An Approximation Method of Nonlinear Mapping for a Modified General Equilibrium and System of Variational Inequality Problems

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Abstract—In this article, we present a new iteration for approximating the solutions to generalized equilibrium problems, fixed point problems of \(\kappa\)-strictly pseudononsprading mapping, and modifications of the system of variational inequality. The variational inequality problem and the general split feasibility problem were then solved using our primary theorem. Additionally, we provide a numerical example to support our primary theorem.

Index Terms—fixed points, variational inequalities, equilibrium problems, \(\kappa\)-strictly pseudononsprading mapping, inverse-strongly monotone.

I. INTRODUCTION

Let \(K\) be a nonempty closed convex subset of \(H\) and let \(H\) be a real Hilbert space for the purposes of this article. Assume \(O : K \rightarrow H\). Making a point \(j \in K\) to the extent that

\[(Oj, h - j) \geq 0\]  
(1)

for all \(h \in K\) referred to be the variational inequality problem (VIP).

The set of solutions to (1) is indicated by

\[\{\lambda_1D_1e^* + l^* - e^*, l - e^*\} \geq 0, \forall l \in K,\]

which is called a system of variational inequalities problem (SVIP) where \(D_1, D_2 : K \rightarrow H\) are mappings and parameters \(\lambda_1, \lambda_2 > 0\). In the case of \(\lambda_1 = \lambda_2, D_1 = D_2, l^* = e^*\), VIP is reduced to VIP.

After that, Kangtunyakarn [3] modified (2) for finding \((l^*, e^*) \in K \times K\) such that

\[
\begin{align*}
(l^* - (I - \lambda_1D_1)(bl^* + (1 - b)e^*), l - e^*) & \geq 0, \forall l \in K, \\
(e^* - (I - \lambda_2D_2)e^*, l - e^*) & \geq 0, \forall l \in K,
\end{align*}

\]

which is called a system of variational inequalities problem (SVIP) where \(D_1, D_2 : K \rightarrow H\) are mappings and parameters \(\lambda_1, \lambda_2 > 0\). In the case of \(\lambda_1 = \lambda_2, D_1 = D_2, l^* = e^*\), VIP is reduced to VIP.

II. MATHEMATICAL BACKGROUND

Lemma 1.1: Let \(D_1, D_2 : K \rightarrow H\) be mappings. For every \(\lambda_1, \lambda_2 > 0\) and \(b \in [0, 1]\), the statements that follow are interchangeable:

1) \((l^*, z^*) \in K \times K\) is a solution of problem (3),
2) \(l^*\) is a fixed point of the mapping \(U : K \rightarrow K\), i.e.,

\[U(l) = P_K(I - \lambda_1D_1)(bl + (1 - b)P_K(I - \lambda_2D_2)t),\]

and solutions of (3) are interchangeable:

\[\langle r - m, (bP + (1 - b)B)m \rangle \geq 0\]  
(5)

for all \(m, r \in K\) and \(b \in (0, 1)\). The set of CSIP is denoted by \(\Phi \cap \psi\), where \(\Phi\) and \(\psi\) are fixed points of the mappings \(\Phi\) and \(\psi\), respectively.
Let $L : K \times K \to R$ be a bifunction. The equilibrium problem for $L$ is to determine its equilibrium points and denote

$$EP(L) = \{x \in K : L(r, m) \geq 0, \forall m \in K\}$$

(6)
as the set of all solutions to the equilibrium problem.

From (1) and (6), we have the following generalized equilibrium problem, i.e., find $g \in K$ such that

$$L(g, m) + \langle Og, m - g \rangle \geq 0$$

(7)
for all $m \in K$. The set of such $g \in K$ is denoted by $EP(L, O)$. When $O \equiv 0$, $EP(L, O)$ is represented by $EP(L)$. In the case of $L \equiv 0$, $EP(L, O)$ is also denoted by $VI(K, O)$.

Blum and Oettli [6] introduced equilibrium problems in 1994. Solving $EP(L)$ is a reduction of several optimization and economics problems (see [6]). The iterative approach for identifying a common element between the set of solutions to the equilibrium problems and the set of solutions to the fixed point problem has garnered attention from several writers recently [7].

In 2008, Takahashi and Takahashi [8] introduced a general iterative method for finding a common element between $EP(L, O)$ and $F(J)$. They defined $\{y_n\}$ in the following way:

$$h, y_n \in K, \text{ arbitrarily};$$
$$L(t_n, m) + \langle Oy_n, m - t_n \rangle + \frac{1}{n} \langle m - t_n, t_n - y_n \rangle \geq 0,$$
$$y_{n+1} = \eta_n y_n + (1 - \eta_n)J(t_n h + (1 - \eta_n)t_n),$$

(8)
for all $m \in K$ and $n \in N$ with $O$ being an $\nu$-ism mapping of $K$ into $H$ with positive real number $\nu$ and $\{t_n\} \in [0, 1]$, $\{\eta_n\} \in [0, 1]$, $\{\rho_n\} \subset [0, 2\alpha]$, and proved strong convergence of the scheme (8) to $t \in F(J) \cap EP(L, O)$, where $t = P_{F(J) \cap EP(L, O)}$, for given suitable constraints on $\{t_n\}$, $\{\eta_n\}$, $\{\rho_n\}$ and $L$.

The following theorem was shown by Inchan [9] by modification of the viscosity approximation method:

**Theorem 1.3:** Let $K \subseteq K \subset K$, and let $J : K \to H$ be a $\nu$-strictly pseudo-contractive mapping with a fixed point for some $0 \leq \nu < 1$. Let $V$ be a strongly positive bounded linear operator on $K$ with coefficient $\beta$ and $J : K \to K$ be a contraction with the strong constant $0 < \theta < 1$ such that $0 < \eta < \frac{\beta}{\theta}$. Let $\{o_n\}$ be generated by

$$o_{n+1} = \eta_n Jf(o_n) + (1 - \eta_n)o_n + ((1 - \eta_n)I - \eta_n V)P_{K}So_n,$$

(9)
where $o_1 \in K$ and $S : K \to H$ is a mapping defined by

$$So = \nu o + (1 - \nu)Jo$$

(10)
If the control sequence $\{\theta_n\}, \{\delta_n\} \subset (0, 1)$ satisfying

(i) $\lim_{n \to \infty} \theta_n = 0$ and $\lim_{n \to \infty} \delta_n = 0$,
(ii) $\sum_{n=1}^{\infty} \theta_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |o_{n+1} - o_n| < \infty$, $\sum_{n=1}^{\infty} |o_{n+1} - s_n| < \infty$.

Then $\{o_n\}$ converges strongly to a fixed point $l$ of $J$, which solves the following solution of VIP:

$$\langle (V - \eta f), l - o \rangle \leq 0$$

for all $o \in F(J)$.

Furthermore, from (5) and (7), we introduce a problem relative to CVIP and equilibrium problems, i.e., find $m \in K$ such that

$$L(m, g) + \langle (\alpha A + (1 - \alpha)B)m, g - m \rangle \geq 0$$

(11)
for all $g \in K$ and $\alpha \in (0, 1)$. The set of all solutions to such problems is denoted by $EP(L, (\alpha A + (1 - \alpha)B))$.

Remember that a mapping $O : K \to K$ is deemed nonexpansive if $\|Om - Os\| \leq \|m - s\|$ for all $m, s \in K$.

The nonspresing mapping in $H$ was presented by Kohsaka and Takahashi [10] in 2008. It is defined as follows:

$$2\|Om - Os\|^2 \leq \|m - s\|^2 + \|m - Os\|^2$$

(12)
for all $m, s \in K$. In 2011, Oslilike and Isiogugu [12] introduced, using terminology from Browder and Petryshyn [11], that a mapping $O : K \to K$ is a $\nu$-strictly pseudononspresing mapping if there exists $\nu \in [0, 1)$ such that

$$\|Om - Os\|^2 \leq \|m - s\|^2 + \nu\|(I - O)m - (I - O)s\|^2$$

(13)
$$+ 2(m - Jm, s - Js)$$

for all $m, s \in K$. It is evident that each nonspresing mapping is $\nu$-strictly pseudononspresing.

A point $h \in C$ is called a fixed point of $J$ if $Jh = h$. The set of fixed points of $J$ is denoted by

$$F(J) = \{h \in K : Jh = h\}.$$

A mapping $R : K \to H$ is called $\tau$-inverse strongly monotone (ism), if there exists a positive real number $\tau$ such that

$$\langle m - s, Rm - Rs \rangle \geq \tau \|Rm - Rs\|^2$$

(14)
for all $m, s \in K$.

Inspired and motivated by Theorem 1.2, (11) and the similar trend of research, we prove a strong convergence theorem for the MSVIP, generalized equilibrium problems and fixed point problems of $\kappa$-strictly pseudononspresing mapping. In addition, we applied our main result to solving the VIP and the general split feasibility problem. In conclusion, we present a numerical example to validate our primary finding.

**II. PRELIMINARIES**

Let $P_K$ be the metric projection of $H$ onto $K$ i.e., for $m \in H$, $P_Km$ satisfies the property

$$\|m - P_Km\| = \min_{s \in K} \|m - s\|.$$

The following characterizes the projection $P_Km$.

**Lemma 2.1 ([13]):** Given $m \in H$ and $s \in K$. Then $P_Km = s$ if and only if there holds the inequality

$$(m - s, s - r) \geq 0$$

for all $r \in K$.

**Lemma 2.2 ([13]):** Let $M$ be a mapping of $K$ into $H$. Let $y \in K$. Then for $\nu > 0$,

$$y = P_K(1 - \nu M)y \Leftrightarrow y \in VI(K, M),$$

where $P_K$ is the metric projection of $H$ onto $K$.

**Lemma 2.3 ([14]):** Let $\{r_n\}$ be a sequence of nonnegative real number satisfying

$$r_{n+1} \leq (1 - \xi_n)r_n + \eta_n, \forall n \geq 0,$$
where \( \{\xi_n\} \) is a sequence in \((0, 1)\) and \( \{\eta_n\} \) is a sequence such that
\[
1) \sum_{n=1}^{\infty} \xi_n = \infty, \\
2) \limsup_{n \to \infty} \frac{\eta_n}{\xi_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\eta_n| < \infty.
\]
Then \( \lim_{n \to \infty} r_n = 0. \)

**Lemma 2.4 ([14]):** Let \( \{r_n\} \) be a sequence of nonnegative real number satisfying
\[
r_{n+1} \leq (1 - \xi_n)s_n + \xi_n\eta_n, \quad \forall n \geq 0,
\]
where \( \{\xi_n\}, \{\eta_n\} \) satisfy the conditions:
\[
1) \{\xi_n\} \subset [0, 1], \sum_{n=1}^{\infty} \xi_n = \infty; \\
2) \limsup_{n \to \infty} \eta_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |\eta_n\xi_n| < \infty.
\]
Then \( \lim_{n \to \infty} r_n = 0. \)

**Lemma 2.5 ([15]):** Let \( W \) be a uniformly convex Banach space, \( K \) be a nonempty closed convex subset of \( W \) and \( D : K \to K \) be a nonexpansive mapping. Then, \( I - D \) is demiclosed at zero.

For solving the equilibrium problem for a bifunction \( F : K \times K \to R \) satisfying (G1) - (G4), let \( L \) be a function satisfying (G1) - (G4), \( g \) be a nonexpansive mapping. Then, \( \{\xi_n\} \) and \( \{\eta_n\} \) be the sequences generated by \( g_1, h \in K \) and
\[
F(h_n, d) + \left( w\tilde{A} + (1-w)\tilde{B} \right) g_n - h_n - \delta_K(h_n, h_n) \geq 0, \quad \forall d \in D, \\
g_{n+1} = \eta_n h + \xi_n Gg_n + \gamma_n P_K(I - \delta_n(I - J)) h_n, \quad \forall n \in N,
\]
where \( \{\eta_n\}, \{\xi_n\}, \{\gamma_n\} \subset [0, 1], \xi_n \in (0, 1), \eta_n + \xi_n + \gamma_n = 1, \forall n \in N \) and \( \{p_n\} \subset [0, 2\gamma], \gamma = \min(\alpha, \beta) \) satisfies:
\[
(i) \sum_{n=1}^{\infty} |\eta_n| = \infty, \lim_{n \to \infty} |\eta_n| = 0, \sum_{n=1}^{\infty} |\xi_n| < \infty; \\
(ii) 0 < \xi_n \leq \eta_n \leq 1, 0 < q \leq p_n \leq m < 2\gamma; \\
(iii) \lim_{n \to \infty} |\xi_{n+1} - \xi_n| = 0; \\
(iv) \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty, \sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty.
\]
Then \( \{g_n\} \) converges strongly to \( e_0 = P_F h. \)

**Proof:** There are seven steps in our proof: Step 1. We’ll demonstrate that \( Y = w\tilde{A} + (1-w)\tilde{B} \) is \( \gamma \)-ism. Let \( g, d \in K \), we have
\[
\langle Yg - Yd, g - d \rangle = \langle w\tilde{A} + (1-w)\tilde{B}, g - d \rangle - \langle w\tilde{A} + (1-w)\tilde{B}, g - d \rangle \\
= \langle \tilde{A}g - \tilde{A}d, g - d \rangle + (1-w)\langle \tilde{B}g - \tilde{B}d, g - d \rangle \\
\geq w\|\tilde{A}g - \tilde{A}d\|^2 + (1-w)\|\tilde{B}g - \tilde{B}d\|^2 \\
= \gamma w\|\tilde{A}g - \tilde{A}d\|^2 + (1-w)\|\tilde{B}g - \tilde{B}d\|^2 \\
\geq \|w\tilde{A}g - w\tilde{A}d + (1-w)\tilde{B}g - \tilde{B}d\|^2 \\
= \|Yg - Yd\|^2.
\]
For each \( n \in N \), we obtain \( I - p_n Y \) as a nonexpansive mapping by applying the same technique as [5].

Step 2. For every \( b \in (0, 1), \) we’ll demonstrate that \( \{g_n\} \) is bounded. Let \( e \in F. \) Deriving from Lemma 2.7, we possess
\[
h_n = T_{p_n}(I - p_n Y) g_n \
\]
and
\[
e = T_{p_n}(I - p_n Y) e
\]
for all \( n \in N. \) Based on Lemma 2.2 and Lemma 2.8, we may obtain
\[
e = P_K(I - \xi_n(I - J)) e
\]
for all \( n \in N. \) By the nonexpansiveness of (16), we have
\[
\|P_K(I - \xi_n(I - J)) h_n - e\| \\
= \|P_K(I - \xi_n(I - J)) h_n - P_K(I - \xi_n(I - J)) e\| \\
\leq \|(I - \xi_n(I - J)) h_n - (I - \xi_n(I - J)) e\|.
\]

### III. Main Result

In this section, we prove strong convergence theorem for approximating the solution of the modification of system variational inequality, generalized equilibrium problems, and fixed point problems of \( \kappa \)-strictly pseudononspreading mapping by modifying Halpern iterative method.
Since $J$ is $\kappa$-strictly pseudononsing mapping and let $E = I - J$, we have

$$
\|Jh_n - Je\|^2 = \|(I - E) h_n - (I - E) e\|^2 \\
= \|(h_n - e) - (Eh_n - Ee)\|^2 \\
= \|h_n - e\|^2 - 2\langle h_n - e, Eh_n \rangle + \|Eh_n\|^2 \\
\leq \|h_n - e\|^2 + \kappa \|Eh_n\|^2,
$$

it suggests that

$$(1 - \kappa) \|Eh_n\|^2 \leq 2 \langle h_n - e, Eh_n \rangle. \tag{18}$$

From (18), we have

$$
\|(I - \xi_n E) h_n - (I - \xi_n E) e\|^2 \\
= \|h_n - e - \xi_n (Eh_n - Ee)\|^2 \\
= \|h_n - e\|^2 - 2\xi_n \langle h_n - e, Eh_n \rangle + \xi_n^2 \|Eh_n\|^2 \\
\leq \|h_n - e\|^2 - \xi_n (1 - \kappa) \|Eh_n\|^2 + \xi_n^2 \|Eh_n\|^2 \\
= \|h_n - e\|^2 - \xi_n (1 - \kappa - \xi_n) \|Eh_n\|^2 \\
\leq \|h_n - e\|^2. \tag{19}
$$

From (17) and (19), we can imply that

$$
\|P_{K^n} (I - \xi_n (I - J)) h_n - e\| \leq \|h_n - e\|. \tag{20}
$$

Since $e \in F$, we have $e = G(e) = P_{K^n} (I - \xi_n A^n (be + (1 - b) P_{K^n} (I - \xi_n B^n)) e)$. Put $M_n = b g_n + (1 - b) P_{K^n} (I - \xi_n B^n) g_n$. Then, we have $g_n g_n = P_{K^n} (I - \xi_n A^n) M_n$. From definition of $g_n$, (20), and nonexpansiveness of $G$, we have

$$
\|g_{n+1} - e\|^2 = \|\eta_n (h_n - e) + \zeta_n (G g_n - e) + \gamma_n (P_{K^n} (I - \xi_n (I - J)) h_n - e)\|^2 \\
\leq \|\eta_n (h_n - e) + \zeta_n (G g_n - e) + \gamma_n (P_{K^n} (I - \xi_n (I - J)) h_n - e)\|^2 \\
\leq \|\eta_n (h_n - e) + \zeta_n (G g_n - e) + \gamma_n (P_{K^n} (I - \xi_n (I - J)) h_n - e)\|^2 \\
\leq \max \{|g_{n+1} - e|, |h_{n+1} - e|\}.
$$

We can demonstrate by induction that both $\{g_n\}$ and $\{h_n\}$ have bounded.

Step 3. We’ll demonstrate that $\lim_{n \to \infty} \|g_{n+1} - g_n\| = 0$. Putting $l_n = g_n - p_n Y g_n$, we obtain $h_n = T_{p_n^n} (g_n - p_n Y g_n) = T_{p_n^n} l_n$. From definition of $h_n$, we obtain

$$
F(h_n, d) + \frac{1}{p_n} \langle d - h_n, h_n - l_n \rangle \geq 0, \quad \forall d \in K \tag{21}
$$

and

$$
F(h_n, h_{n+1}) + \frac{1}{p_n} \langle d - h_n h_{n+1} - l_{n+1} \rangle \geq 0, \quad \forall d \in K. \tag{22}
$$

Instead of $d$ by $h_{n+1}$ and $h_n$ in (21) and (22), correspondingly, we have

$$
F(h_n, h_{n+1}) + \frac{1}{p_n} \langle h_{n+1} - h_n, h_n - l_n \rangle \geq 0 \tag{23}
$$

and

$$
F(h_{n+1}, h_n) + \frac{1}{p_n} \langle h_{n+1} - h_n, h_{n+1} - l_{n+1} \rangle \geq 0. \tag{24}
$$

Adding (23) and (24) and using (G2), we obtain

$$
0 \leq \frac{1}{p_n} \langle h_{n+1} - h_n, h_n - l_n \rangle \\
+ \frac{1}{p_{n+1}} \langle h_n - h_{n+1}, h_{n+1} - l_{n+1} \rangle \\
= \frac{1}{p_n} \langle h_{n+1} - h_n, h_n - l_n \rangle \\
+ \frac{1}{p_{n+1}} \langle h_n - h_{n+1}, h_{n+1} - l_{n+1} \rangle \\
= \frac{1}{p_n} \langle h_{n+1} - h_n, h_n - l_n \rangle \\
- \frac{1}{p_{n+1}} \langle h_n - h_{n+1}, h_{n+1} - l_{n+1} \rangle.
$$

It implies that

$$
0 \leq \frac{1}{p_n} \langle h_{n+1} - h_n, h_n - l_n \rangle - \frac{p_n}{p_{n+1}} \langle h_{n+1} - l_{n+1} \rangle \\
= \frac{1}{p_n} \langle h_{n+1} - h_n, h_n - h_{n+1} + h_{n+1} - l_n \rangle \\
- \frac{p_n}{p_{n+1}} \langle h_{n+1} - l_{n+1} \rangle.
$$

From (25), we obtain

$$
\|h_{n+1} - h_n\|^2 \\
\leq \langle h_{n+1} - h_n, h_{n+1} - l_n \rangle - \frac{p_n}{p_{n+1}} \langle h_{n+1} - l_{n+1} \rangle \\
= \langle h_{n+1} - h_n, h_{n+1} - l_n \rangle \\
- \frac{p_n}{p_{n+1}} \langle h_{n+1} - l_{n+1} \rangle \\
\leq \|h_{n+1} - h_n\| \|h_{n+1} - l_n\| \\
+ \frac{1}{p_n} \langle p_{n+1} - p_n \| h_{n+1} - l_{n+1} \| \rangle.
$$

Hence

$$
\|h_{n+1} - h_n\| \leq \|h_{n+1} - l_n\| \\
+ \frac{1}{q} \langle p_{n+1} - p_n \| h_{n+1} - l_{n+1} \| \rangle. \tag{25}
$$

Since $l_n = g_n - p_n Y g_n$, we obtain

$$
\|l_{n+1} - l_n\| \\
= \| (g_{n+1} - p_{n+1} Y g_{n+1}) - (g_n - p_n Y g_n) \| \\
= \| (I - p_{n+1} Y) g_{n+1} - (I - p_n Y) g_n \| \\
+ \| (I - p_{n+1} Y) g_n - (I - p_n Y) g_n \| \\
\leq \| (I - p_{n+1} Y) g_{n+1} - (I - p_n Y) g_n \| \\
+ \| (p_{n+1} - p_n) Y g_n \| \\
\leq \| g_{n+1} - g_n \| + |p_{n+1} - p_n| \| Y g_n \|.
$$

Substitute (26) into (25), we obtain

$$
\|h_{n+1} - h_n\| \\
\leq \|l_{n+1} - l_n\| + \frac{1}{q} |p_{n+1} - p_n| \| h_{n+1} - l_{n+1} \| \\
\leq \|g_{n+1} - g_n\| + |p_{n+1} - p_n| \| Y g_n \| \\
+ \frac{1}{q} |p_{n+1} - p_n| \| h_{n+1} - l_{n+1} \| \\
\leq \|g_{n+1} - g_n\| + |p_{n+1} - p_n| L + \frac{1}{q} |p_{n+1} - p_n| L.
$$
\[ L = \max_{n \in N} \{ \| Y g_n \|, \| h_n - l_n \| \}. \]

From definition of \( g_n \) and let \( E = I - J \), we obtain

\[
\| g_{n+1} - g_n \| = \| \eta_n h + \zeta_n \| G g_n + \gamma_n P_K(I - \xi_n E) h_n - \eta_{n-1} h \| \leq \frac{1}{2} \left( \| g_n - e \| + \| h_n - e \| \right)^2.
\]

It implies that

\[
\| g_{n+1} - g_n \|^2 \leq \frac{1}{2} \| g_n - e \|^2 + \frac{1}{2} \| h_n - e \|^2 + \frac{1}{2} \| p^*_n \|^2 \| Y g_n - Y e \|^2 + 2p_n \| (g_n - h_n, Y g_n - Y e) \|.
\]

Using the same technique as [5] and the nonexpansiveness of \( T_{p_n} \), we obtain

\[
\| g_{n+1} - e \|^2 \leq \| g_n - e \|^2.
\]

From definition of \( g_n \), (20) and (31), we obtain

\[
\| g_{n+1} - e \|^2 = \| \eta_n (h - e) + \zeta_n (g_n - e) + p_n \| Y g_n - Y e \|^2 + \frac{1}{2} \left( \| g_n - e \| + \| h_n - e \| \right)^2 - \frac{1}{2} \| g_n - e \|^2 - \frac{1}{2} \| h_n - e \|^2 + \frac{1}{2} \| p^*_n \|^2 \| Y g_n - Y e \|^2 + 2p_n \| (g_n - h_n, Y g_n - Y e) \|.
\]

which implies that

\[
\zeta_n \gamma_n \| P_K(I - \xi_n(I - J)) h_n - G g_n \|^2 \leq \| \eta_n (h - e) \|^2 + \| g_n - e \|^2 - \| g_{n+1} - e \|^2 \| h_n - e \|^2 + \| p^*_n \|^2 \| Y g_n - Y e \|^2 + 2p_n \| (g_n - h_n, Y g_n - Y e) \|.
\]

From (29), (32), condition (i) and (ii), we obtain

\[
\lim_{n \to \infty} \| P_K(I - \xi_n(I - J)) h_n - G g_n \| = 0.
\]

Since

\[
\lim_{n \to \infty} \| g_{n+1} - P_K(I - \xi_n(I - J)) h_n \| = 0.
\]
Since
\[ \| g_n - P_K(I - \xi_n(I - J))h_n \| \]
\[ \leq \| g_n - g_{n+1} \|
+ \| g_{n+1} - P_K(I - \xi_n(I - J))h_n \|, \]
(29) and (34), we obtain
\[ \lim_{n \to \infty} \| g_n - P_K(I - \xi_n(I - J))h_n \| = 0. \]  
(35)

Since
\[ \zeta_n \| Gg_n - g_n \|
\leq \| g_{n+1} - g_n \| + \eta_n \| h - g_n \|
+ \gamma_n \| P_K(I - \xi_n(I - J))h_n - g_n \|, \]
(29), (35), condition (i) and (ii), we obtain
\[ \lim_{n \to \infty} \| Gg_n - g_n \| = 0. \]  
(36)

Step 5. We’ll demonstrate that \( \lim_{n \to \infty} \| h_n - g_n \| = 0. \) Using the same technique as [5] and the nonexpansiveness of \( T_{p_n} \), we have
\[ h_n - e \| = \| T_{p_n}(I - p_nY)x_n - T_{p_n}(I - p_nY)e \| \]
\[ \leq \| g_n - e \| ^2
- p_{n}(2\gamma - p_n) \| Yg_n - Ye \| ^2. \]  
(37)

From (20) and (37), we have
\[ \| g_{n+1} - e \| ^2
= \| \eta_n(h - e) + \zeta_n(Gg_n - e)
+ \gamma_n(P_K(I - \xi_n(I - J))h_n - e) \| ^2
\leq \eta_n \| h - e \| ^2 + \zeta_n \| Gg_n - e \| ^2
+ \gamma_n \| P_K(I - \xi_n(I - J))h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \zeta_n \| g_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \zeta_n \| g_n - e \| ^2
+ \gamma_n \| (2\gamma - p_n) \| Yg_n - Ye \| ^2
\| Yg_n - Ye \| ^2
\leq \eta_n \| h - e \| ^2 + (1 - \eta_n) \| g_n - e \| ^2
- p_{n}\gamma_n(2\gamma - p_n) \| Yg_n - Ye \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2
- p_{n}\gamma_n(2\gamma - p_n) \| Yg_n - Ye \| ^2. \]  
(38)

It implies that
\[ p_{n}\gamma_n(2\gamma - p_n) \| Yg_n - Ye \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 - \| g_{n+1} - e \| ^2
= \eta_n \| h - e \| ^2
+(\| g_n - e \| + \| g_{n+1} - e \| ) \| g_{n+1} - g_n \|. \]  
(39)

From (29), (39), condition (i) and (ii), we obtain
\[ \lim_{n \to \infty} \| Yg_n - Ye \| = 0. \]  
(40)

From definition of \( g_n \) and (30), we obtain
\[ \| g_{n+1} - e \| ^2
= \| \eta_n(h - e) + \zeta_n(Gg_n - e)
+ \gamma_n(P_K(I - \xi_n(I - J))h_n - e) \| ^2
\leq \eta_n \| h - e \| ^2 + \zeta_n \| Gg_n - e \| ^2
+ \gamma_n \| P_K(I - \xi_n(I - J))h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \zeta_n \| g_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \zeta_n \| g_n - e \| ^2 + \gamma_n \| \| g_n - e \| ^2
\| + \eta_n \| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\| h_n - e \| ^2 + \gamma_n \| h_n - e \| ^2
\leq \eta_n \| h - e \| ^2 + \| g_n - e \| ^2 + \| g_{n+1} - e \| ^2
\langle e_t - h_n, Y e_t \rangle \\
\geq \langle e_t - h_n, Y e_t \rangle - \langle e_t - h_n_k, Y g_n_k \rangle \\
- \langle e_t - h_n, h_n - g_n \rangle + F(e_t, h_n) \\
= \langle e_t - h_n, Y e_t - Y h_n \rangle \\
+ \langle e_t - h_n, Y h_n - Y g_n \rangle \\
- \langle e_t - h_n, h_n - g_n \rangle + F(e_t, h_n). \tag{45}
\]
Since \( \| h_n - g_n \| \to 0 \), we have \( \| Y h_n - Y g_n \| \to 0 \).
Further, from monotonicity of \( Y \), we obtain
\[ \langle e_t - h_n, Y e_t, Y h_n \rangle \geq 0. \]
So, from (G4) we have
\[ \langle e_t - \omega, Y e_t \rangle \geq F(e_t, \omega) \quad \text{as} \quad k \to \infty. \tag{46} \]
From (G1),(G4) and (46), we also have
\[
0 = F(e_t, e_t) \\
\leq tF(e_t, d) + (1 - t)F(e_t, \omega) \\
\leq tF(e_t, d) + (1 - t)\| e_t - \omega, Y e_t \| \\
= tF(e_t, d) + (1 - t)\| d - \omega, Y e_t \|. 
\]
Letting \( t \to 0^+ \), we obtain
\[ 0 \leq F(\omega, d) + \| d - \omega, Y \omega \| \quad \forall d \in K. \tag{47} \]
Therefore
\[ \omega \in EP(F, Y), \tag{48} \]
where \( Y = w\tilde{A} + (1 - w)\tilde{B} \) for all \( w \in [0, 1] \). Since
\[
\| P_K(I - \xi_n(I - J))h_n - h_n \| \\
\leq \| P_K(I - \xi_n(I - J))h_n - g_n \| + \| g_n - h_n \|, \\
\] (35) and (42), we obtain
\[
\lim_{n \to \infty} \| P_K(I - \xi_n(I - J))h_n - h_n \| = 0. \tag{49} \]
From Remark 2.9, we have \( F(J) = F(P_K(I - \xi_n(I - J))) \).
Assume that \( \omega \neq P_K(I - \xi_n(I - J))\omega \). Since \( h_n \to \omega \) as \( k \to \infty \), Opial’s property, (49) and condition (i), we obtain
\[
\lim_{k \to \infty} \| h_n - \omega \| \\
< \lim_{k \to \infty} \| h_n_k - P_K(I - \xi_n(I - J))\omega \| \\
< \lim_{k \to \infty} \| h_n - P_K(I - \xi_n(I - J))h_n \| \\
+ \| P_K(I - \xi_n(I - J))h_n \| \\
- P_K(I - \xi_n(I - J))\omega \| \\
\leq \lim_{k \to \infty} \| h_n - P_K(I - \xi_n(I - J))h_n \| \\
+ \| h_n - \omega \| + \| \xi_n \| (I - J)h_n - (I - J)\omega \| \\
= \lim_{k \to \infty} \| h_n - \omega \|. \tag{50} \]
This is a contradiction. Then
\[ \omega \in F(J). \tag{51} \]
From (36), we obtain
\[ \lim_{k \to \infty} \| Gg_n_k - g_n_k \| = 0. \]
From the nonexpansiveness of \( G, g_n \to \omega \) as \( k \to \infty \) and Lemma 2.5, we obtain
\[ \omega \in F(G). \tag{52} \]
From (48), (51), and (52), we have \( \omega \in F \). Since \( g_n \to \omega \) as \( k \to \infty \) and \( \omega \in F \), we obtain
\[
\lim_{n \to \infty} \sup \{ h - e_0, g_n - e_0 \} = \lim_{k \to \infty} \langle h - e_0, g_n - e_0 \rangle \\
= \langle h - e_0, \omega - e_0 \rangle \\
\leq 0. \tag{53} \]
Step 7. Finally, we show that \( \{g_n \} \) converges strongly to \( e_0 = P_F h \). From definition of \( g_n \), (20) and let \( E = I - J \), we obtain
\[ \| g_{n + 1} - e_0 \|^2 \\
= \| \eta_n (h - e_0) + \xi_n (Gg_n - e_0) + \gamma_n (P_K(I - \xi_n E)h_n - e_0) \|^2 \\
\leq \| \xi_n (Gg_n - e_0) + \gamma_n (P_K(I - \xi_n E)h_n - e_0) \|^2 \\
+ 2\eta_n \langle h - e_0, g_{n + 1} - e_0 \rangle \\
\leq \xi_n \| Gg_n - e_0 \|^2 + \gamma_n \| P_K(I - \xi_n E)h_n - e_0 \|^2 \\
+ 2\eta_n \langle h - e_0, g_{n + 1} - e_0 \rangle \\
\leq \xi_n \| g_n - e_0 \|^2 + 2\eta_n \| P_K(I - \xi_n E)h_n - e_0 \|^2 \\
+ 2\eta_n \langle h - e_0, g_{n + 1} - e_0 \rangle \\
\leq (1 - \eta_n) \| g_n - e_0 \|^2 + 2\eta_n \langle h - e_0, g_{n + 1} - e_0 \rangle. \tag{54} \]
From (53) and Lemma 2.4, we have \( \{g_n \} \) converges strongly to \( e_0 = P_F h \). The proof is finished with this.

Remark 3.1: From Theorem 3.1, putting \( F(G) = VI(K, \bar{A}) \cap VI(K, \bar{B}) \), we have \( \{g_n \} \) converges strongly to \( e_0 = P_F h \).

IV. Applications

We derive Theorems 4.5 and 4.6 in this section, which provide solutions to the general split feasibility problem and the variational inequality problem.
Let \( H_1 \) and \( H_2 \) be real Hilbert spaces and \( K, M \) be nonempty closed convex subsets of \( H_1 \) and \( H_2 \), respectively. Let \( \tilde{A}, \tilde{B} : H_1 \to H_2 \) be bounded linear operators with \( \tilde{A}^*, \tilde{B}^* \) are adjoint of \( \tilde{A} \) and \( \tilde{B} \), correspondingly.
Finding a point \( q \in K \) and \( \tilde{A}q \in M \) is the split feasibility problem (SFP). Censor and Elfving [17] introduced this problem. \( A = \{ g \in K : \tilde{A}g \in M \} \) represents the set of all SFP solutions. The split feasibility problem has been thoroughly studied as a very potent tool in many different domains, including resolution enhancement, signal processing, sensor networks, medical image reconstruction, and computer tomography (see [18]).

Many authors utilize the lemma proposed by Ceng, Ansari, and Yao [19] in 2012 to support their findings while solving SFP (see [20]).

After that Kangtunyakarn [21] modified SFP, he introduce the general split feasibility problem (GSP) which is to find a point \( g^* \in K \) and \( \tilde{A}g^*, \tilde{B}g^* \in M \). The set of this solution is denoted by \( A = \{ g \in K : \tilde{A}g, \tilde{B}g \in M \} \). In the case of
Theorem 4.5: Let $K$ be a closed convex subset of Hilbert space $H$ and let $F : K \times K \to R$ be a function satisfying $(G1)$ - $(G4)$, let $A^*, B^* : K \to H$ be nonexpansive mapping and let $A^*, B^* : K \to H$ be $\alpha^*, \beta^*$-ism, correspondingly. Define $G : K \to G$ by $Gg = P_K(I - \xi_1A^*)(wg + (1 - w)P_K(I - \xi_2B^*)g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha^n)$ and $\xi_2 \in (0, 2\beta^n)$. Let $J : K \to K$ be $\kappa$-strictly pseudononsparing mapping with $F = F(J) \cap F(G) \cap F(S) \neq \phi$ for all $w \in (0, 1)$. Then $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$g_{n+1} = \eta_n h + \zeta_n Gg_n + \gamma_n S_n g_n, \quad \forall n \geq 1 \tag{55}$$

where $S_n = P_K(I - \xi_n(I - J))P_K(I - p_n(w\tilde{\Phi} + (1 - w))\tilde{B})$ and $\{\eta_n, \{\xi_n, \{\gamma_n \mid 0, 1\}, \zeta_n \in (0, 1 - \kappa), \gamma_n + \zeta_n = 1, \forall n \in N, \{p_n \in [0, 2], \gamma = \min(\alpha, \beta) \} \}$ satisfy:

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\liminf_n \eta_n = 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $0 < \xi_n \leq p < 1$, $0 < q \leq p_n \leq m < 2\gamma$;

(iii) $\liminf_n |p_{n+1} - p_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\xi_n + \zeta_n| < \infty$, $\sum_{n=1}^{\infty} |\zeta_n - 1| < \infty$.

Then $\{g_n\}$ converges strongly to $c_0 = P_Fh$.

Proof: Using $F \equiv 0$ from (12) in Theorem 3.1, we obtain

$$h_n = P_K(I - p_n Y)g_n, \quad \forall d \in K,$$

where $Y = w\tilde{\Phi} + (1 - w)\tilde{B}, \quad \forall w \in [0, 1]$ From Lemma 2.1, we have

$$h_n = P_K(I - p_n Y)g_n \tag{56}$$

Then, we have (55). Based on Theorem 3.1, we may arrive to the intended result.

Theorem 4.6: Let $K, M$ be a closed convex subset of Hilbert space $H_1, H_2$ respectively and let $F : K \times K \to R$ be a function satisfying $(G1)$ - $(G4)$, let $A^*, B^* : K \to H_1$ be $\alpha^*, \beta^*$-ism, correspondingly. Let $A_i, B_i : H_2 \to H_1$ be bounded linear operator with $A_i, B_i$ are adjoint of $A_i$ and $B_i$, correspondingly and $L = \max\{L_{A_i}, L_{B_i}\}$ where $L_{A_i}$ and $L_{B_i}$ are spectral radius of $A_i, B_i$.

Define $G : K \to G$ by $Gg = P_K(I - \xi_1A^*)(wg + (1 - w)P_K(I - \xi_2B^*)g)$ for all $g \in K$ with $\xi_1 \in (0, 2\alpha^n)$ and $\xi_2 \in (0, 2\beta^n)$. Let $J : K \to K$ be $\kappa$-strictly pseudononsparing mapping with $F = F(J) \cap F(G) \cap F(S) \neq \phi$ for all $b \in (0, 1)$, Let $\{g_n\}$ and $\{h_n\}$ be the sequences generated by $g_1, h \in K$ and

$$g_{n+1} = \eta_n h + \zeta_n Gg_n + \gamma_n W_n g_n, \quad \forall n \geq 1 \tag{59}$$

where $W_n = P_K(I - \xi_n(I - J))P_K(I - p_n(W_1 + W_2)),$

$$W_1 = w(I - P_K(I - w\tilde{\Phi}(I - p_{n1}W_1 + W_2)),$$

$$W_2 = (1 - w)(I - P_K(I - w\tilde{\Phi}(I - p_{n2}W_1 + W_2)) + \frac{\tilde{\Phi}^2(I - p_{n2}W_1 + W_2))}{2}$$

and $\{\eta_n, \{\xi_n, \{\gamma_n \mid 0, 1\}, \zeta_n \in (0, 1 - \kappa), \gamma_n + \zeta_n = 1, \forall n \in N, \{p_n \in [0, 2], \gamma = \min(\alpha, \beta) \} \}$ satisfy:

(i) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\liminf_n \eta_n = 0$, $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $0 < \xi_n \leq p < 1$, $0 < q \leq p_n \leq m < 2\gamma$;

(iii) $\liminf_n |p_{n+1} - p_n| = 0$;

(iv) $\sum_{n=1}^{\infty} |\xi_n + \zeta_n| < \infty$, $\sum_{n=1}^{\infty} |\zeta_n - 1| < \infty$.

Then $\{g_n\}$ converges strongly to $c_0 = P_Fh$. Proof: The result is obtained by using Lemma 4.2 and Theorem 4.3.
Then \( \{g_n\} \) converges strongly to \( c_0 = Prh \).

**Proof:** We obtain the required result by applying Lemma 4.1 and Theorem 4.5.

V. EXAMPLE AND NUMERICAL RESULTS

We provide a numerical example in this section to bolster our primary theorem.

**Example 5.1:** Let \( R \) be the set of real numbers, \( K = [-50, 50] \), and \( H = R \). Let \( F : K \times K \to R \) defined by \( F(x,y) = -5x^2 + xy + 4y^2 \) for all \( x, y \in K \). Let \( A, B, A', B' : K \to H \) defined by \( Ax = x + \frac{1}{3}Bx = x - \frac{1}{3}Bx, B'x = \frac{3x^2 - 7}{2}, A'x = \frac{3x^2 - 7}{2} \) for all \( x \in K \). Define \( G : K \to K \) by \( Gx = P_K(I - \frac{1}{4}A') (\frac{1}{2}x + \frac{1}{2}P_K(I - \frac{1}{4}B')x) \) for all \( x \in K \). Let \( J : K \to K \) defined by \( Jx = x \) for all \( x \in K \). It is easy to show that \( A, B, A', B' \) are 1-sm, \( F \) is satisfied (G1) - (G4), and \( J \) is \( \frac{1}{4} \)-strictly pseudononsprading.

It is clear that \( F(J) \cap F(G) \cap EP(F, wA + (1 - w)B) = \{0\} \). Let \( \{g_n\} \) and \( \{h_n\} \) be the sequences generated by (12). By the definition of \( F \) and choose \( w = \frac{1}{2} \in (0, 1) \), we have

\[
0 \leq F(h_n, d) + (wA + (1 - w)B)g_n, d - h_n) + \frac{1}{p_n}(d - h_n, g_n - g_n) = (-5h_n^2 + h_n d + 4d^2) + (g_n)(d - h_n) + \frac{1}{p_n}(d - h_n, g_n - g_n) = (-5h_n^2 + h_n d + 4d^2) + (g_n)(d - g_n) + \frac{1}{p_n}(h_n d - g_n d - h_n^2 + g_n g_n)
\]

\[
0 \leq p_n(-5h_n^2 + h_n d + 4d^2) + p_n(g_n(d - g_n)) + (h_n d - g_n d - h_n^2 + g_n g_n)
\]

Let \( Q(y) = 4p_n d^2 + (p_n h_n + p_n g_n + h_n - g_n) d - 5p_n h_n^2 - p_n g_n h_n - h_n^2 + g_n g_n. \) Then \( Q(d) \) is quadratic function of \( d \) with coefficient \( a = 4p_n, b = p_n h_n + p_n g_n + h_n - g_n, c = -5p_n h_n^2 - p_n g_n h_n - h_n^2 + g_n g_n. \) Determine the discriminant \( \Delta \) of \( Q \) as follows:

\[
\Delta = b^2 - 4ac = (p_n h_n + p_n g_n + h_n - g_n)^2 - 4(4p_n)(-5p_n h_n^2 - p_n g_n h_n - h_n^2 + g_n g_n)
\]

\[
= p_n^2 h_n^2 + p_n^2 g_n h_n + p_n h_n^2 - p_n g_n h_n + 2p_n g_n h_n + p_n^2 h_n^2 - p_n g_n h_n + 2p_n g_n h_n + p_n^2 h_n^2 - p_n g_n h_n + p_n^2 h_n^2 - g_n h_n + 2g_n^2 + 16p_n(-5p_n h_n^2 - p_n g_n h_n - h_n^2 + g_n g_n)
\]

\[
= h_n^2 + 18p_n h_n^2 + 8p_n^2 h_n^2 + 18p_n^2 g_n h_n - 2h_n g_n - 16p_n h_n g_n + 2p_n g_n^2 + g_n^2 + 8p_n^2 h_n^2 + 16p_n^2 g_n h_n + 16p_n^2 g_n h_n + 2p_n^2 h_n^2 + 2p_n^2 g_n h_n + 2p_n^2 g_n h_n - 2h_n g_n - 16p_n h_n g_n + 2p_n g_n^2 + g_n^2
\]

\[
= (h_n + 9p_n h_n)^2 + 2(h_n + 9p_n h_n)(p_n - 1) + (p_n - 1)g_n)^2
\]

For any \( y \) in \( R \), we know that \( Q(d) \geq 0 \). If \( R \) has just one solution, then \( \Delta \leq 0 \), leading to the following result:

\[
h_n = \frac{1 - p_n}{9p_n} g_n.
\]

VI. CONCLUSION

1. We derive Remark 3.1 from Theorem 3.1.
2) We get a new method for solve the combination of variational inequality problem and equilibrium problem.
3) Applying our main result to solve the general split feasibility problem.
4) The sequences \( \{g_n\} \) and \( \{h_n\} \) converge to 0, as Table I. Figure 1, and Figure 2 demonstrate. Here, \( \{0\} = F(J) \cap F(G) \cap EP(F, w \tilde{A} + (1-w) \tilde{B}) \).
5) In Example 5.1, the convergence of \( \{g_n\} \) and \( \{h_n\} \) is ensured by Theorem 3.1.

REFERENCES