Automata Field Analysis: Operations, Substructures, and Homomorphisms

G. Sathiyasorubini and R. Venkatesan

Abstract—**This paper introduced the automata field concept and studied its characteristics. In particular, we define new operations by converting finite automata to finite automata field. Additionally, we examine the properties of finite automata field and their relation to field. We also define substructures of the automata field, including sub-automata field and automata field homomorphism.**

Index Terms—**Automata field, Sub-automata field, Automata integral domain, Automata field Homomorphism.**

I. Introduction

THE exploration of algebraic structures through the lens of automata theory has garnered increasing attention, emerging as a pivotal concept within abstract algebra. Numerous theories have evolved in tandem with this field, as documented in the scholarly work by Johnson [6]. This paper aims to present a ground-breaking methodology that combines algebra and automata theory, as put forth by Harris and Miller [5]. The intersection of these disciplines is manifest in various articles elucidating the role of automata in automata groups [1], [2], [3], [8], [12], [13], and [14]. Luminaries like John von Neumann, whose contributions have left an indelible mark on the convergence of algebra and automata theory, have significantly advanced the field of automata theory, which has its roots in the annals of theoretical computer science. At its core, automata theory investigates abstract machines and the computational challenges surmountable through them, establishing profound connections with mathematical logic. Of particular note are finite automata, pivotal in discerning string membership within a specified language, thereby facilitating string acceptance. An illustrative example is the Watson-Crick automaton, a type operating on dual standard tapes through complementary relationships. Finite automata's application in group theory is noteworthy, enhancing the comprehension of group properties and interactions. The research by Y.S. Gang et al. on finite automata over $\mathbf{Z}_n$ and $\mathbf{Z}_n \times \mathbf{Z}_n$ is a big step in the right direction. They found a link between the Cayley table and the picture representation of finite automata. K. Muthukumaran's team of researchers has looked into the intersections between automata and finite groups. They have looked into finite abelian groups, associates, and commutative finite binary automata, quotient automata, isomorphic automata, homomorphisms, and automaton groups [13]. They posit that if a finite group automaton accepts a language L, then the quotient is finite group automata also accept L [9]. F.W. Heng et al. delve into various aspects of automata for subgroups, permutation groups in automata diagrams, and automata representation for abelian groups [4]. Hellen and Mridul Dutta were the first people to talk about automata on rings. They defined and studied different types of automata rings, including sub-automata, commutative automata, zero divisors of automata, integral domain, and homomorphism [10]. John Kasper et al. contributes to the discourse by exploring lattice automata and their languages, elucidating theoretical and practical applications in query checking, abstraction methods, and verification. Their work on lattice grammar, showing closure properties, and defining equivalence between different grammar types and regular expressions give the field a lot more depth [7]. Sarsengan Abdymanapov et al. give information on how to make polynomials for finite groups using field extension operations. They suggest a third-degree cyclic polynomials over $\mathbb{F}^2$ and make irreducible polynomials of degree $2n$ [11]. These contributions extend the theoretical foundations and practical applications of automata theory in the realm of finite fields. This paper embarks on an exploration of finite automata fields, unravelling their characteristics and underpinning them with illustrative examples to foster comprehension. A meticulous examination of these fields' properties promise to yield a profound understanding of their intricacies and potential applications. The paper unfolds methodically, with Section 2 introducing essential terminologies are vital for comprehension. Section 3 discusses the automata field in detail and provides pertinent examples. Section 4 meticulously delineates the key features of the automata field, while Section 5 succinctly summarises the salient concepts discussed throughout the paper, and finally conclusion.

II. Preliminaries

This section will cover the basic definitions of the field, an essential topic in abstract algebra and automata ring structures. Thoroughly grasping the definitions related to algebra and automata theory is crucial to building a strong foundation for further analysis and study on these topics. One can better understand the underlying concepts of these subjects by clearly understanding their properties. This knowledge is crucial for excelling in these areas and contributing significantly to the automata field. By utilising this knowledge, individuals can develop innovative solutions and advancements within the field of automata.

Definition 2.1: A deterministic finite automata is defined by the quintuple

$$M = \{Q\ , \sum\ , \delta\ , q_0\ , F\},$$

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where,
Q is a finite set of internal states,
$\sum$ is a finite set of symbols called as the input alphabet,
$\delta : Q \times \sum \to Q$ is the transition function,
$q_0 \in Q$ is the initial state,
$F \subseteq Q$ is a set of final states.

Definition 2.2: Let us consider the automata,
$\sum = (Q,\ A,\ \delta,\ F,\ G)$ with $o\ (Q) = n$ and $o\ (A) = m$.
Let $T = \{F\ (q_a,\ x_i) : q_a \in Q,\ x_i \in A,\ a,\ i \in \mathbb{N} \cup \{0\}\}$.
Let us define the maps, $\mu : T \times T \to T$ as
$\mu\ (F\ (q_a,\ x_i),\ F\ (q_b,\ x_j)) = F\ (q_{a \oplus_n b},\ x_{i \oplus_m j})$ and
$\nu : T \times T \to T$ as
$\nu\ (F\ (q_a,\ x_i),\ F\ (q_b,\ x_j)) = F\ (q_{a \otimes_n b},\ x_{i \otimes_m j})$,
where, $\oplus_n$ denotes addition modulo n and $\otimes_m$ denotes multiplication modulo m.
Then the triple $(T,\ \mu,\ \nu)$ is an automata ring if it satisfies the following axioms:
a) $(T,\ \mu)$ is an abelian group.
b) $(T,\ \nu)$ is a semigroup.
c) Distributive property.

Definition 2.3: A field is a commutative ring with unity in which every nonzero element is a unit.

Definition 2.4: The characteristic of a ring R is the least positive integer n such that $nx = 0\ \forall\ x \in R$. If no such integer exists, we say that R has characteristic 0. The characteristic of R is denoted by char$R$.

Theorem 2.1: A finite integral domain is a field.

Theorem 2.2: The characteristic of an integral domain is 0 or prime.

Definition 2.5: A field F is called perfect if F has characteristic 0 or if F has characteristic p and $F^p = \{a^p \mid a \in F\} = F$.

Theorem 2.3: Every finite field is perfect.

Definition 2.6: A commutative automata ring T is called an automata integral domain if $\forall\ G\ (p_a,\ y_i),\ G\ (p_b,\ y_j) \in T$, $\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = G\ (p_e,\ y_e)$. This implies $G\ (p_a,\ y_i) = G\ (p_e,\ y_e)$ or $G\ (p_b,\ y_j) = G\ (p_e,\ y_e)$.

III. Construction of Automata field

This section explores the relationship between automata theory and algebraic structures. While Section 2 provides concise definitions of the finite automata field, this section aims to improve comprehension and facilitate visual understanding of these abstract structures through carefully crafted illustrations. By fusing the fundamentals of automata theory with the framework of algebraic fields, we build a solid theoretical foundation for investigating the characteristics and uses of finite automata field. This section is an excellent resource for those interested in learning more about the fascinating area where automata theory meets algebraic structures. Algebraic automata theory uses algebraic techniques to explore and solve problems relating to abstract machines. It also facilitates the efficient and precise analysis of the behaviour of discrete systems.

Definition 3.1: Let us consider the automata,
$\sum = (Q,\ A,\ \delta,\ G,\ H)$ choose $o\ (Q) = n$ and $o\ (A) = m$ where o denotes the order or number of elements in the sets Q and A such that cardinality of T is p^n for some prime p and $n \in \mathbb{N}$.
Let $T = \{G\ (p_a,\ y_i) \mid p_a \in Q,\ y_i \in A,\ a,\ i \in \mathbb{N} \cup \{0\}\}$.
Let us define the maps $\mu : T \times T \to T$ as
$\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = G\ (p_{a \oplus_{|T|} b},\ y_{i \oplus_{|T|} j})$ and
$\nu : T \times T \to T$ as
$\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = G\ (p_{a \otimes_{|T|} b},\ y_{i \otimes_{|T|} j})$
where $\oplus_{|T|}$ denotes addition modulo $|T|$ and $\otimes_{|T|}$ denotes multiplication modulo $|T|$, $|T|$ is the cardinality of the set T.
Then the triplet $(T,\ \mu,\ \nu)$ is an automata field if it satisfies the following axioms:

(a) $(T,\ \mu)$ is an abelian group, i.e.
(i) For any state transition function $G\ (p_a,\ y_i),\ G\ (p_b,\ y_j) \in T$, then $\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) \in T$.
(ii) For any state transition function $G\ (p_a,\ y_i), G\ (p_b,\ y_j), G\ (p_c,\ y_k) \in T$, then $\mu\ (G\ (p_a,\ y_i),\ \mu\ (G\ (p_b,\ y_j),\ G\ (p_c,\ y_k))) = \mu\ (\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)),\ G\ (p_c,\ y_k))$.
(iii) There exists a state transition $G\ (p_e,\ y_e) \in T$ such that $\mu\ (G\ (p_a,\ y_i),\ G\ (p_e,\ y_e)) = G\ (p_a,\ y_i) = \mu\ (G\ (p_e,\ y_e),\ G\ (p_a,\ y_i))$.
(iv) For every state transition function $G\ (p_a,\ y_i) \in T$ there exists $G\ (p_a^{-1},\ y_i^{-1}) \in T$ such that $\mu\ (G\ (p_a,\ y_i),\ G\ (p_a^{-1},\ y_i^{-1})) = G\ (p_e,\ y_e) = \mu\ (G\ (p_a^{-1},\ y_i^{-1},\ G\ (p_a,\ y_i))$.
(v) For any two state transition function $G\ (p_a,\ y_i),\ G\ (p_b,\ y_j) \in T$, then $\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = \mu\ (G\ (p_b,\ y_j),\ G\ (p_a,\ y_i))$.

(b) $(T,\ \nu)$ is an abelian group, i.e.
(i) For any state transition function $G\ (p_a,\ y_i),\ G\ (p_b,\ y_j) \in T$, then $\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) \in T$.
(ii) For any state transition function $G\ (p_a,\ y_i), G\ (p_b,\ y_j), G\ (p_c,\ y_k) \in T$, then $\nu\ (G\ (p_a,\ y_i),\ \nu\ (G\ (p_b,\ y_j),\ G\ (p_c,\ y_k))) = \nu\ (\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)),\ G\ (p_c,\ y_k))$.
(iii) There exists a state transition $G\ (p_e',\ y_e') \in T$ such that $\nu\ (G\ (p_a,\ y_i),\ G\ (p_e',\ y_e')) = G\ (p_a,\ y_i) = \nu\ (G\ (p_e',\ y_e'),\ G\ (p_a,\ y_i))$.
(iv) For every non zero state transition function $G\ (p_a,\ y_i) \in T$ there exists $G\ (p_a^{-1},\ y_i^{-1}) \in T$ such that $\nu\ (G\ (p_a,\ y_i),\ G\ (p_a^{-1},\ y_i^{-1})) = G\ (p_e',\ y_e') = \nu\ (G\ (p_a^{-1},\ y_i^{-1}),\ G\ (p_a,\ y_i))$.
(v) For any two states transition function $G\ (p_a,\ y_i),\ G\ (p_b,\ y_j) \in T$, then $\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = \nu\ (G\ (p_b,\ y_j),\ G\ (p_a,\ y_i))$.

(c) Distributive property

For any state transition function $G\ (p_a,\ y_i),\ G\ (p_b,\ y_j),$ $G\ (p_c,\ y_k) \in T$, then $\nu\ (G\ (p_a,\ y_i),\ \mu\ (G\ (p_b,\ y_j),\ G\ (p_c,$ $y_k))) = \mu\ (\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)),\ \nu(G\ (p_a,\ y_i),\ G(p_c,$ $y_k)))$ and $\nu\ (\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)),\ G\ (p_c,\ y_k))$ = $\mu\ (\nu\ (G\ (p_a,\ y_i),\ G\ (p_c,\ y_k)),\ \nu\ (G\ (p_b,\ y_j),\ G\ (p_c,\ y_k)))$.

Definition 3.2: A non-empty subset S_T of T is said to be a sub-automata field if $(S_T,\ \mu,\ \nu)$ itself an automata field under the operations μ and ν.

Definition 3.3: The Characteristic of an automata field is defined to be a smallest positive integer n such that $\mu\ (G^n\ (p_a,\ y_i)) = G\ (p_e,\ y_e)\ \forall\ G\ (p_a,\ y_i) \in T$ and is defined to be 0 otherwise.

Definition 3.4: Let $(T,\ \mu,\ \nu)$ and $(T',\ \mu,\ \nu)$ be two automata fields. A mapping $f : T \to T'$ is called automata field homomorphism $\forall\ G\ (p_a,\ y_i),\ G\ (p_b,\ y_j) \in T$:

(i) $f\ (\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j))) = \mu\ (f\ (G\ (p_a,\ y_i)),$ $f\ (G\ (p_b,\ y_j)))$.

(ii) $f\ (\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j))) = \nu\ (f\ (G\ (p_a,\ y_i)),$ $f\ (G\ (p_b,\ y_j)))$.

(iii) $f\ (G\ (p_e,\ y_e)) = G\ (p'_e,\ y'_e)$.

Definition 3.5: An automata field T is called perfect if T has characteristic 0 or if T has characteristic p and T^p = $\{\nu\ (G^p\ (p_a,\ y_i))\mid G\ (p_a,\ y_i) \in T\} = T$.

In order to better understand the difference between automata and non-automata fields, it is helpful to examine specific examples. This examination identifies the key characteristics that distinguish an automata field from other algebraic structures. This type of analysis enables researchers to compare and contrast the criteria and standards necessary to classify a field as an automata field. Furthermore, studying these examples can enhance theoretical knowledge and provide valuable insights into the practical applications of the automata field.

Example 3.1: Let us examine a finite state automaton denoted by $\sum = (Q,\ A,\ \delta,\ G,\ H)$ with $A = \{y_0,\ y_1,\ y_2,\ y_3,$ $y_4,\ y_5,\ y_6,\ y_7,\ y_8,\ y_9,\ y_{10},\ y_{11},\ y_{12},\ y_{13},\ y_{14},\ y_{15},\ y_{16}\}$, $Q = \{p_0\}$, and a state transition function G defined by the Table I and the state diagram shown in Fig. 1.

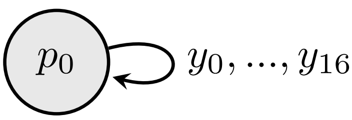

Fig. 1. State Diagram Of $\sum$

TABLE I
STATE TRANSITION TABLE OF $\sum$

G	p_0
y_0	p_0
y_1	p_0
y_2	p_0
y_3	p_0
y_3	p_0
y_4	p_0
y_5	p_0
y_6	p_0
y_7	p_0
y_8	p_0
y_9	p_0
y_{10}	p_0
y_{11}	p_0
y_{12}	p_0
y_{13}	p_0
y_{14}	p_0
y_{15}	p_0
y_{16}	p_0

From the definition of the automata field, let us consider $T = \{G\ (p_0,\ y_0),\ G\ (p_0,\ y_1),\ G\ (p_0,\ y_2), G\ (p_0,\ y_3),$ $G\ (p_0,\ y_4),\ G\ (p_0,\ y_5),\ G\ (p_0,\ y_6),\ G\ (p_0,\ y_7),\ G\ (p_0,\ y_8),$ $G\ (p_0,\ y_9),\ G\ (p_0,\ y_{10}),\ G\ (p_0,\ y_{11}),\ G\ (p_0,\ y_{12}),$ $G\ (p_0,\ y_{13}), G\ (p_0,\ y_{14}),\ G\ (p_0,\ y_{15}),\ G\ (p_0,\ y_{16})\}$.

Two maps μ and ν on T are defined as follows:

$\mu : T \times T \to T$ as

$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_1)) = G\ (p_0,\ y_1)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_2)) = G\ (p_0,\ y_2)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_3)) = G\ (p_0,\ y_3)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_1,\ y_4)) = G\ (p_0,\ y_4)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_5)) = G\ (p_0,\ y_5)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_6)) = G\ (p_0,\ y_6)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_7)) = G\ (p_0,\ y_7)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_8)) = G\ (p_0,\ y_8)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_9)) = G\ (p_0,\ y_9)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_{10})) = G\ (p_0,\ y_{10})$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_{11})) = G\ (p_0,\ y_{11})$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_{12})) = G\ (p_0,\ y_{12})$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_{13})) = G\ (p_0,\ y_{13})$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_{14})) = G\ (p_0,\ y_{14})$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_{15})) = G\ (p_0,\ y_{15})$

$$\mu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_{16}\right)$$

$$\mu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_1\right)$$
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$$\mu\left(G\left(p_0,\ y_3\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_{16}\right)$$
$$\mu\left(G\left(p_0,\ y_3\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_0\right)$$
$$\mu\left(G\left(p_0,\ y_3\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_1\right)$$
$$\mu\left(G\left(p_0,\ y_3\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_2\right)$$

$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_4\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_5\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_6\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_7\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_1,\ y_4\right)\right)=G\left(p_0,\ y_8\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_9\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{10}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_{11}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_{12}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_{13}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_{14}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_{15}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_{16}\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_0\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_1\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_2\right)$$
$$\mu\left(G\left(p_0,\ y_4\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_3\right)$$

$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_5\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_6\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_7\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_8\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_1,\ y_4\right)\right)=G\left(p_0,\ y_9\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_{10}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{11}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_{12}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_{13}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_{14}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_{15}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_{16}\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_0\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_1\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_2\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_3\right)$$
$$\mu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_4\right)$$

$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_6\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_7\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_8\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_9\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_1,\ y_4\right)\right)=G\left(p_0,\ y_{10}\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_{11}\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{12}\right)$$
$$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_{13}\right)$$

$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_5\right)$

$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_6\right)$

$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_7\right)$

$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_8\right)$

$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_9\right)$

$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_{10}\right)$

$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_{11}\right)$

$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_{12}\right)$

$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_{13}\right)$

$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{15}\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_5\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_6\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_7\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_8\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_9\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_{11}\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_{12}\right)$
$\mu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{13}\right)$
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$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_{16}\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_0\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_1\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_2\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_1,\ y_4\right)\right) = G\left(p_0,\ y_3\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_4\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_5\right)$
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$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_{11}\right)$
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$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_{13}\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{14}\right)$
$\mu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_{15}\right)$

It implies that $(T,\ \mu)$ forms an abelian group with $G\left(p_0,\ y_0\right)$ as its identity element.

$\nu : T \times T \to T$ as
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_4\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_0\right)$

$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_0\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_0\right)$

$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_{10}\right)$
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$\nu\left(G\left(p_0,\ y_1\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_{16}\right)$

$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_2\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{12}\right)$
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$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_7\right)$

$\nu\left(G\left(p_0,\ y_5\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_{12}\right)$

$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_6\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_{11}\right)$

$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_7\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_{10}\right)$

$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_8\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_9\right)$

$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_9\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_8\right)$

$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_8\right)\right)=G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_9\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{10}\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{11}\right)\right)=G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{12}\right)\right)=G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{13}\right)\right)=G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{14}\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{15}\right)\right)=G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{10}\right),\ G\left(p_0,\ y_{16}\right)\right)=G\left(p_0,\ y_7\right)$

$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_0\right)\right)=G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_1\right)\right)=G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_2\right)\right)=G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_3\right)\right)=G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_4\right)\right)=G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_5\right)\right)=G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_6\right)\right)=G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_7\right)\right)=G\left(p_0,\ y_9\right)$

$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{11}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_6\right)$

$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_4\right)\right) = G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{12}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_5\right)$

$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_4\right)\right) = G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{13}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_4\right)$

$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_4\right)\right) = G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_{14}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_3\right)$

$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_4\right)\right) = G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_1\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_{15}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_2\right)$

$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_0\right)\right) = G\left(p_0,\ y_0\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_1\right)\right) = G\left(p_0,\ y_{16}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_2\right)\right) = G\left(p_0,\ y_{15}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_3\right)\right) = G\left(p_0,\ y_{14}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_4\right)\right) = G\left(p_0,\ y_{13}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_5\right)\right) = G\left(p_0,\ y_{12}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_6\right)\right) = G\left(p_0,\ y_{11}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_7\right)\right) = G\left(p_0,\ y_{10}\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_8\right)\right) = G\left(p_0,\ y_9\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_9\right)\right) = G\left(p_0,\ y_8\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{10}\right)\right) = G\left(p_0,\ y_7\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{11}\right)\right) = G\left(p_0,\ y_6\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{12}\right)\right) = G\left(p_0,\ y_5\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{13}\right)\right) = G\left(p_0,\ y_4\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{14}\right)\right) = G\left(p_0,\ y_3\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{15}\right)\right) = G\left(p_0,\ y_2\right)$
$\nu\left(G\left(p_0,\ y_{16}\right),\ G\left(p_0,\ y_{16}\right)\right) = G\left(p_0,\ y_1\right)$

It implies $(T,\ \nu)$ is an abelian group under the binary map ν with $G\ (p_0,\ y_1)$ as its identity element. Also, it satisfies distributive property.
Thus, ν is distributive with respect to μ.
Hence, we can say that $(T,\ \mu,\ \nu)$ is an automata field.

Example 3.2: Let us examine a finite state automaton denoted by $\sum = (Q,\ A,\ \delta,\ G,\ H)$ with $A = \{y_0\}$, $Q = \{p_0,\ p_1,\ p_2,\ p_3,\ p_4\}$, and a state transition function G defined by the Table II and the state diagram shown in Fig. 2.

TABLE II
STATE TRANSITION TABLE OF $\sum$

G	y_0
p_0	p_1
p_1	p_2
p_2	p_3
p_3	p_4
p_4	p_0

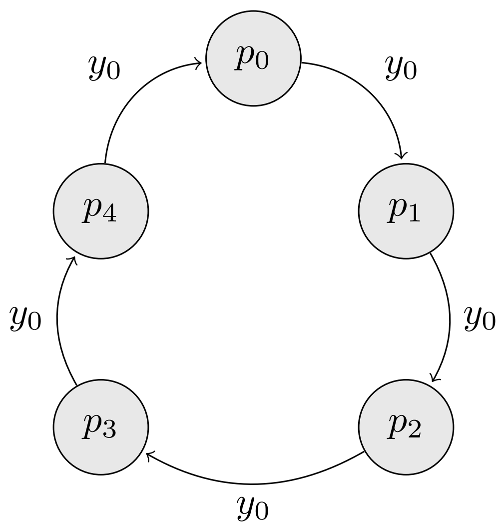

Fig. 2. State Diagram Of $\sum$

From the definition of the automata field, $T = \{G\ (p_0,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_4,\ y_0)\}$.
Two maps μ and ν on T are defined as follows:
$\mu : T \times T \to T$ as
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_1,\ y_0)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_2,\ y_0)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_3,\ y_0)$
$\mu\ (G\ (p_0,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_4,\ y_0)$

$\mu\ (G\ (p_1,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_1,\ y_0)$
$\mu\ (G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_2,\ y_0)$
$\mu\ (G\ (p_1,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_3,\ y_0)$
$\mu\ (G\ (p_1,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_4,\ y_0)$
$\mu\ (G\ (p_1,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_0,\ y_0)$

$\mu\ (G\ (p_2,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_2,\ y_0)$
$\mu\ (G\ (p_2,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_3,\ y_0)$
$\mu\ (G\ (p_2,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_4,\ y_0)$
$\mu\ (G\ (p_2,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_0,\ y_0)$
$\mu\ (G\ (p_2,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_1,\ y_0)$

$\mu\ (G\ (p_3,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_3,\ y_0)$
$\mu\ (G\ (p_3,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_4,\ y_0)$
$\mu\ (G\ (p_3,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_0,\ y_0)$
$\mu\ (G\ (p_3,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_1,\ y_0)$
$\mu\ (G\ (p_3,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_2,\ y_0)$

$\mu\ (G\ (p_4,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_4,\ y_0)$
$\mu\ (G\ (p_4,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_0,\ y_0)$
$\mu\ (G\ (p_4,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_1,\ y_0)$
$\mu\ (G\ (p_4,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_2,\ y_0)$
$\mu\ (G\ (p_4,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_3,\ y_0)$

Here $(T,\ \mu)$ forms an abelian group with $G\ (p_0,\ y_0)$ as its identity element.

$\nu : T \times T \to T$ as
$\nu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_0,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_0,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_0,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_0,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_0,\ y_0)$

$\nu\ (G\ (p_1,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_1,\ y_0)$
$\nu\ (G\ (p_1,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_2,\ y_0)$
$\nu\ (G\ (p_1,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_3,\ y_0)$
$\nu\ (G\ (p_1,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_4,\ y_0)$

$\nu\ (G\ (p_2,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_2,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_2,\ y_0)$
$\nu\ (G\ (p_2,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_4,\ y_0)$
$\nu\ (G\ (p_2,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_1,\ y_0)$
$\nu\ (G\ (p_2,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_3,\ y_0)$

$\nu\ (G\ (p_3,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_3,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_3,\ y_0)$
$\nu\ (G\ (p_3,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_1,\ y_0)$
$\nu\ (G\ (p_3,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_4,\ y_0)$
$\nu\ (G\ (p_3,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_2,\ y_0)$

$\nu\ (G\ (p_4,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$
$\nu\ (G\ (p_4,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_4,\ y_0)$
$\nu\ (G\ (p_4,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_3,\ y_0)$
$\nu\ (G\ (p_4,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_2,\ y_0)$
$\nu\ (G\ (p_4,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_1,\ y_0)$

Thus, ν is distributive with respect to μ.
Hence, we can say that $(T,\ \mu,\ \nu)$ is an automata field.

Example 3.3: Let us consider a finite state automata $\sum = (Q,\ A,\ B,\ G,\ H)$ with $Q = \{p_0,\ p_1,\ p_2\}$ and $A = \{y_0,\ y_1,\ y_2\}$ and the state function G is defined by Table III and state diagram of $\sum$ in Fig. 3.

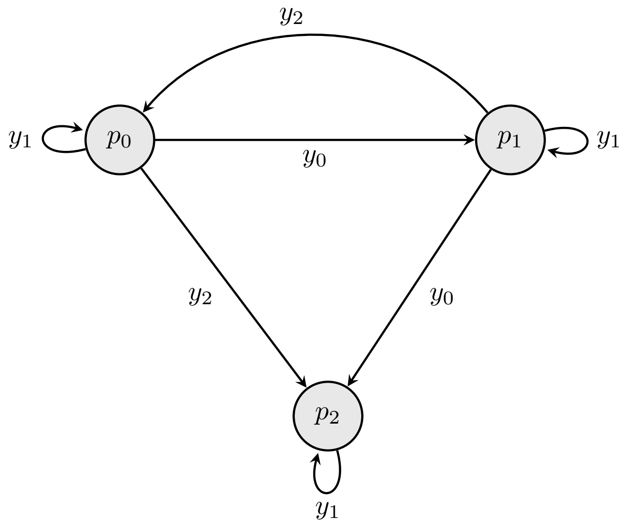

Fig. 3. State Diagram Of $\sum$

By the definition of the automata field, suppose $T = \{G\ (p_0,\ x_0),\ G\ (p_0,\ y_1),\ G(p_1,\ y_1),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_1)\}$.
But $\mu\ (G\ (p_0,\ y_1),\ G\ (p_1,\ y_1)) = G\ (p_1,\ y_0) \notin T$. It does not satisfy the closure property.
So $(T,\ \mu,\ \nu)$ is not an automata field.

TABLE III
STATE TRANSITION TABLE OF $\sum$

G	y_0	y_1	y_2
p_0	p_1	p_0	p_2
p_1	p_2	p_1	p_0
p_2	$-$	p_2	p_1

Example 3.4: Consider the automata field in example 3.2.
Let us take $S_T = \{G\ (p_0,\ y_0)\}$.
This set is a sub-automata field. Since $(S_T,\ \mu)$ and $(S_T,\ \nu)$ are abelian group with $G\ (p_0,\ y_0)$ identity of $(S_T,\ \mu)$,
It is trivial automata field. By the definition of maps μ and ν,

$\mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$

$\nu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$

Example 3.5: Consider the automata field in example 3.2.
Let us calculate the characteristic of the automata field.
According to the definition of the characteristic of automata field $\mu\ (G^n\ (p_a,\ y_i)) = G\ (p_e,\ y_e)\ \forall\ G\ (p_a,\ y_i) \in T$ is 5.

Below is the computation.
$\mu\ (G^5\ (p_0,\ y_0)) = \mu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)),\ G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$.
$\mu\ (G^5\ (p_1,\ y_0)) = \mu\ (G\ (p_1,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)) = \mu\ (G\ (p_2,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)\ G\ (p_1,\ y_0)) = \mu\ (G\ (p_3,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)) = \mu\ (G\ (p_4,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_0,\ y_0)$.

$\mu\ (G^5\ (p_2,\ y_0)) = \mu\ (G\ (p_2,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_0)) = \mu\ (G\ (p_4,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_0)\ G\ (p_2,\ y_0)) = \mu\ (G\ (p_1,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_0)) = \mu\ (G\ (p_3,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_0,\ y_0)$.
$\mu\ (G^5\ (p_3,\ y_0)) = \mu\ (G\ (p_3,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_3,\ y_0)) = \mu\ (G\ (p_1,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_3,\ y_0)\ G\ (p_3,\ y_0)) = \mu\ (G\ (p_4,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_3,\ y_0)) = \mu\ (G\ (p_2,\ y_0),\ G\ (p_3,\ y_0)) = G\ (p_0,\ y_0)$.
$\mu\ (G^5\ (p_4,\ y_0)) = \mu\ (G\ (p_4,\ y_0),\ G\ (p_4,\ y_0),\ G\ (p_4,\ y_0),\ G\ (p_4,\ y_0),\ G\ (p_4,\ y_0)) = \mu\ (G\ (p_3,\ y_0),\ G\ (p_4,\ y_0),\ G\ (p_4,\ y_0)\ G\ (p_4,\ y_0)) = \mu\ (G\ (p_2,\ y_0),\ G\ (p_4,\ y_0),\ G\ (p_4,\ y_0)) = \mu\ (G\ (p_1,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_0,\ y_0)$.
So, the characteristic of the automata field is 5.

Example 3.6: Let $(T,\ \mu,\ \nu)$ and $(T',\ \mu,\ \nu)$ be two automata fields. Consider $f : T \to T'$ defined as $f\ (t) = t$, i.e identity map.
(i)$f\ (\mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j))) = \mu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = G\ (p_{a \oplus_{|T|} b},\ y_{i \oplus_{|T|} j})$
$\mu\ (f\ (G\ (p_a,\ y_i)),\ f\ (G\ (p_b,\ y_j))) = \mu\ (G\ (p_a,\ y_i),\ G\ (p_b\ ,\ y_j)) = G\ (p_{a \oplus_{|T|} b},\ y_{i \oplus_{|T|} j})$
(ii)$f\ (\nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j))) = \nu\ (G\ (p_a,\ y_i),\ G\ (p_b,\ y_j)) = \nu\ (f\ (G\ (p_a,\ y_i)),\ f\ (G\ (p_b,\ y_j)))$.
(iii) $f\ (G\ (p_e,\ y_e)) = G\ (p_e,\ y_e) = G\ (p'_e,\ y'_e)$.
This results that f is an automata field homomorphism.

Example 3.7: Let us examine the example 3.2.
According to the example, the automata field has the characteristic 5. We aim to prove it is perfect by computing $\nu\ (G^5\ (p_a,\ y_i))$.
$\nu\ (G^5\ (p_0,\ y_0)) = \nu\ (G\ (p_0,\ y_0),\ G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)),\ G\ (p_0,\ y_0),\ G\ (p_0,\ y_0)) = G\ (p_0,\ y_0)$.
$\nu\ (G^5\ (p_1,\ y_0)) = \nu\ (G\ (p_1,\ y_0),\ G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)),$

$G\ (p_1,\ y_0),\ G\ (p_1,\ y_0)) = G\ (p_1,\ y_0)$.
$\nu\ (G^5\ (p_2,\ y_0)) = \nu\ (G\ (p_2,\ y_0),\ G\ (p_2,\ y_0),\ G\ (p_2,\ y_0),$
$G\ (p_2,\ y_0),\ G\ (p_2,\ y_0)) = \nu\ (G\ (p_4,\ y_0),\ G\ (p_2,\ y_0),$
$G\ (p_2,\ y_0)\ G\ (p_2,\ y_0)) = \nu\ (G\ (p_3,\ y_0),\ G\ (p_2,\ y_0),$
$G\ (p_2,\ y_0)) = \nu\ (G\ (p_1,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_2,\ y_0)$.
$\nu\ (G^5\ (p_3,\ y_0)) = \nu\ (G\ (p_3,\ y_0),\ G\ (p_3,\ y_0),\ G\ (p_3,\ y_0),$
$G\ (p_3,\ y_0),\ G\ (p_3,\ y_0)) = \nu\ (G\ (p_4,\ y_0),\ G\ (p_3,\ y_0),$
$G\ (p_3,\ y_0)\ G\ (p_3,\ y_0)) = \nu\ (G\ (p_2,\ y_0),\ G\ (p_3,\ y_0),$
$G\ (p_3,\ y_0)) = \nu\ (G\ (p_1,\ y_0),\ G\ (p_2,\ y_0)) = G\ (p_3,\ y_0)$.
$\nu\ (G^5\ (p_4,\ y_0)) = \nu\ (G\ (p_4,\ y_0),\ G\ (p_4,\ y_0),\ G\ (p_4,\ y_0),$
$G\ (p_4,\ y_0),\ G\ (p_4,\ y_0)) = \nu\ (G\ (p_1,\ y_0),\ G\ (p_4,\ y_0),$
$G\ (p_4,\ y_0)\ G\ (p_4,\ y_0)) = \nu\ (G\ (p_4,\ y_0),\ G\ (p_4,\ y_0),$
$G\ (p_4,\ y_0)) = \nu\ (G\ (p_1,\ y_0),\ G\ (p_4,\ y_0)) = G\ (p_4,\ y_0)$.
This implies $T^5 = \{\nu\ (G^5\ (p_a,\ y_i))\mid G\ (p_a,\ y_i)\in T\} = T$.
So, the automata field is perfect.

IV. Characteristics of automata field

This section will examine the essential theorems developed by studying the automata field. The application of field properties in finite automata is what enables this achievement. These theorems provide valuable insights into the characteristics and properties of algebraic structures, helping to reveal their fundamental nature. By using automata theory, we have found interesting connections between abstract algebra and computational models. These theorems deepen our understanding of the automata field and set the groundwork for future advancements. We aim to establish a strong framework for studying algebraic automata theory.

Theorem 4.1: Let T be an automata field.
(i) The additive identity in T is unique.
(ii) The additive inverse in T is unique.
(iii) The Multiplicative identity in T is unique.
(iv) The multiplicative inverse in T is unique.

Proof: (i) Let us assume the existence of two state functions, $G(p_e, y_e)$ and $G_1(p_e, y_e)$, belonging to the automata field T. These functions satisfy the conditions:

$$\mu(G(p_a, y_i), G(p_e, y_e)) = G(p_a, y_i) = \mu(G(p_e, y_e), G_1(p_a, y_i)) \text{ for all } G(p_a, y_i), \text{ and}$$
$$\mu(G(p_a, y_i), G_1(p_e, y_e)) = G(p_a, y_i) = \mu(G_1(p_e, y_e), G(p_a, y_i)) \text{ for all } G(p_a, y_i).$$

It implies that

$$\mu(G(p_a, y_i), G(p_e, y_e)) = \mu(G(p_a, y_i), G_1(p_e, y_e)) = G(p_a, y_i).$$

Therefore,

$$\mu(G(p_a, y_i), G(p_e, y_e)) = \mu(G(p_a, y_i), G_1(p_e, y_e)) = G(p_a, y_i).$$

As T is a deterministic finite automaton, we can conclude that $G(p_e, y_e) = G_1(p_e, y_e)$. Thus, we have shown that the additive identity in T is unique.

(ii) Let us consider two state functions, $G((p_a)^{-1}, (y_i)^{-1})$ and $G_1((p_a)^{-1}, (y_i)^{-1})$, both belonging to the automata field T. These functions satisfy the conditions:

$$\mu(G(p_a, y_i), G((p_a)^{-1}, (y_i)^{-1})) = G(p_e, y_e) = \mu(G((p_a)^{-1}, (y_i)^{-1}), G_1(p_a, y_i)) \quad \text{for all } G(p_a, y_i), \text{ and}$$
$$\mu(G(p_a, y_i), G_1((p_a)^{-1}, (y_i)^{-1})) = G(p_e, y_e) = \mu(G_1((p_a)^{-1}, (y_i)^{-1}), G(p_a, y_i)) \quad \text{for all } G(p_a, y_i).$$

It implies that

$$\mu(G(p_a, y_i), G((p_a)^{-1}, (y_i)^{-1})) = \mu(G(p_a, y_i), G_1((p_a)^{-1}, (y_i)^{-1})) = G(p_e, y_e).$$

Therefore,

$$\mu(G(p_a, y_i), G((p_a)^{-1}, (y_i)^{-1})) = \mu(G(p_a, y_i), G_1((p_a)^{-1}, (y_i)^{-1})) = G(p_e, y_e).$$

As T is a deterministic finite automaton, we can conclude that $G((p_a)^{-1}, (y_i)^{-1}) = G_1((p_a)^{-1}, (y_i)^{-1})$. Thus, we have shown that the additive inverse in T is unique.

(iii) Let $G'(p_e, y_e)$ and $G''(p_e, y_e)$ be two state functions in the automata field T such that

$$\nu(G(p_a, y_i), G'(p_e, y_e)) = G(p_a, y_i) = \nu(G'(p_e, y_e), G(p_a, y_i)) \quad \text{for all } G(p_a, y_i), \text{ and}$$
$$\nu(G(p_a, y_i), G'_1(p_e, y_e)) = G(p_a, y_i) = \nu(G'_1(p_e, y_e), G(p_a, y_i)) \quad \text{for all } G(p_a, y_i).$$

This implies that

$$\nu(G(p_a, y_i), G'(p_e, y_e)) = \nu(G(p_a, y_i), G'_1(p_e, y_e)) = G(p_a, y_i).$$

Therefore,

$$\nu(G(p_a, y_i), G'(p_e, y_e)) = \nu(G(p_a, y_i), G'_1(p_e, y_e)) = G(p_a, y_i).$$

As T is a deterministic finite automaton, we can conclude that $G'(p_e, y_e) = G'_1(p_e, y_e)$. Thus, we have shown that the multiplicative identity in T is unique.

(iv) Suppose there exist two state functions $G'((p_a)^{-1}, (y_i)^{-1})$ and $G'_1((p_a)^{-1}, (y_i)^{-1})$ in the automata field T such that

$$\nu(G(p_a, y_i), G'((p_a)^{-1}, (y_i)^{-1})) = G'(p_e, y_e) = \nu(G'((p_a)^{-1}, (y_i)^{-1}), G(p_a, y_i)) \quad \text{for all } G(p_a, y_i), \text{ and}$$
$$\nu(G(p_a, y_i), G'_1((p_a)^{-1}, (y_i)^{-1})) = G(p_e, y_e) = \nu(G'_1((p_a)^{-1}, (y_i)^{-1}), G(p_a, y_i)) \quad \text{for all } G(p_a, y_i).$$

It implies that

$$\nu(G(p_a, y_i), G'((p_a)^{-1}, (y_i)^{-1})) = \nu(G(p_a, y_i), G'_1((p_a)^{-1},$$

$$(y_i)^{-1})) = G'(p_e, y_e).$$

So,

$$\nu(G(p_a, y_i), G'((p_a)^{-1}, (y_i)^{-1})) = \nu(G(p_a, y_i), G'_1((p_a)^{-1}, (y_i)^{-1})) = G'(p_e, y_e).$$

As T is a deterministic finite automaton, we can conclude that $G'((p_a)^{-1}, (y_i)^{-1}) = G'_1((p_a)^{-1}, (y_i)^{-1})$. Thus, we have shown that the multiplicative inverse in T is unique. It completes the proof.

Theorem 4.2: In the automata field T, the only ideals are $\{G(p_0, y_0)\}$ and T.

Proof: Suppose there exists a non-zero ideal of the automata field T other than T, denoted as $A \subset T$. It must contain a non-zero element that is also a unit, as T is an automata field. Since A is ideal, this non-zero element and its inverse will form a unity, i.e., $G(p_1, y_1)$ and $G(p_1, y_1) \in A$. Therefore, for all a and i, $\nu(G(p_1, y_1), G(p_a, y_i)) = G(p_a, y_i)$. It implies that $A = T$. However, this contradicts the assumption that A is a non-zero ideal distinct from T. Therefore, no such non-zero ideal A exists, and the only ideals of the automata field T are $\{G(p_0, y_0)\}$ and T.

Theorem 4.3: An automata field is an automata integral domain.

Proof: Let T be an automata field with zero element $G(p_e, y_e)$ and unity element $G'(p_e, y_e)$. Assume there exist two distinct elements $G(p_a, y_i)$ and $G(p_b, y_j)$ in T such that their convolution, denoted as $\nu(G(p_a, y_i), G(p_b, y_j))$, equals $G(p_e, y_e)$.
Now, assume $G(p_a, y_i) \neq G(p_e, y_e)$. According to the definition of an automata field, the inverse element $G((p_a)^{-1}, (y_i)^{-1})$ exists in T, and we have $G(p_b, y_j) = \nu(G'(p_e, y_e), G(p_b, y_j))$.
This convolution can be expressed as follows:

$$\begin{aligned}
&\nu(\nu(G(p_a, y_i), G((p_a)^{-1}, (y_i)^{-1})), G(p_b, y_j)) \\
&= \nu(G((p_a)^{-1}, (y_i)^{-1}), \nu(G(p_a, y_i), G(p_b, y_j))) \\
&= \nu(G((p_a)^{-1}, (y_i)^{-1}), G(p_e, y_e)) \\
&= G(p_e, y_e).
\end{aligned}$$

Similarly, if $G(p_b, y_j) \neq G(p_e, y_e)$, we can show that $G((p_b)^{-1}, (y_j)^{-1})$ exists in T and $G(p_a, y_i) = \nu(G'(p_e, y_e), G(p_a, y_i))$ equals $G(p_e, y_e)$.
This leads to a contradiction to our initial assumption. Therefore, we conclude that $G(p_a, y_i) = G(p_e, y_e)$ and $G(p_b, y_j) = G(p_e, y_e)$.

Theorem 4.4: Every finite automata integral domain is also an automata field.

Proof: The only requirement for a finite automata integral domain to be an automata field is to establish the existence of a multiplicative inverse for a state function that is not equal to $G(p_e, y_e)$.
Consider the sequence $G(p_a, y_i)$, $\nu(G(p_a, y_i), G(p_a, y_i))$, and so forth. Due to the finiteness of elements, there must exist integers m and n such that $m < n$ and $\nu(G^m(p_a, y_i)) = \nu(G^n(p_a, y_i))$.
This implies that $G(p_e, y_e)$ can be expressed as $\nu(G^m(p_a, y_i)) - \nu(G^n(p_a, y_i))$, which further simplifies to $\nu(G^m(p_a, y_i))(G(p'_e, y'_e) - \nu(G^{n-m}(p_a, y_i))$.
Given that there are no zero-divisors, we can deduce that $\nu(G^m(p_a, y_i) \neq G(p_e, y_e)$. Consequently, we find that $G(p'_e, y'_e) - \nu(G^{n-m}(p_a, y_i)) = G(p_e, y_e)$, leading to $G(p'_e, y'_e) = G(p_a, y_i)(\nu(G^{n-m-1}(p_a, y_i)))$.
It establishes the existence of a multiplicative inverse for $G(p_a, y_i)$, thereby confirming that the finite automata integral domain is indeed an automata field.

Theorem 4.5: If T is a finite automata field, then the characteristic of T is a prime number p.

Proof: Suppose T has characteristic $n = ab$, where a and b are integers greater than 1. By the distributive property, we have $\mu(G^n(p_1, y_1)) = \nu(\mu(G^a(p_1, y_1)\mu(G^b(p_1, y_1))))$. Consequently, $\mu(G^a(p_1, y_1)) = 0$ or $\mu(G^b(p_1, y_1)) = 0$, which contradicts the minimal property of characteristic.
This contradiction arises from the assumption that n can be expressed as the product of two integers greater than 1, concluding that the characteristic of T must be a prime number.

Theorem 4.6: If T is a finite automata field of characteristic p, then the map $\phi : T \to T$ defined by $G(p_a, y_i) \to G^p(p_a, y_i)$ is an injective homomorphism.

Proof: (i) Consider the expression $\phi(\mu(G(p_a, y_i), G(p_b, y_j)))$. By the definition of ϕ, this is equivalent to $(\mu(G(p_a, y_i), G(p_b, y_j))^p$. This further simplifies to $\mu((G^p(p_a, y_i)), (G^p(p_b, y_j)))$. By the properties of automata fields, this is equal to $\mu(\phi(G(p_a, y_i)), \phi(G(p_b, y_j)))$.

(ii) Similarly, consider $\phi(\nu(G(p_a, y_i), G(p_b, y_j)))$. This is equal to $\nu(\phi(G(p_a, y_i)), \phi(G(p_b, y_j)))$.

(iii) Consider $\phi(G(p_e, y_e))$. By the definition of ϕ, this is equal to $G^p(p_e, y_e)$. By the characteristic property of the automata field, this is equivalent to $\nu(G(p_e, y_e), G(p_e, y_e), ..., G(p_e, y_e))$, which further simplifies to $G(p_e, y_e)$.

The injective property follows from the fact that if $G^p(p_a, y_i) = G(p_e, y_e)$, then $G(p_a, y_i) = G(p_e, y_e)$. It establishes the injective homomorphism property of ϕ.

Corollary 4.6: Let T be a finite automata field of characteristic p. If ϕ is an automorphism of T, then ϕ preserves the field structure of T.

Proof: We only need to prove the surjection of ϕ. Since T is a finite automaton field, we know ϕ is onto because it is 1-1. Therefore, ϕ is an automorphism of T.

Theorem 4.7: If $\phi : T \to T'$ is an automata field homomorphism, then the kernel of ϕ is given by:

$$\ker(\phi) = \{G(p_0, y_0)\} \quad \text{or} \quad \ker(\phi) = T$$

Proof: Consider an automata field homomorphism $f : T \to T'$. We want to establish that the kernel of f, denoted as

$ker f$, is either $G(p_0, y_0)$ or the entire field T.
Firstly, we note that $f(G(p_e, y_e)) = G(p'_e, y'_e)$. This implies that if $G(p_e, y_e)$ is in $ker f$, then $f(G(p_e, y_e))$ equals the identity element in T', and consequently, $ker f$ is nonempty. Assume there exist elements $G(p_a, y_i)$ and $G(p_b, y_j)$ in $ker f$. It implies that

$$f(G(p_a, y_i)) = G(p'_e, y'_e) \text{ and } f(G(p_b, y_j)) = G(p'_e, y'_e).$$

Consequently, considering the multiplication operation in T', we observe that

$$f(\mu(G(p_a, y_i), G(p_b^{-1}, y_j^{-1}))) = \mu(f(G(p_a, y_i)), f(G(p_b^{-1}, y_j^{-1}))) = G(p'_e, y'_e).$$

It implies that $\mu(G(p_a, y_i), G(p_b^{-1}, y_j^{-1}))$ is also in $ker f$. Similarly, $\nu(G(p_a, y_i), G(p_r, y_r))$ and $\nu(G(p_r, y_r), G(p_a, y_i))$ are in $ker f$. It establishes that $ker f$ is an ideal of T.
Now, the ideals of an automata field are $G(p_0, y_0)$ and the automata field T, we conclude that $ker f$ must be either $G(p_0, y_0)$ or T. Thus, the kernel of an automata field homomorphism from T to T' is either $G(p_0, y_0)$ or the automata field T.

Theorem 4.8: Let $\phi : G \to H$ be an automata field homomorphism. Then, ϕ is injective.

Proof: Indeed, assuming ϕ is an automata field homomorphism implies that it is also an automata ring homomorphism. It is worth noting that the kernel of an automata ring homomorphism is an automata ideal, and an automata field G only possesses two automata ideals: $G(p_e, y_e)$ and G.
Furthermore, by the definition of an automata field homomorphism, $\phi(G'(p_e, y_e)) = G'(p_e, y_e)$ holds. Consequently, $G'(p_e, y_e)$ is not in the kernel of the map, implying that the kernel must be equal to $G(p_e, y_e)$. This fact, coupled with the limited options for automata ideals in G, leads to the conclusion that the kernel of ϕ is $G(p_e, y_e)$. Therefore, we have the injectivity of ϕ.

Theorem 4.9: Every finite automata field is perfect.

Proof: Assume T is a finite automata field with characteristic p. Let $\phi : T \to T$ be defined as

$$\phi(G(p_a, y_i)) = G^p(p_a, y_i) \text{ for all } G(p_a, y_i) \in T.$$

We claim that ϕ is an automata field automorphism.
Firstly, observe that

$$\phi(\mu(G(p_a, y_i), G(p_b, y_j))) = \mu(G(p_a, y_i), G(p_b, y_j))^p = G^p(p_{a\oplus_n b}, y_{i\oplus_m j}) = G^p(p_a, y_i) \oplus_n G^p(p_b, y_j) = \mu(\phi(G(p_a, y_i)), \phi(G(p_b, y_j))).$$

Moreover, $\phi(\nu(G(p_a, y_i), G(p_b, y_j))) = \nu(G(p_a, y_i), G(p_b, y_j))^p = G^p(p_{a\otimes_n b}, y_{i\otimes_m j}) = G^p(p_a, y_i) \otimes_n G^p(p_b, y_j) = \nu(\phi(G(p_a, y_i)), \phi(G(p_b, y_j)))$.

Finally, as $G^p(p_a, y_i) \neq G(p_e, y_e)$ when $G(p_a, y_i) \neq G(p_e, y_e)$, it follows that $ker\phi = \{G(p_e, y_e)\}$. Therefore, ϕ is injective. Since T is finite, ϕ is also surjective, implying $T^p = T$.
Hence, every finite automata field is perfect.

V. Structure of Automata Field

In this section, we explore the complexities of the automata field structure, which can be classified into two distinct types. Within the context of this discussion, the organisational framework that defines the scope of automata studies is the primary focus. These two categories of structures have a significant influence on the dynamics of the field as well as the way it is understood. The purpose of this investigation is to provide a comprehensive and illuminating examination with the goal of elucidating the fundamental components that contribute to the overall framework of automata research. The purpose of this examination is to provide insights into the fundamental principles that govern the field, thereby providing a solid understanding for those who are exploring the various aspects of automata theory.

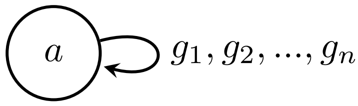

Fig. 4. Structure/type-1

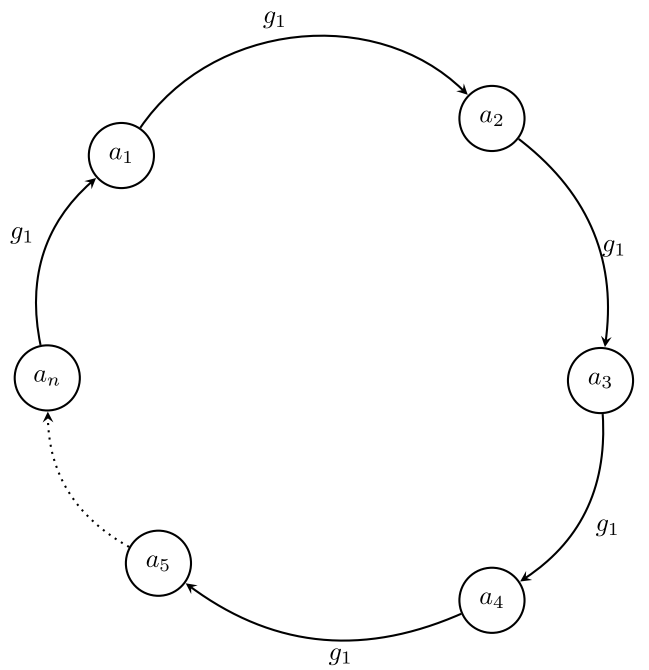

Fig. 5. Structure/type-2

Theorem 5.1: If the set T which has the structure that $o(Q) = p$ and $o(A) = 1$ or $o(Q) = 1$ and $o(A) = p$ with cardinality equals p for some prime p. Then the set T is an automata field.

Proof: By the definition of T we have $T = \{G\ (p_a,\ y_i) \mid p_a \in Q, y_i \in A, a, i \in \mathbb{N} \cup \{0\}\}$. Since we have $o(Q) = p$ and $o(A) = 1$ it follows that $T = \{G\ (p_0,\ y_0), ..., G\ (p_p,\ y_0)\}$ implying $|T| = p$. Notably, the set $\{0, 1, 2, ..., p\} = \mathbb{Z}_p$ is a field. With the operations defined for automata field, it can be deduced that T satisfies the axioms of automata field. Therefore, T is indeed an automata field.

Theorem 5.2: For a set T with cardinality $|T| = p^n$ where $n \neq 1$, T is not an automata field.

Proof: The definition of an automata field specifies that the cardinality of T must be p to satisfy the conditions for an automata field. Specifically, for $|T| = p$, the set T forms an automata field. However, for other forms of p^n, where $n \neq 1$, the set T fails to meet the axioms of an automata field. Consider the elements $G(p_a, y_0)$ and $G(p_0, y_i)$ in T. For these elements, the computation of ν yields:

$$\nu(G(p_a, y_0), G(p_0, y_i)) = G(p_0, y_0),$$
$$\nu(G(p_0, y_0), G(p_a, y_0)) = G(p_0, y_0),$$
$$\nu(G(p_0, y_0), G(p_0, y_i)) = G(p_0, y_0).$$

This implies that for the order of T other than p, the set T does not form an integral domain, and consequently, it does not qualify as an automata field. Therefore, the theorem holds true.

Theorem 5.3: In this automata field, the only sub-automata fields are $S_T = \{G(p_0, y_0)\}$ and T, both of which are trivial sub-automata fields.

Proof: Given that $|T| = p$, by Lagrange's theorem, the order of a subfield must divide the order of the field. The only divisors of the prime number p are 1 and p itself. Consequently, the only sub-automata fields that satisfy the axioms of an automata field are $S_T = \{G(p_0, y_0)\}$ and T. Therefore, these are the only sub-automata fields, and both are trivial.

VI. Conclusion

The researchers have already established the definition of automata rings and have presented theorems and illustrations pertaining to automata rings. We formulated the definition for the automata field as an extension. We have analysed and identified the structures and characteristics of the automata field. We have demonstrated the theorems and corollaries pertaining to the automata field, drawing upon the fundamental theorems and corollaries associated with the algebraic concept of a field. Thus, from the constructed automata field, one can analyse the characteristics and properties of the field with prime order p. In the future, this work has the potential to be expanded to include extension fields, Galois fields, and other related areas.

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