\
& +\frac{1}{q}|p_{n-1} - p_n|O + \xi_n\|Eh_n - Eh_{n-1}\| \\
& +|\xi_n - \xi_{n-1}|\,\|Eh_{n-1}\| \\
& +|\gamma_n - \gamma_{n-1}|\,\|P_K(I-\xi_{n-1}E)h_{n-1}\| \\
\le\ & |\eta_n - \eta_{n-1}|O + (1-\eta_n)\|g_n - g_{n-1}\| \\
& +|\zeta_n - \zeta_{n-1}|O + |p_{n-1} - p_n|O + \frac{1}{q}|p_{n-1} - p_n|O \\
& +\xi_n O + |\xi_n - \xi_{n-1}|O + |\gamma_n - \gamma_{n-1}|O \qquad (28)
\end{aligned}$$

where

$$\begin{aligned}
O = \max_{n\in N}\big\{ & \|h\|, \|Gg_{n-1}\|, \|Eh_n - Eh_{n-1}\|, \\
& \|Eh_{n-1}\|, \|P_K(I-\xi_{n-1}E)h_{n-1}\|, \\
& \|Yg_n\|, \|h_n - l_n\|\big\}.
\end{aligned}$$

From Lemma 2.3, (28), condition (i),(iii), and (iv), we obtain

$$\lim_{n\to\infty}\|g_{n+1} - g_n\| = 0. \qquad (29)$$

Step 4. We'll demonstrate that $\lim_{n\to\infty}\|g_n - Gg_n\| = 0$. Since $h_n = T_{p_n}(g_n - p_n Yg_n)$, we obtain

$$\begin{aligned}
\|h_n - e\|^2 &= \|T_{p_n}(I-p_nY)g_n - T_{p_n}(I-p_nY)e\|^2 \\
&\le \langle (I-p_nY)g_n - (I-p_nY)e, h_n - e\rangle \\
&= \frac{1}{2}(\|(I-p_nY)g_n - (I-p_nY)e\|^2 \\
&\quad + \|h_n - e\|^2 - \|(I-p_nY)g_n \\
&\quad -(I-p_nY)z - h_n + e\|^2) \\
&\le \frac{1}{2}(\|g_n - e\|^2 + \|h_n - e\|^2 \\
&\quad - \|(g_n - h_n) - p_n(Yg_n - Ye)\|^2) \\
&\le \frac{1}{2}(\|g_n - e\|^2 + \|h_n - e\|^2 \\
&\quad - \|g_n - h_n\|^2 - p_n^2\|Yg_n - Ye\|^2 \\
&\quad +2p_n\langle g_n - h_n, Yg_n - Ye\rangle).
\end{aligned}$$

It implies that

$$\begin{aligned}
\|h_n - e\|^2 \le\ & \|g_n - e\|^2 - \|g_n - h_n\|^2 \\
& -p_n^2\|Yg_n - Ye\|^2 \\
& +2p_n\langle g_n - h_n, Yg_n - Ye\rangle. \qquad (30)
\end{aligned}$$

Using the same technique as [5] and the nonexpansiveness of T_{p_n}, we obtain

$$\|h_n - e\|^2 \le \|g_n - e\|^2. \qquad (31)$$

From definition of g_n, (20) and (31), we obtain

$$\begin{aligned}
&\| g_{n+1} - e \|^2 \\
=\ & \| \eta_n(h-e) + \zeta_n(Gg_n - e) \\
& +\gamma_n(P_K(I-\xi_n(I-J))h_n - e) \|^2 \\
\le\ & \eta_n \| h - e \|^2 + \zeta_n \| Gg_n - e \|^2 \\
& +\gamma_n \| P_K(I-\xi_n(I-J))h_n - e \|^2 \\
& -\beta_n\gamma_n \| P_K(I-\xi_n(I-J))h_n - Gg_n \|^2 \\
\le\ & \eta_n \| h - e \|^2 + \zeta_n \| Gg_n - e \|^2 + \gamma_n \| h_n - e \|^2 \\
& -\zeta_n\gamma_n \| P_K(I-\xi_n(I-J))h_n - Gg_n \|^2 \\
\le\ & \eta_n \| h - e \|^2 + \zeta_n \| g_n - e \|^2 + \gamma_n \| g_n - e \|^2 \\
& -\zeta_n\gamma_n \| P_K(I-\xi_n(I-J))h_n - Gg_n \|^2 \\
=\ & \eta_n \| h - e \|^2 + (1-\eta_n) \| g_n - e \|^2 \\
& -\zeta_n\gamma_n \| P_K(I-\xi_n(I-J))h_n - Gg_n \|^2 \\
\le\ & \eta_n \| h - e \|^2 + \| g_n - e \|^2 \\
& -\zeta_n\gamma_n \| P_K(I-\xi_n(I-J))h_n - Gg_n \|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\zeta_n\gamma_n \| P_K(I-\xi_n(I-J))h_n - Gg_n \|^2 \\
&\quad\le \eta_n \| h - e \|^2 + \| g_n - e \|^2 - \| g_{n+1} - e \|^2 \\
&\quad\le \eta_n \| h - e \|^2 \qquad (32) \\
&\qquad +(\| g_n - e \| + \| g_{n+1} - e \|) \| g_{n+1} - g_n \|.
\end{aligned}$$

From (29), (32), condition (i) and (ii), we obtain

$$\lim_{n\to\infty} \| P_K(I-\xi_n(I-J))h_n - Gg_n \| = 0. \qquad (33)$$

Since

$$\begin{aligned}
&\| g_{n+1} - P_K(I-\xi_n(I-J))h_n \| \\
&\quad= \| \eta_n(h - P_K(I-\xi_n(I-J))h_n) \\
&\qquad +\zeta_n(Gg_n - P_K(I-\xi_n(I-J))h_n) \| \\
&\quad\le \eta_n \| h - P_K(I-\xi_n(I-J))h_n \| \\
&\qquad +\zeta_n \| Gg_n - P_K(I-\xi_n(I-J))h_n \|,
\end{aligned}$$

(33) and condition (i), we obtain

$$\lim_{n\to\infty} \| g_{n+1} - P_K(I-\xi_n(I-J))h_n \| = 0. \qquad (34)$$

Since

$$\begin{aligned}\| g_n - P_K(I-\xi_n(I-J))h_n \| &\\ \leq\ & \| g_n - g_{n+1} \| \\ & + \| g_{n+1} - P_K(I-\xi_n(I-J))h_n \|,\end{aligned}$$

(29) and (34), we obtain

$$\lim_{n\to\infty} \| g_n - P_K(I-\xi_n(I-J))h_n \| \ = \ 0. \quad (35)$$

Since

$$\begin{aligned}\zeta_n \| Gg_n - g_n \| &\\ \leq\ & \| g_{n+1} - g_n \| + \eta_n \| h - g_n \| \\ & + \gamma_n \| P_K(I-\xi_n(I-J))h_n - g_n \|,\end{aligned}$$

(29), (35), condition (i) and (ii), we obtain

$$\lim_{n\to\infty} \| Gg_n - g_n \| \ = \ 0. \quad (36)$$

Step 5. We'll demonstrate that $\lim_{n\to\infty} \| h_n - g_n \| = 0$. Using the same technique as [5] and the nonexpansiveness of T_{p_n}, we have

$$\begin{aligned}\| h_n - e \|^2 &= \| T_{p_n}(I-p_nY)x_n - T_{p_n}(I-p_nY)e \|^2 \\ &\leq \| g_n - e \|^2 \\ &\quad -p_n(2\gamma - p_n) \| Yg_n - Ye \|^2 . \quad (37)\end{aligned}$$

From (20) and (37), we have

$$\begin{aligned}\| g_{n+1} - e \|^2 &\\ =\ & \| \eta_n(h-e) + \zeta_n(Gg_n - e) \\ & + \gamma_n(P_K(I-\xi_n(I-J)h_n - e) \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| Gg_n - e \|^2 \\ & + \gamma_n \| P_K(I-\xi_n(I-J))h_n - e \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| g_n - e \|^2 + \gamma_n \| h_n - e \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| g_n - e \|^2 \\ & + \gamma_n(\| g_n - e \|^2 - p_n(2\gamma - p_n) \| Yg_n - Ye \|^2) \\ =\ & \eta_n \| h-e \|^2 + (1-\eta_n) \| g_n - e \|^2 \\ & - p_n\gamma_n(2\gamma - p_n) \| Yg_n - Ye \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \| g_n - e \|^2 \\ & - p_n\gamma_n(2\gamma - p_n) \| Yg_n - Ye \|^2 . \quad (38)\end{aligned}$$

It implies that

$$\begin{aligned}p_n\gamma_n(2\gamma - p_n) \| Yg_n - Ye \|^2 &\\ \leq\ & \eta_n \| h-e \|^2 + \| g_n - e \|^2 - \| g_{n+1} - e \|^2 \\ =\ & \eta_n \| h-e \|^2 \quad (39) \\ & + (\| g_n - e \| + \| g_{n+1} - e \|) \| g_{n+1} - g_n \| .\end{aligned}$$

From (29), (39), condition (i) and (ii), we obtain

$$\lim_{n\to\infty} \| Yg_n - Ye \| \ = \ 0. \quad (40)$$

From definition of g_n and (30), we obtain

$$\begin{aligned}\| g_{n+1} - e \|^2 &\\ =\ & \| \eta_n(h-e) + \zeta_n(Gg_n - e) \\ & + \gamma_n(P_K(I-\xi_n(I-J))h_n - e) \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| Gg_n - e \|^2 \\ & + \gamma_n \| P_K(I-\xi_n(I-J))h_n - e \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| g_n - e \|^2 + \gamma_n \| h_n - e \|^2 \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| g_n - e \|^2 + \gamma_n\big(\| g_n - e \|^2 \\ & - \| g_n - h_n \|^2 - p_n^2 \| Yg_n - Ye \|^2 \\ & + 2p_n\langle g_n - h_n, Yg_n - Ye\rangle\big) \\ \leq\ & \eta_n \| h-e \|^2 + \zeta_n \| g_n - e \|^2 + \gamma_n \| g_n - e \|^2 \\ & - \gamma_n \| g_n - h_n \|^2 \\ & + 2p_n\gamma_n \| g_n - h_n \| \| Yg_n - Ye \| \\ =\ & \eta_n \| h-e \|^2 + (1-\eta_n) \| g_n - e \|^2 \\ & - \gamma_n \| g_n - h_n \|^2 \\ & + 2p_n\gamma_n \| g_n - h_n \| \| Yg_n - Ye \| \\ \leq\ & \eta_n \| h-e \|^2 + \| g_n - e \|^2 - \gamma_n \| g_n - h_n \|^2 \\ & + 2p_n\gamma_n \| g_n - h_n \| \| Yg_n - Ye \|,\end{aligned}$$

it suggests that

$$\begin{aligned}\gamma_n \| g_n - h_n \|^2 &\\ \leq\ & \eta_n \| h-e \|^2 + \| g_n - e \|^2 - \| g_{n+1} - e \|^2 \\ & + 2p_n\gamma_n \| g_n - h_n \| \| Yg_n - Ye \| \\ \leq\ & \eta_n \| h-e \|^2 \quad (41) \\ & + (\| g_n - e \| + \| g_{n+1} - e \|) \| g_{n+1} - g_n \| \\ & + 2p_n\gamma_n \| g_n - h_n \| \| Yg_n - Ye \| .\end{aligned}$$

From (29), (40), (41), condition (i) and (ii), we obtain

$$\lim_{n\to\infty} \| g_n - h_n \| \ = \ 0. \quad (42)$$

Step 6. We'll demonstrate that $\limsup_{n\to\infty}\langle h - e_0, g_n - e_0\rangle \leq 0$, where $e_0 = P_F h$. To show this equality, take a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that

$$\limsup_{n\to\infty}\langle h - e_0, g_n - e_0\rangle = \lim_{k\to\infty}\langle h - e_0, g_{n_k} - e_0\rangle. \quad (43)$$

Without loss of generality, we may assume that $g_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in K$. First, we demonstrate that $\omega \in EP(F,Y)$, where $Y = w\widetilde{A} + (1-w)\widetilde{B}$ for all $w \in [0,1]$. From (42), we have $h_{n_k} \rightharpoonup \omega$ as $k \to \infty$. From (14), we obtain

$$F(h_n, d) + \langle Yg_n, d - h_n\rangle + \frac{1}{p_n}\langle d - h_n, h_n - g_n\rangle \geq 0,\ \forall d \in K.$$

From (G2), we have

$$\langle Yg_n, d - h_n\rangle + \frac{1}{p_n}\langle d - h_n, h_n - g_n\rangle \geq F(d, h_n).$$

Then

$$\langle Yg_{n_k}, d - h_{n_k}\rangle + \frac{1}{p_{n_k}}\langle d - h_{n_k}, h_{n_k} - g_{n_k}\rangle \geq F(d, h_{n_k}) \quad (44)$$

for all $d \in K$. Put $e_t = td + (1-t)\omega$ for all $t \in (0,1]$ and $d \in K$. Then, we have $e_t \in K$. So, from (44), we have

$$\begin{aligned}
&\langle e_t - h_{n_k}, Ye_t\rangle \\
&\quad\geq \langle e_t - h_{n_k}, Ye_t\rangle - \langle e_t - h_{n_k}, Yg_{n_k}\rangle \\
&\qquad -\langle e_t - h_{n_k}, \frac{h_{n_k}-g_{n_k}}{p_{n_k}}\rangle + F(e_t, h_{n_k}) \\
&\quad= \langle e_t - h_{n_k}, Ye_t - Yh_{n_k}\rangle \\
&\qquad +\langle e_t - h_{n_k}, Yh_{n_k} - Yg_{n_k}\rangle \\
&\qquad -\langle e_t - h_{n_k}, \frac{h_{n_k}-g_{n_k}}{p_{n_k}}\rangle + F(e_t, h_{n_k}). \quad (45)
\end{aligned}$$

Since $\| h_{n_k} - g_{n_k} \| \to 0$, we have $\| Yh_{n_k} - Yg_{n_k} \| \to 0$. Further, from monotonicity of Y, we obtain

$$\langle e_t - h_{n_k}, Ye_t, Yh_{n_k}\rangle \geq 0.$$

So, from (G4) we have

$$\langle e_t - \omega, Ye_t\rangle \geq F(e_t, \omega) \quad as \quad k \to \infty. \quad (46)$$

From (G1),(G4) and (46), we also have

$$\begin{aligned}
0 &= F(e_t, e_t) \\
&\leq tF(e_t, d) + (1-t)F(e_t, \omega) \\
&\leq tF(e_t, d) + (1-t)\langle e_t - \omega, Ye_t\rangle \\
&= tF(e_t, d) + (1-t)t\langle d - \omega, Ye_t\rangle
\end{aligned}$$

hence

$$0 \leq F(e_t, d) + (1-t)\langle d - \omega, Ye_t\rangle.$$

Letting $t \to 0^+$, we obtain

$$0 \leq F(\omega, d) + \langle d - \omega, Y\omega\rangle \ \forall d \in K. \quad (47)$$

Therefore

$$\omega \in EP(F, Y), \quad (48)$$

where $Y = w\widetilde{A} + (1-w)\widetilde{B}$ for all $w \in [0, 1]$. Since

$$\begin{aligned}
&\| P_K(I - \xi_n(I-J))h_n - h_n \| \\
&\quad\leq \| P_K(I - \xi_n(I-J))h_n - g_n \| + \| g_n - h_n \|,
\end{aligned}$$

(35) and (42), we obtain

$$\lim_{n\to\infty} \| P_K(I - \xi_n(I-J))h_n - h_n \| = 0. \quad (49)$$

From Remark 2.9, we have $F(J) = F(P_K(I - \xi_{n_k}(I-J)))$. Assume that $\omega \neq P_K(I - \xi_{n_k}(I-J))\omega$. Since $h_{n_k} \rightharpoonup \omega$ as $k \to \infty$, Opial's property, (49) and condition (i), we obtain

$$\begin{aligned}
&\liminf_{k\to\infty} \| h_{n_k} - \omega \| \\
&\quad< \liminf_{k\to\infty} \| h_{n_k} - P_K(I - \xi_{n_k}(I-J))\omega \| \\
&\quad\leq \liminf_{k\to\infty} \big(\| h_{n_k} - P_K(I - \xi_{n_k}(I-J))h_{n_k} \| \\
&\qquad + \| P_K(I - \xi_{n_k}(I-J))h_{n_k} \\
&\qquad - P_K(I - \xi_{n_k}(I-J))\omega \| \big) \\
&\quad\leq \liminf_{k\to\infty} \big(\| h_{n_k} - P_K(I - \xi_{n_k}(I-J))h_{n_k} \| \\
&\qquad + \| h_{n_k} - \omega \| + \xi_{n_k} \| (I-J)h_{n_k} - (I-J)\omega \| \big) \\
&\quad= \liminf_{k\to\infty} \| h_{n_k} - \omega \| . \quad (50)
\end{aligned}$$

This is a contradiction. Then

$$\omega \in F(J). \quad (51)$$

From (36), we obtain

$$\lim_{k\to\infty} \| Gg_{n_k} - g_{n_k} \| = 0.$$

From the nonexpansiveness of $G, g_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and Lemma 2.5, we obtain

$$\omega \in F(G). \quad (52)$$

From (48), (51), and (52), we have $\omega \in F$. Since $g_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in F$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle h - e_0, g_n - e_0\rangle &= \lim_{k\to\infty} \langle h - e_0, g_{n_k} - e_0\rangle \\
&= \langle h - e_0, \omega - e_0\rangle \\
&\leq 0. \quad (53)
\end{aligned}$$

Step 7. Finally, we show that $\{g_n\}$ converges strongly to $e_0 = P_F h$. From definition of g_n, (20) and let $E = I - J$, we obtain

$$\begin{aligned}
&\| g_{n+1} - e_0 \|^2 \\
&\quad= \| \eta_n(h - e_0) + \zeta_n(Gg_n - e_0) \\
&\qquad + \gamma_n(P_K(I - \xi_n E)h_n - e_0) \|^2 \\
&\quad\leq \| \zeta_n(Gg_n - e_0) + \gamma_n(P_K(I - \xi_n E)h_n - e_0) \|^2 \\
&\qquad + 2\eta_n\langle h - e_0, g_{n+1} - e_0\rangle \\
&\quad\leq \zeta_n \| Gg_n - e_0 \|^2 + \gamma_n \| P_K(I - \xi_n E)h_n - e_0 \|^2 \\
&\qquad + 2\eta_n\langle h - e_0, g_{n+1} - e_0\rangle \\
&\quad\leq \zeta_n \| g_n - e_0 \|^2 + \gamma_n \| h_n - e_0 \|^2 \quad (54) \\
&\qquad + 2\eta_n\langle h - e_0, g_{n+1} - e_0\rangle \\
&\quad= \zeta_n \| g_n - e_0 \|^2 + \gamma_n \| T_{p_n}(I - p_n D)g_n - e_0 \|^2 \\
&\qquad + 2\eta_n\langle h - e_0, g_{n+1} - e_0\rangle \\
&\quad\leq (1 - \eta_n) \| g_n - e_0 \|^2 + 2\eta_n\langle h - e_0, g_{n+1} - e_0\rangle.
\end{aligned}$$

From (53) and Lemma 2.4, we have $\{g_n\}$ converges strongly to $e_0 = P_F h$. The proof is finished with this. ■

Remark 3.1: From Theorem 3.1, putting $F(G) = VI(K, \widetilde{A}) \cap VI(K, \widetilde{B})$, we have $\{g_n\}$ converges strongly to $e_0 = P_F h$.

IV. Applications

We derive Theorems 4.5 and 4.6 in this section, which provide solutions to the general split feasibility problem and the variational inequality problem.

Let H_1 and H_2 be real Hilbert spaces and K, M be nonempty closed convex subsets of H_1 and H_2, correspondingly. Let $\widetilde{A}, \widetilde{B} : H_1 \to H_2$ be bounded linear operators with $\widetilde{A}^*, \widetilde{B}^*$ are adjoint of $\widetilde{A}$ and $\widetilde{B}$, correspondingly.

Finding a point $g \in K$ and $\widetilde{A}g \in M$ is the split feasibility problem (SFP). Censor and Elfving [17] introduced this problem. $\Lambda = \{g \in K : \widetilde{A}g \in M\}$ represents the set of all SFP solutions. The split feasibility problem has been thoroughly studied as a very potent tool in many different domains, including resolution enhancement, signal processing, sensor networks, medical image reconstruction, and computer tomography (see [18]).

Many authors utilize the lemma proposed by Ceng, Ansari, and Yao [19] in 2012 to support their findings while solving SFP (see [20]).

After that Kangtunyakarn [21] modified SFP, he introduce the general split feasibility problem (GSFP) which is to find a point $g^* \in K$ and $\widetilde{A}g^*, \widetilde{B}g^* \in M$. The set of this solution is denoted by $\Lambda = \{g \in K : \widetilde{A}g, \widetilde{B}g \in M\}$. In the case of

$\widetilde{A} \equiv \widetilde{B}$, GSFP can be reduced to SFP. In addition, he also proved the following property of GSFP problem,

Lemma 4.1 ([21]): Let $\Lambda \neq \phi$. Then the followings are equivalent.

(i) $g^* \in \Lambda$,

(ii) $P_K(I - a(\frac{\widetilde{A}^*(I-P_M)\widetilde{A}}{2} + \frac{\widetilde{B}^*(I-P_M)\widetilde{B}}{2}))g^* = g^*$,

for all $a > 0$ and $L_{\widetilde{A}}, L_{\widetilde{B}}$ are spectal redius of $\widetilde{A}^*\widetilde{A}$ and $\widetilde{B}^*\widetilde{B}$, correspondingly with $a \in (0, \frac{2}{L})$ and $L = max\{L_{\widetilde{A}}, L_{\widetilde{B}}\}$.

We derive Theorem 4.6 from these findings, and we require the following Lemma in order to demonstrate Theorem 4.5.

Lemma 4.2: Let K be a nonempty closed convex subset of H. Let $J : K \to K$ be a nonexpansive mapping with $F(J) \neq \phi$. Then $F(J) = VI(K, (I - J))$.

Theorem 4.3: Let K be a closed convex subset of Hilbert space H and let $F : K \times K \to R$ be a function satisfying (G1) - (G4), let $\widetilde{A}, \widetilde{B}, A'', B'' : K \to H$ be $\widetilde{\alpha}, \widetilde{\beta}, \alpha'', \beta''$-ism, correspondingly. Define $G : K \to K$ by $Gg = P_K(I - \xi_1 A'')(wg + (1 - w)P_K(I - \xi_2 B'')g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha'')$ and $\xi_2 \in (0, 2\beta'')$. Let $J : K \to K$ be κ-strictly pseudononspreading mapping with $F = F(J) \cap F(G) \cap VI(K, \widetilde{A}) \cap VI(K, \widetilde{B}) \neq \phi$ for all $w \in (0, 1)$. Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$g_{n+1} = \eta_n h + \zeta_n G g_n + \gamma_n S_n g_n, \quad \forall n \geq 1. \tag{55}$$

where $S_n = P_K(I - \xi_n(I - J))P_K(I - p_n(w\widetilde{A} + (1 - w)\widetilde{B}))$ and $\{\eta_n\}, \{\zeta_n\}, \{\gamma_n\} \subset [0, 1], \xi_n \in (0, 1 - \kappa), \eta_n + \zeta_n + \gamma_n = 1, \forall n \in N$, $\{p_n\} \subset [0, 2\gamma], \gamma = min\{\widetilde{\alpha}, \widetilde{\beta}\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\lim_{n\to\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $0 < o \leq \zeta_n \leq p < 1$, $0 < q \leq p_n \leq m < 2\gamma$;

(iii) $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$.

Then $\{g_n\}$ converges strongly to $e_0 = P_F h$.

Proof: Using $F \equiv 0$ from (12) in Theorem 3.1, we obtain

$$\langle h_n - (I - p_n Y)g_n, d - h_n \rangle \;\geq\; 0, \quad \forall d \in K,$$

where $Y = w\widetilde{A} + (1 - w)\widetilde{B}$, $\forall w \in [0, 1]$. From Lemma 2.1, we have

$$h_n = P_K(I - p_n Y)g_n. \tag{56}$$

Then, we have (55). Based on Theorem 3.1, we may arrive to the intended result.

■

Theorem 4.4: Let K be a closed convex subset of Hilbert space H and let $F : K \times K \to R$ be a function satisfying (G1) - (G4), let $\widetilde{A}, \widetilde{B}, A'', B'' : K \to H$ be $\widetilde{\alpha}, \widetilde{\beta}, \alpha'', \beta''$-ism, correspondingly. Define $G : K \to K$ by $Gg = P_K(I - \xi_1 A'')(bg + (1 - b)P_K(I - \xi_2 B'')g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha'')$ and $\xi_2 \in (0, 2\beta'')$. Let $J : K \to K$ be κ-strictly pseudononspreading mapping with $F = F(J) \cap F(G) \cap EP(F, \widetilde{A}) \neq \phi$ for all $b \in (0, 1)$. Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$\begin{cases} F(h_n, d) + \langle \widetilde{A} g_n, d - h_n \rangle + \frac{1}{p_n}\langle d - h_n, h_n - g_n \rangle \geq 0, \\ g_{n+1} = \eta_n h + \zeta_n G g_n + \gamma_n P_K(I - \xi_n(I - J))h_n, \end{cases} \tag{57}$$

for all $d \in K$ and $n \geq 1$ with $\{\eta_n\}, \{\zeta_n\}, \{\gamma_n\} \subset [0, 1], \xi_n \in (0, 1 - \kappa), \eta_n + \zeta_n + \gamma_n = 1, \forall n \in N$ and $\{p_n\} \subset [0, 2\gamma], \gamma = min\{\widetilde{\alpha}, \widetilde{\beta}\}$ satisfy;

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\lim_{n\to\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $0 < o \leq \zeta_n \leq p < 1$, $0 < q \leq p_n \leq m < 2\gamma$;

(iii) $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$.

Then $\{g_n\}$ converges strongly to $e_0 = P_F h$.

Proof: By using $\widetilde{A} \equiv \widetilde{B}$, we may get the intended result from Theorem 3.1.

■

Theorem 4.5: Let K be a closed convex subset of Hilbert space H and let $F : K \times K \to R$ be a function satisfying (G1) - (G4), let $S, S' : K \to K$ be nonexpansive mapping and let $A'', B'' : K \to H$ be α'', β''-ism, correspondingly. Define $G : K \to K$ by $Gg = P_K(I - \xi_1 A'')(wg + (1 - w)P_K(I - \xi_2 B'')g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha'')$ and $\xi_2 \in (0, 2\beta'')$. Let $J : K \to K$ be κ-strictly pseudononspreading mapping with $F = F(J) \cap F(G) \cap F(S) \cap F(S') \neq \phi$ for all $w \in (0, 1)$. Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$g_{n+1} = \eta_n h + \zeta_n G g_n + \gamma_n K_n g_n, \forall n \geq 1. \tag{58}$$

where $K_n = P_K(I - \xi_n(I - T))P_K(I - p_n(b(I - S) + (1 - b)(I - S')))$ and $\{\eta_n\}, \{\zeta_n\}, \{\gamma_n\} \subset [0, 1], \xi_n \in (0, 1 - \kappa), \eta_n + \zeta_n + \gamma_n = 1, \forall n \in N$, $\{p_n\} \subset [0, 2\gamma], 0 < \gamma < \frac{1}{2}$ satisfy;

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\lim_{n\to\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $0 < o \leq \zeta_n \leq p < 1$, $0 < q \leq p_n \leq m < 2\gamma$;

(iii) $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$.

Then $\{g_n\}$ converges strongly to $e_0 = P_F h$.

Proof: The result is obtained by using Lemma 4.2 and Theorem 4.3.

■

Theorem 4.6: Let K, M be a closed convex subset of Hilbert space H_1, H_2 respectively and let $F : K \times K \to R$ be a function satisfying (G1) - (G4) , let $A'', B'' : K \to H_1$ be α'', β''-ism, correspondingly. Let $A_i, B_i : H_1 \to H_2$ be bounded linear operator with A_i^*, B_i^* are adjoint of A_i and B_i, correspondingly and $L = max\{L_{A_i}, L_{B_i}\}$ where L_{A_i} and L_{B_i} are spectal radius of $A_i^* A_i$ and $B_i^* B_i$ with $i = 1, 2$. Define $G : K \to K$ by $Gg = P_K(I - \xi_1 A'')(wg + (1 - w)P_K(I - \xi_2 B'')g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha'')$ and $\xi_2 \in (0, 2\beta'')$. Let $J : K \to K$ be κ-strictly pseudononspreading mapping. Assume that $F = F(J) \cap F(G) \cap \Lambda_1 \cap \Lambda_2 \neq \phi$, where $\Lambda_i = \{g \in K : A_i g, B_i g \in M\}$ for all $i = 1, 2$ and $w \in (0, 1)$. Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$g_{n+1} \;=\; \eta_n h + \zeta_n G g_n + \gamma_n W_n g_n, \forall n \geq 1. \tag{59}$$

where $W_n = P_K\big(I - \xi_n(I - J)\big)P_K\big(I - p_n(W_1 + W_2)\big)$, $W_1 = w(I - P_K(I - w(\frac{A_1^*(I-P_M)A_1}{2} + \frac{B_1^*(I-P_M)B_1}{2})))$, $W_2 = (1 - w)(I - P_K(I - w(\frac{A_2^*(I-P_M)A_2}{2} + \frac{B_2^*(I-P_M)B_2}{2})))$ and $\{\eta_n\}, \{\zeta_n\}, \{\gamma_n\} \subset [0, 1], \xi_n \in (0, 1 - \kappa), \eta_n + \zeta_n + \gamma_n = 1, \forall n \in N$, $\{p_n\} \subset [0, 2\gamma], 0 < \gamma < \frac{1}{2}$ satisfy;

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\lim_{n\to\infty} \eta_n = 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $0 < o \leq \zeta_n \leq p < 1$, $0 < q \leq p_n \leq m < 2\gamma$;

(iii) $\lim_{n\to\infty} |p_{n+1} - p_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$, $\sum_{n=1}^{\infty} |\zeta_{n+1} - \zeta_n| < \infty$.

Then $\{g_n\}$ converges strongly to $e_0 = P_F h$.

Proof: We obtain the required result by applying Lemma 4.1 and Theorem 4.5. ■

V. Example and Numerical Results

We provide a numerical example in this section to bolster our primary theorem.

Example 5.1: Let R be the set of real numbers, $K = [-50, 50]$, and $H = R$. Let $F : K \times K \to R$ defined by $F(x,y) = -5x^2 + xy - 4y^2$ for all $x, y \in K$. Let $\widetilde{A}, \widetilde{B}, A'', B'' : K \to H$ defined by $\widetilde{A}x = x + \frac{2}{3}, \widetilde{B}x = x - \frac{4}{3}, A''x = \frac{2x+1}{2}, B''x = \frac{3x-7}{3}$ for all $x \in K$. Define $G : K \to K$ by $Gx = P_K(I - \frac{1}{2}A'')(\frac{1}{2}x + \frac{1}{2}P_K(I - \frac{3}{7}B'')x)$ for all $x \in K$. Let $J : K \to K$ defined by $Jx = x$ for all $x \in K$. It is easy to show that $\widetilde{A}, \widetilde{B}, A'', B''$ are 1-ism, F is satisfied (G1) - (G4), and J is $\frac{1}{5}$-strictly pseudononspreading. It is clear that $F(J) \cap F(G) \cap EP(F, w\widetilde{A} + (1-w)\widetilde{B}) = \{0\}$. Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by (12). By the definition of F and choose $w = \frac{1}{2} \in (0,1)$, we have

$$\begin{aligned}
0 &\leq F(h_n, d) + \langle (w\widetilde{A} + (1-w)\widetilde{B})g_n, d - h_n\rangle \\
&\quad + \frac{1}{p_n}\langle d - h_n, h_n - g_n\rangle \\
&= (-5h_n^2 + h_n d + 4d^2) + (g_n)(d - h_n) \\
&\quad + \frac{1}{p_n}(d - h_n)(h_n - g_n) \\
&= (-5h_n^2 + h_n d + 4d^2) + (g_n d - g_n h_n) \\
&\quad + \frac{1}{p_n}(h_n d - g_n d - h_n^2 + h_n g_n)
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
0 &\leq p_n(-5h_n^2 + h_n d + 4d^2) + p_n(g_n d - g_n h_n) \\
&\quad + (h_n d - g_n d - h_n^2 + h_n g_n) \\
&= -5p_n h_n^2 + p_n h_n d + 4p_n d^2 + p_n g_n d \\
&\quad - p_n g_n h_n + h_n d - g_n d - h_n^2 + h_n g_n \\
&= 4p_n d^2 + (p_n h_n + p_n g_n + h_n - g_n)d \\
&\quad - 5p_n h_n^2 - p_n g_n h_n - h_n^2 + h_n g_n.
\end{aligned}$$

Let $Q(y) = 4p_n d^2 + (p_n h_n + p_n g_n + h_n - g_n)d - 5p_n h_n^2 - p_n g_n h_n - h_n^2 + h_n g_n$. Then $Q(d)$ is quadratic function of d with coefficient $a = 4p_n,\ b = p_n h_n + p_n g_n + h_n - g_n,\ c = -5p_n h_n^2 - p_n g_n h_n - h_n^2 + h_n g_n$. Determine the discriminant Δ of Q as follow:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (p_n h_n + p_n g_n + h_n - g_n)^2 \\
&\quad - 4(4p_n)(-5p_n h_n^2 - p_n g_n h_n - h_n^2 + h_n g_n) \\
&= p_n^2 h_n^2 + p_n^2 g_n h_n + p_n h_n^2 - p_n g_n h_n \\
&\quad + p_n^2 g_n h_n + p_n^2 g_n^2 + p_n g_n h_n - p_n g_n^2 + p_n h_n^2 \\
&\quad + p_n g_n h_n + h_n^2 - h_n g_n - p_n g_n h_n - p_n g_n^2 \\
&\quad - g_n h_n + g_n^2 \\
&\quad - 16p_n(-5p_n h_n^2 - p_n g_n h_n - h_n^2 + h_n g_n) \\
&= p_n^2 h_n^2 + p_n^2 g_n h_n + p_n h_n^2 + p_n^2 g_n h_n + p_n^2 g_n^2 \\
&\quad - p_n g_n^2 + p_n h_n^2 + h_n^2 - h_n g_n - p_n g_n^2 - g_n h_n \\
&\quad + g_n^2 + 80p_n^2 h_n^2 + 16p_n^2 g_n h_n \\
&\quad + 16p_n h_n^2 - 16p_n h_n g_n \\
&= h_n^2 + 18p_n h_n^2 + 81p_n^2 h_n^2 + 18p_n^2 g_n h_n - 2h_n g_n \\
&\quad - 16p_n h_n g_n + p_n^2 g_n^2 - 2p_n g_n^2 + g_n^2 \\
&= (h_n + 9p_n h_n)^2 + 2(h_n + 9p_n h_n)(p_n - 1)(g_n) \\
&\quad + \big((p_n - 1)g_n\big)^2 \\
&= (h_n + 9p_n h_n + (p_n - 1)g_n)^2 \\
&= (h_n + 9p_n h_n + p_n x_n - g_n)^2.
\end{aligned}$$

For any y in R, we know that $Q(d) \geq 0$. If R has just one solution, then $\Delta \leq 0$, leading to the following result:

$$h_n = \frac{1 - p_n}{1 + 9p_n} g_n. \qquad (60)$$

Put $\eta_n = \frac{1}{3n}, \zeta_n = \frac{3n-2}{3n}, \gamma_n = \frac{1}{3n}, \xi_n = \frac{1}{n(n+1)}, p_n = \frac{n}{n+1}$ $\forall n \in N$. For every $n \in N$, from (60) we rewrite (12) as follows:

$$\begin{aligned}
g_{n+1} &= \frac{1}{3n}h + \frac{3n-2}{3n}Gg_n \\
&\quad + \frac{1}{3n}P_K\Big(I - \frac{1}{n(n+1)}(I - J)\Big)\frac{1-p_n}{1+9p_n}g_n, \forall n \geq 1.
\end{aligned}$$

It is clear that the sequences $\{\eta_n\}, \{\zeta_n\}, \{\gamma_n\}, \{\xi_n\}$ and $\{p_n\}$ satisfy all the conditions of Theorem 3.1. The sequences $\{g_n\}$ and $\{h_n\}$ converge strongly to 0, as shown by Theorem 3.1.

TABLE I

The values of the $hn = un$ sequence and the $xn = gn$ sequence with initial values $h = g1 = -10$ and $h = g1 = 10$ with $n = 30$.

n	$h = g1 = -10$		$h = g1 = 10$	
	hn	gn	hn	gn
1	−0.90909	−10.00000	0.	10.00000
2	−0.15615	−3.27922	0.30045	6.30952
3	−0.04558	−1.41284	0.09027	2.79847
⋮	⋮	⋮	⋮	⋮
15	−0.00000	−0.00003	0.00000	0.00005
⋮	⋮	⋮	⋮	⋮
28	−0.00000	−0.00000	0.00000	0.00000
29	−0.00000	−0.00000	0.00000	0.00000
30	−0.00000	−0.00000	0.00000	0.00000

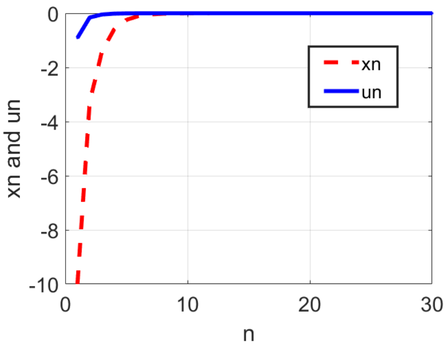

Fig. 1. The convergence of $\{hn\}$ and $\{gn\}$ with initial values $h = g1 = -10$ and $n = 30$.

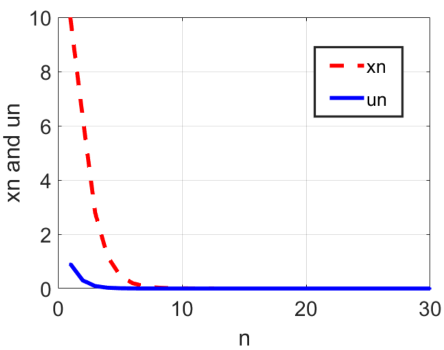

Fig. 2. The convergence of $\{hn\}$ and $\{gn\}$ with initial values $h = g1 = 10$ and $n = 30$.

VI. Conclusion

1) We derive Remark 3.1 from Theorem 3.1.

2) We get a new method for solve the combination of variational inequality problem and equilibrium problem.
3) Applying our main result to solve the general split feasibility problem.
4) The sequences $\{g_n\}$ and $\{h_n\}$ converge to 0, as Table I, Figure 1, and Figure 2 demonstrate. Here, $\{0\} = F(J) \cap F(G) \cap EP(F, w\widetilde{A} + (1-w)\widetilde{B})$.
5) In Example 5.1, the convergence of $\{g_n\}$ and $\{h_n\}$ is ensured by Theorem 3.1.

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