An Iterative Algorithm for Numerical Solution of Nonlinear Fractional Differential Equation using Legendre Wavelet Method

Bharti Thakur, Sandipan Gupta

Abstract— **In this article, an efficient iterative technique has been developed to solve the nonlinear fractional order differential equation based on the Legendre wavelet. The primary objective of the proposed scheme is to convert the nonlinear problem into a system of linear algebraic equations. The efficiency and accuracy of the present scheme have been investigated through various numerical examples and the obtained results demonstrate superior accuracy compared to existing approaches. The solution of the present scheme improves as the Legendre wavelet parameter or number of iterations increases. The convergence of the iterative technique is investigated through numerical examples.**

Index Terms — **Legendre wavelet, fractional integration, Collocation method, Laplace Transformation.**

I. INTRODUCTION

Fractional calculus has a long mathematical history, introduced by L'Hospital in 1695. It is a branch of calculus that extends the definition of the derivative of a function to non-integer orders. In the past decades, this subject has gained widespread and diverse applications in science, engineering, finance, and other critical fields [1], [2], [3], [4]. The integer order differential equation has local behaviour, while on another side the fractional differential equation (FDE) has non-local behaviour. Due to the extensive application of FDE, many researchers have developed efficient numerical methods [5], [6], [7], [8], [9], [10] to obtain approximate numerical solutions.

In the present work, an innovative iterative technique is developed to address the non-linear fractional order initial value problem (IVP)

$$D^{\alpha}u(t) = g\big(t, u(t)\big), 0 < \alpha \leq 1 \quad (1)$$

with

$$u(0) = b. \quad (2)$$

Nowadays many researchers are interested in solving these types of problems and have also proposed many numerical techniques to approximate their solutions. A polynomial least squares method (PLSM) is applied to solve fractional Riccati-type differential equation, as addressed in [11]. Saeed et al. [12] developed a numerical technique to solve the fractional delay type equation using the method of steps and the shifted Chebyshev wavelet method. A Bernoulli wavelet based new function is introduced by Rahimkhani et al [13] to solve FDEs using operational matrix (OM) of fractional integration (FI). The author used collocation method, to build up the given problem into a system of algebraic equations (AE). Yang et al. [14] applied a Jacobi spectral collocation-based numerical method to solve fractional pantograph DDE. The author first converted the given problem into a nonlinear Volterra integral equation and then solutions for these integral equations were derived using a numerical technique. Kumbinarasaiah and Raghunatha [15] used the Hermite wavelet method (HWM) for solving the nonlinear heat transfer problem. The fractional order Legendre collocation spectral method has been used by Mdallal and Omer [16], where the author constructed a system of non-linear equations to solve second-order fractional IVP. Banihashemi et al. [17] applied Legendre collocation method to approximate the solution of non-linear fractional stochastic DDE. Mohammadi and Cattani [18] proposed a numerical scheme to solve non-linear fractional differential equations that are based on the OM and the typical Tau method. There are several methods [19], Bernstein Polynomials Collocation Method (BPCM) [20], [21], [22], [23] that exist in the literature in which the researcher solved the non-linear problem by converting it to a nonlinear system of an AE. Solving the system of the nonlinear AE is the main drawback of these existing methods. Keeping in view the major drawback of these methods, in our study Legendre wavelet (LW) based iterative process has been proposed in the current article. Here the solution to the non-linear problem is approximated as a linear combination of fractional integrals of the LW basis function. In this process, each iterative step reduces the non-linear AE of wavelet coefficients to a system of linear AE whose solutions are used in the next iteration (iter) for the improvement of the solution of the problem. The exact FI and Legendre wavelet method (LWM) have been used in the present work. To the best of our knowledge, the idea presented in this paper has not been previously addressed in existing literature.

The remaining article is structured in the following manner: In section II, important results and essential definitions related to the present work have been introduced. Section III is associated with the proposed technique for solving the IVP based on the LWM. The error analysis is presented in Section IV. The proposed scheme's reliability and efficiency are demonstrated through numerical examples in Section V, culminating in a conclusive presentation in Section VI.

II. PRELIMINARIES

In this section, various properties and definitions pertaining to fractional calculus are presented.

Definition 1. [4] C_μ and C_μ^n spaces are defined on the domain $[0, \infty]$ as follows

$$C_\mu = \{f(t) \in \mathbb{R}: p(> \mu) \in \mathbb{R}, f_1(t) \in C[0, \infty], f(t) = t^p f_1(t)\}$$

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and $C_\mu^n = \{f \in C_\mu : f^{(n)}, exists\}$.

Definition 2. [1] The Riemann-Liouville (RL) fractional order integration I^α of α which is greater than equal to zero of a function $f \in C_\mu$ is defined

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, & \alpha > 0 \\ f(t), & \alpha = 0 \end{cases}$$

Definition 3. [1] The RL differential operator ${}^R D_t^\alpha f(t)$ of order $\alpha > 0$, $f \in C_1^n$ is defined as

$${}^R D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-s-1} f(s)ds, & n-1 < \alpha < n \\ \frac{d^n f(t)}{dt^n}, & \alpha = n \end{cases}$$

where $n \in \mathbb{Z}$.

Definition 4. [1] The Caputo fractional differential operator D_*^α of α which is greater than zero of a function $f \in C_1^n$ is

$$D_*^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n-1 < \alpha \leq n, t > 0$ and n is an integer.

Some important results of the Caputo derivative are used in this paper

$$D^\alpha t^\beta = \begin{cases} 0 & , \quad \beta < \alpha \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & otherwise, \end{cases}$$

where $\beta \in N_0$.

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} f^i(0)\frac{t^i}{i!}.$$

$$D^\alpha I^\alpha f(t) = f(t).$$

Definition 5. [4] The unit step function is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

and the shifted unit step function is defined as

$$u(t-c) = \begin{cases} 0, & t < c \\ 1, & t > c \end{cases}$$

Definition 6. [1] The convolution of $f(t)$ and $g(t)$ for $t > 0$, is given by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

The Laplace transformation of convolution is

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

Theorem 1. [24] (Second Translation Theorem) If $\mathcal{L}\{f(t)\} = F(s)$, then,

$$\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s), c \geq 0. \quad (3)$$

Lemma 1. [25] Let $z: [0,1] \to \mathbb{R}$ and $I^\alpha(.)$ represents RL fractional integral operator. Then,

$$\|I^\alpha z(t)\|_\infty \leq \frac{1}{\Gamma(\alpha+1)}\|z(t)\|_\infty.$$

Wavelets and Legendre wavelets:

Wavelets constitute a family of functions constituted from the dilation and translation of a single function called mother wavelet is defined by

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), a \neq 0, a, b \in \mathbb{R}$$

where a is the dilation parameter and the translation parameter b.

The LW $\psi_{n,m}(t) = \psi_{n,m}(k, \hat{n}, m, t)$ defined within [0,1) as

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\left(m+\frac{1}{2}\right)} 2^{\frac{k}{2}} L_m(2^k t - \hat{n}), \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k} \\ 0 \qquad , \qquad otherwise \end{cases}$$

where $\hat{n} = 2n-1$, $n = 1,2,\ldots,2^{k-1}, k \in \mathbb{N}$, and t is the normalized time. $L_m(t)$ represents the Legendre polynomial of order $m = 0,1,\ldots,M-1$, where M is a fixed positive integer. These polynomial are orthogonal w.r.t. the weight function $w(t) = 1$over $[-1,1]$, satisfying a following recursive formula

$$L_0(t) = 1, L_1(t) = t,$$

$$L_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t L_m(t) - \left(\frac{m}{m+1}\right) L_{m-1}(t),$$

A function $f(t)$ defined over [0,1] may be expressed by LW as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \quad (4)$$

where $c_{nm} = \langle f(t), \psi(t) \rangle$. The above equation is truncated and can be written as

$$f(t) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\psi(t) \quad (5)$$

where $C^T$and $\psi(t)$ are column matrices of order $2^{k-1}M \times 1, (\hat{m} = 2^{k-1}M)$, given by

$$C = \left[c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}M-1}\right]^T$$

$$\psi(t) = \left[\psi_{10}(t), \ldots, \psi_{1M-1}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M-1}(t)\right]^T \quad (6)$$

Fractional integration of Legendre wavelet:

The FI has already been studied in [26],[27]. The FI of the LW basis function is expressed using the unit step function as follows

$$\psi_{nm}(t) = \sqrt{2m+1}2^{\frac{k-1}{2}}\left[u\left(t-\frac{n-1}{2^{k-1}}\right)L_m(2^k t - 1 - 2(n-1)) - u\left(t-\frac{n}{2^{k-1}}\right)L_m(2^k t + 1 - 2n)\right]$$

Taking Laplace transformation in the above equation and using (3), we have

$$\mathcal{L}\{\psi_{nm}(t)\} = \sqrt{2m+1}2^{\frac{k-1}{2}}\left[e^{-\frac{n-1}{2^{k-1}}s}\mathcal{L}\{L_m(2^k t - 1)\} - e^{-\frac{n}{2^{k-1}}s}\mathcal{L}\{L_m(2^k t + 1)\}\right] \quad (7)$$

Consider

$$L_m(2^k t - 1) = 2^m \sum_{v=0}^{m}\binom{m}{v}\binom{\frac{m+v-1}{2}}{m}(2^k t - 1)^v$$

$$L_m(2^k t - 1) = 2^m \sum_{v=0}^{m}\binom{m}{v}\binom{\frac{m+v-1}{2}}{m}\sum_{\eta=0}^{v}\binom{v}{\eta}(-1)^\eta\, 2^{k(v-\eta)} t^{v-\eta},$$

taking Laplace transformation in the above equation

$$\mathcal{L}\{L_m(2^k t - 1)\} = 2^m \sum_{v=0}^{m}\binom{m}{v}\binom{\frac{m+v-1}{2}}{m}\sum_{\eta=0}^{v}\frac{v!}{\eta!}(-1)^\eta\, 2^{k(v-\eta)}\frac{1}{s^{v-\eta+1}}, \quad (8)$$

similarly

$$\mathcal{L}\{L_m(2^k t + 1)\} = 2^m \sum_{v=0}^{m}\binom{m}{v}\binom{\frac{m+v-1}{2}}{m}\sum_{\eta=0}^{v}\frac{v!}{\eta!}2^{k(v-\eta)}\frac{1}{s^{v-\eta+1}}, \quad (9)$$

substitute the values of (8) and (9) in (7), we get

$\mathcal{L}\{\psi_{nm}(t)\}$
$$= \sqrt{2m+1}\, 2^{\frac{k-1}{2}} 2^m \sum_{v=0}^{m} \binom{m}{v} \binom{\frac{m+v-1}{2}}{m} \sum_{\eta=0}^{v} \frac{v!}{\eta!} 2^{k(v-\eta)} H$$
where $H = \frac{1}{s^{v-\eta+1}}\left[(-1)^\eta e^{-\frac{n-1}{2^{k-1}}s} - e^{-\frac{n}{2^{k-1}}s}\right]$. RL integration is then defined using convolution properties as
$$I^\alpha \psi_{nm}(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * \psi_{nm}(t).$$
Again, taking Laplace's transformation
$$\mathcal{L}\{I^\alpha \psi_{nm}(t)\} = \frac{1}{s^\alpha}\mathcal{L}\{\psi_{nm}(t)\},$$
$\mathcal{L}\{I^\alpha \psi_{nm}(t)\}$
$$= \sqrt{2m+1}\, 2^{\frac{k-1}{2}} 2^m \sum_{v=0}^{m} \binom{m}{v} \binom{\frac{m+v-1}{2}}{m} \sum_{\eta=0}^{v} \frac{v!}{\eta!} 2^{k(v-\eta)} G$$
where $G = \frac{1}{s^\alpha} H$.

Taking the inverse Laplace transformation in the above equation, we have
$$I^\alpha \psi_{nm}(t) = \begin{cases} 0, & t \in [0, \frac{n-1}{2^{k-1}}) \\ \sqrt{2m+1}\, 2^{\frac{k-1}{2}}\, 2^m V, & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}) \\ \sqrt{2m+1}\, 2^{\frac{k-1}{2}}\, 2^m S, & t \in \left[\frac{n}{2^{k-1}}, 1\right] \end{cases} \quad (10)$$
where
$$V = \sum_{v=0}^{m}\sum_{\eta=0}^{v} \binom{m}{v} \binom{\frac{m+v-1}{2}}{m} \frac{v!}{\eta!} \frac{2^{k(v-\eta)}(-1)^\eta J}{\Gamma(v-\eta+1+\alpha)},$$
where $J = \left(t - \frac{n-1}{2^{k-1}}\right)^{v-\eta+\alpha}$ and

S
$$= \sum_{v=0}^{m}\sum_{\eta=0}^{v} \binom{m}{v} \binom{\frac{m+v-1}{2}}{m} \frac{v!}{\eta!} \frac{2^{k(v-\eta)}}{\Gamma(v-\eta+1+\alpha)} \left[(-1)^\eta \left(t - \frac{n-1}{2^{k-1}}\right)^{v-\eta+\alpha} - \left(t - \frac{n}{2^{k-1}}\right)^{v-\eta+\alpha}\right].$$

III. PROPOSED METHOD

This section describes the proposed iterative technique based on LW, employed for solving the non-linear fractional order initial value problem, defined as
$$D^\alpha u(t) = g(t, u(t)), \quad (11)$$
with
$$u(0) = b, 0 < \alpha \le 1 \quad (12)$$
For solving the problem in (11)-(12), let us assume that
$$D^\alpha u(t) \approx \psi^T(t) C = D^\alpha u_{\hat{m}}(t) \quad (13)$$
where C and $\psi(t)$ are $2^{k-1}M \times 1$ order matrices. Applying RL integration operator I^α in (13) and using condition in (12), we have
$$u(t) - u(0) \approx [I^\alpha \psi(t)]^T C$$
$$u(t) \approx [I^\alpha \psi(t)]^T C + u(0) = u_{\hat{m}}(t), \quad (14)$$
substituting values of $u(t)$ and $D^\alpha u(t)$ in (11), we get
$$\psi^T(t) C = g(t, [I^\alpha \psi(t)]^T C + u(0))$$
$$\psi^T(t) C = g(t, u(t)) \quad (15)$$
To implement the iterative process, we have considered the initial approximation of C^0 as a non-zero vector (obtained by solving $\psi^T(t) C^0 = u(0)$, if $u(0) \neq 0$, otherwise any non-zero random vector) which is substituted in R.H.S. of the above expression (15), and C of L.H.S. are considered as modified unknown coefficients which need to be determined. The above equation is written as
$$\psi^T(t) C^{i+1} = g(t, u_i(t)), i = 0,1,2, \dots \quad (16)$$
where $u_i(t) = [I^\alpha \psi(t)]^T C^i + u(0)$.

Then discretizing the above expression using collocation points $t_j = \frac{2j-1}{2^k M}, j = 1,2,3, \dots, 2^{k-1}M$, to obtain a linear system of AE for each $i = 0, 1, 2, \dots$
$$\psi^T(t_j) C^{i+1} = g\left(t_j, u_i(t_j)\right), \quad (17)$$
$$C^{i+1} = B^{-1} F(C^i), \quad (18)$$
where $F(C^i) = g\left(t_j, u_i(t_j)\right)$ and $B = \psi^T(t_j)$ be the non-singular matrix. By solving (18), the unknown coefficient C^{i+1} is obtained. This coefficient will be re-substituted in the R.H.S. of (16) and again discretized to produce a linear system of AE in (18). When this linear system is solved, the improved value of the unknown coefficient C^{i+1} is obtained. This iterative process has been repeated until the final modified unknown coefficient $max|C^{i+1} - C^i| < \varepsilon$ where $\varepsilon > 0$, is attained.

Substituting the final modified unknown coefficient C^{i+1} in (14), we arrive at an estimate for the solution to the specified problem outlined in (11).

The algorithm of this proposed method follows the following steps.

Algorithm

Step 1: Set ε, α, and LW parameters k, M, as input values.
Step 2: Find the LW basis function $\psi(t)$ defined in (6) for input parameters k, M.
Step 3: Calculate the exact FI $I^\alpha \psi(t)$ in equation (10) of section II.
Step 4: Set $i = 0$ in (16) to start the iteration process, and the initial approximate C^0 in R.H.S of this equation as a column matrix $2^{k-1}M \times 1$.
Step 5: Discretize the expression obtained in (16) using the collocation point $t_j = \frac{2j-1}{2^k M}, j = 1,2,3, \dots, 2^{k-1}M$ to convert nonlinear equations to linear equations as mentioned in equation (18).
Step 6: Calculate the modified coefficient C^{i+1} from the expression $C^{i+1} = B^{-1}F(C^i)$ in (18).
Step 7: If $max|C^{i+1} - C^i| < \varepsilon$, proceed to step 10 otherwise go to the next step.
Step 8: Store C^{i+1} in $(i+1)$th column of a matrix U. Set $i = i + 1$ in (16) and substitute the modified coefficient in the R.H.S of (16).
Step 9: Repeat the procedure starting from step 5.
Step 10: Substitute the modified coefficient $C = C^{i+1}$ in $u(t) \approx [I^\alpha \psi(t)]^T C + u(0)$ to acquire the solution of problem in (11).

IV. ERROR ANALYSIS

The error analysis of the devised approach, as outlined in section III, for addressing nonlinear FDE is described in this segment. For non-integer order α the exact solution is unknown in most cases, so the following procedure has been demonstrated to verify the reliability and effectiveness of the present technique.

Let

$$\|\varphi(t)\|_\infty = \begin{matrix} sup|\varphi(t)| \\ t \in [0,1] \end{matrix}$$

Let $u(t)$ satisfies

$$D^\alpha u(t) = g(t, u(t)) \tag{19}$$

and $u_{\hat{m}}(t)$ satisfies

$$D^\alpha u_{\hat{m}}(t) = g\big(t, u_{\hat{m}}(t)\big) - R_{\hat{m}}(t) \tag{20}$$

where $R_{\hat{m}}(t)$ is the residual function due to the substitution of the approximate solution $u_{\hat{m}}(t)$ in (19). Substituting (20) from (19) we have,

$$D^\alpha\big(u(t) - u_{\hat{m}}(t)\big) = g\big(t, u(t)\big) - g\big(t, u_{\hat{m}}(t)\big) - R_{\hat{m}}(t) \tag{21}$$

Denoting the error $e_{\hat{m}}(t) = u(t) - u_{\hat{m}}(t)$, and following the idea [28], the above equation (21) reduces to,

$$D^\alpha\big(e_{\hat{m}}(t)\big) = g\big(t, u(t)\big) - g\big(t, u_{\hat{m}}(t)\big) - R_{\hat{m}}(t) \tag{22}$$

Therefore,

$$e_{\hat{m}}(t) = I^\alpha\left(g\big(t, u(t)\big) - g\big(t, u_{\hat{m}}(t)\big)\right) - I^\alpha R_{\hat{m}}(t) \tag{23}$$

where from (14) $e_{\hat{m}}(t) = 0$. Using the Lemma 1, the above equation reduces to,

$$|e_{\hat{m}}(t)| \le \frac{t^\alpha}{\Gamma(\alpha+1)}\left(\left|g\big(t,u(t)\big) - g\big(t,u_{\hat{m}}(t)\big)\right| + |R_{\hat{m}}(t)|\right)$$

If we assume $u(t)$ and $u_{\hat{m}}(t)$ is bounded in $0 \le t \le 1$ and $\left|g\big(t,u(t)\big) - g\big(t,u_{\hat{m}}(t)\big)\right| \le \rho|u(t) - u_{\hat{m}}(t)|, \rho > 0$, then,

$$|e_{\hat{m}}(t)| \le \frac{t^\alpha}{\Gamma(\alpha+1)}\left(\rho\,|u(t) - u_{\hat{m}}(t)| + |R_{\hat{m}}(t)|\right)$$

$$|e_{\hat{m}}(t)| \le \frac{t^\alpha}{\Gamma(\alpha+1)}\left(\rho \sup_{t\in[0,1]} |u(t) - u_{\hat{m}}(t)| + \sup_{t\in[0,1]} |R_{\hat{m}}(t)|\right)$$

$$\|e_{\hat{m}}(t)\|_\infty \le \frac{1}{\Gamma(\alpha+1)}\left(\rho\|e_{\hat{m}}(t)\|_\infty + \|R_{\hat{m}}(t)\|_\infty\right)$$

$$\|e_{\hat{m}}(t)\|_\infty \le \frac{\|R_{\hat{m}}(t)\|_\infty}{\Gamma(\alpha+1) - \rho}$$

If $\Gamma(\alpha+1) - \rho \ne 0$ and the residual function $R_{\hat{m}}(t)$ approaches to zero as $\hat{m}$ approaches to infinity, then the error $\|e_{\hat{m}}(t)\|_\infty$ approaches to zero.

It is clear from (20) that at the collocation point, $t_j = \frac{2j-1}{2^k M}, j = 1,2,3, \dots, 2^{k-1}M$,

$$R_{\hat{m}}(t_j) = \left|D^\alpha u_{\hat{m}}(t_j) - g\left(t_j, u_{\hat{m}}(t_j)\right)\right| \cong 0 \tag{24}$$

So $\|e_{\hat{m}}(t)\|_\infty \cong 0$.

V. NUMERICAL IMPLEMENTATION

In this section, different types of nonlinear problems will be addressed to illustrate the efficacy of the current scheme.

Example 1. Consider nonlinear fractional order IVP [18]

$$D^\alpha u(t) + u^\gamma(t) = t + \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)^\gamma, \ \gamma \ge 2 \tag{25}$$

with $u(0) = 0, 0 < \alpha \le 1$.

The problem was studied in [18] for $\gamma = 2$. The exact solution of (25) is $u(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$.

$D^\alpha \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} = C^T\psi(t) = t$ is polynomial in t and can be exactly represented by the linear combination of LW basis function $k \ge 1, M \ge 2$. Considering the parameter values $k = 1, M = 2$, we have

$$t = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{3}(2t-1) \end{pmatrix}$$

So, the solution problem $u(t) = u(0) + C^T I^\alpha \psi(t)$ can also be represented exactly as $I^\alpha\psi(t)$ wavelet basis function $k \ge 1, M \ge 2$, considering the parameter values $k = 1, M = 2$,

$$\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \end{pmatrix}\begin{pmatrix} \frac{t^\alpha}{\Gamma(\alpha+1)} \\ \frac{2\sqrt{3}t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{\sqrt{3}t^\alpha}{\Gamma(\alpha+1)} \end{pmatrix}.$$

Since the problem (25) converges to $u(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$ for $k \ge 1, M \ge 2$, the proposed methodology finds the unknown wavelet coefficient by an iterative process to find the approximate solution to the problem for fixed parameter k, M.

For $\gamma = 2$, Table I shows the difference between the exact LW coefficient (C_e) and the wavelet coefficient obtained by the proposed scheme (C_p) for different k, M. This table depicts that the proposed method highly approximates the wavelet coefficient C_p in 120 iterations with the maximum difference 10^{-40}. The approximate solution of problem (25) is plotted in Figure 1 for different α. Table II presents the absolute error for $\alpha = 0.5, 0.8, 1$, and compared with those provided in [13] and [28]. This problem has also been investigated for each γ from 2 to 9.5 with step length 0.5. At $\alpha = 0.5, 0.8, 1$ with wavelet parameter $k = 1, M = 2$, it has been observed that at 120 iterations the maximum absolute error for the solution of (25) in the discretized domain of γ for $\alpha = 0.5$ is 1.15e-40, $\alpha = 0.8$ is 1.03e-40, and $\alpha = 1$ is 6.89e-41. It is clear that as α increases, the absolute error decreases.

TABLE I
COMPARISON OF WAVELET COEFFICIENT C OF EXAMPLE 1 WITH $ITERATION = 120$

C_e	$(C_e - C_p)$ k=1, M=2	$(C_e - C_p)$ k=1, M=3	C_e k=2, M=2	$(C_e - C_p)$ k=2, M=2
$1/2$	0	9.184e-41	$\sqrt{2}/8$	0
$1/2\sqrt{3}$	-1.11e-40	1.837e-40	$\sqrt{6}/24$	2.296e-41
0	-	1.027e-40	$3\sqrt{2}/8$	1.837e-40
-	-	-	$\sqrt{6}/24$	1.378e-40

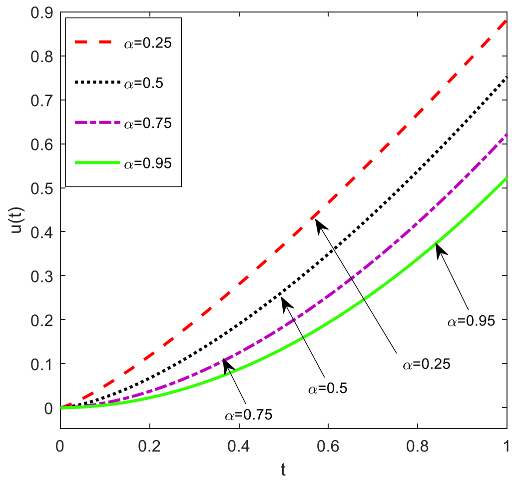


Fig. 1. The approximate solution with $k = 2, M = 4$, and different α of Example 1 at $iteration = 800$.

TABLE II

COMPARISON OF ABSOLUTE ERROR FOR $ITERATION = 120$ WITH DIFFERENT α OF EXAMPLE 1.

t	Present Method $k = 1,\ M = 2$			BWM [13]			GBCM [28]
	$\alpha = 0.5$	$\alpha = 0.8$	$\alpha = 1$	$\beta = 0.5$, $\alpha = 0.5$	$\beta = 0.8$, $\alpha = 0.8$	$\beta = 0.5$, $\alpha = 1$	$\beta = 0.5$, $\alpha = 0.5$
0	0	0	0	8.44e-10	1.27e-05	3.05e-04	-
0.1	4.5918e-41	0	1.1479e-41	1.42e-09	2.00e-06	3.57e-05	8.6736e-16
0.2	9.1835e-41	4.5918e-41	2.2959e-41	1.71e-09	2.94e-06	2.30e-05	1.0131e-15
0.3	9.1835e-41	0	4.5918e-41	1.95e-08	2.86e-06	2.15e-05	1.4794e-14
0.4	9.1835e-41	9.1835e-41	2.2959e-41	1.47e-08	1.51e-06	1.87e-05	8.7930e-14
0.5	0	9.1835e-41	4.5918e-41	1.14e-08	3.97e-04	1.64e-05	2.4547e-13
0.6	6.8877e-41	2.2959e-41	6.8877e-41	8.98e-09	3.60e-04	1.02e-05	5.0115e-13
0.7	5.7397e-41	0	0	7.47e-09	3.24e-04	1.91e-05	8.6287e-13
0.8	4.5918e-41	0	2.2959e-41	6.98e-09	2.85e-04	1.65e-04	1.3342e-12
0.9	4.5918e-41	0	1.1479e-41	7.78e-09	2.40e-04	7.00e-04	1.9169e-12
1	1.8367e-40	4.5918e-41	0	1.02e-08	1.76e-04	2.25e-03	2.6106e-11

Example 2. Consider the nonlinear fractional order IVP

$$D^{\alpha}u(t) = t^3u^2(t) - 2t^4u(t) + t^5 + 1 \qquad (26)$$

with

$$u(0) = 0, 0 < \alpha \leq 1.$$

The exploration and solution of this problem were earlier accomplished by [28] through the utilization of the generalized Bell collocation method (GBCM).

The exact solution of (26) is $u(t) = t$, for $\alpha = 1$. $D^{\alpha}t = C^T\psi(t) = 1$ is polynomial of t^0 and can be exactly represented by the linear combination of the LW basis function for $k \geq 1, M \geq 1$. For $k \geq 1, M \geq 1$, the solution has been investigated and is seen as the number of iterations reaches to 4, the difference between the coefficients C_p and C_e is zero. For $k = 1$, $M = 1$ the difference between coefficients C_e and C_p are tabulated in Table III with different number of iterations. It is clear from the table that when the iteration reaches to 4, the wavelet coefficient C_p approximates exactly as the coefficient C_e . The absolute errors are listed in Table IV and also compared with results from [29], [30], [28]. The given table illustrates that the achieved results are highly accurate as compared to the existing method. The numerical results for $\alpha = 0.2, 0.7, 1$ are depicted in Figure 2 and the zoomed in portion of these solutions is also included in this figure.

TABLE IV

COMPARISON OF ABSOLUTE ERROR WITH DIFFERENT METHODS FOR EXAMPLE 2

t	IRKHSM [29]	JCM [30]	GBCM [28]	Present Method $M = 1, k = 1$		
				iter = 1	*iter = 2*	*iter = 3*
0.1	5.59e-07	7.45e-07	8.80e-15	3.4e-07	3.3e-23	6.4e-29
0.2	1.11e-06	8.51e-07	1.45e-14	6.8e-07	6.6e-23	1.3e-28
0.3	1.67e-06	9.30e-07	1.05e-14	1.0e-06	9.9e-23	1.9e-28
0.4	2.23e-06	1.08e-06	5.01e-15	1.4e-06	1.3e-22	2.6e-28
0.5	2.79e-06	1.14e-06	9.75e-16	1.7e-06	1.6e-22	3.2e-28
0.6	3.30e-06	1.14e-06	3.62e-14	2.0e-06	2.0e-22	3.8e-28
0.7	3.95e-06	1.21e-06	1.81e-13	2.4e-06	2.3e-22	4.5e-28
0.8	4.56e-06	1.04e-06	5.90e-13	2.7e-06	2.6e-22	5.1e-28
0.9	5.24e-06	1.13e-06	1.51e-12	3.1e-06	3.0e-22	5.8e-28
1	6.07e-06	4.84e-07	3.32e-12	3.4e-06	3.3e-22	6.4e-28

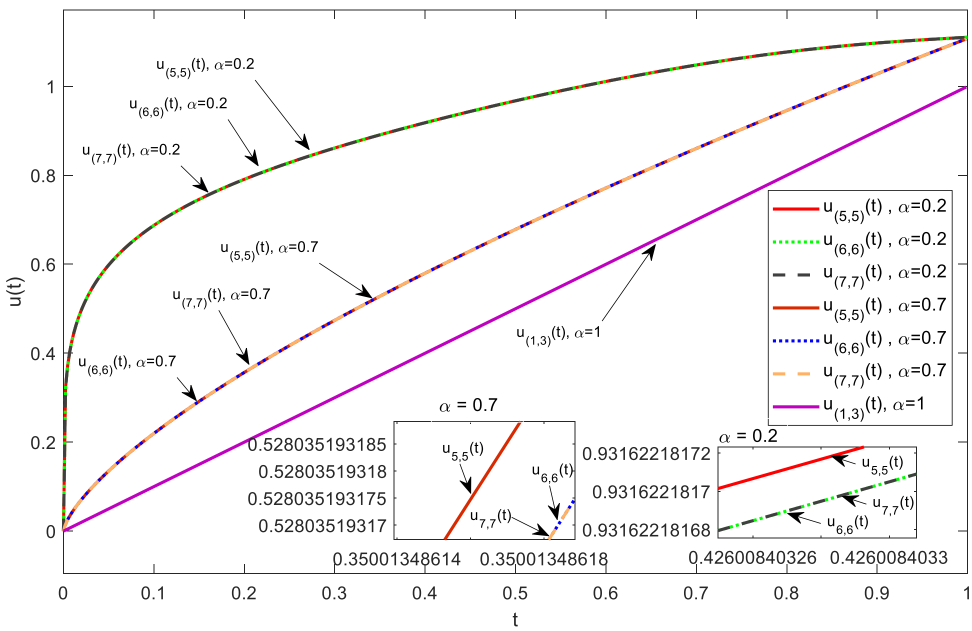


Fig. 2. Comparison of approximate solution with various values of parameters k, M of LW and α of Example 2 at $iteration = 4$.

Table III
COMPARISON OF WAVELET COEFFICIENT C AT $\alpha = 1$ AND $k = 1, M = 1$ FOR EXAMPLE 2.

C_e	$C_e - C_p$			
iter	1	2	3	4
1	-2.4038e-09	-1.8057e-19	-1.1020e-39	0

Example 3. Consider the nonlinear FDE

$$D^\alpha u(t) = \frac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - \frac{3\Gamma\left(5-\frac{\alpha}{2}\right)}{\Gamma\left(5-\frac{\alpha}{2}\right)} t^{\left(4-\frac{\alpha}{2}\right)} + \frac{9}{4}\Gamma(\alpha+1) + \left(\frac{3}{2}t^{\frac{\alpha}{2}} - t^4\right)^3 - (u(t))^{\frac{3}{2}},$$

with $0 < \alpha \leq 1$ and $u(0) = 0$.

$u(t) = t^8 - 3t^{\left(4+\frac{\alpha}{2}\right)} + \frac{9}{4}t^\alpha$ is the exact solution. $D^\alpha u(t)$ is not a polynomial form, so the proposed scheme's reliability is evaluated by increasing the LW parameter k, M. For various values of parameter k of LW, the absolute errors are tabulated in Table V at iteration 54, and observed that it is stable after iteration 54. From the given table it is clear that as k increases, the error decreases. For $k = 5$ and $M = 7$, the approximate solution is plotted in Figure 3 with different α. For $\alpha = 0.5$, the residual error is shown in Figure 4. It is clear from Figure 4 that as k increases, the residual error decreases.

TABLE V
ABSOLUTE ERROR WITH VARIOUS VALUES OF k AT $M = 7$, ITERATION = 54, AND $\alpha = 0.5$ FOR EXAMPLE 3.

t	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
0.1	3.64e-10	4.88e-12	3.15e-13	1.34e-14	5.02e-16
0.2	5.02e-11	4.12e-12	1.86e-13	7.32e-15	2.54e-16
0.3	5.46e-11	2.99e-12	1.24e-13	4.78e-15	1.55e-16
0.4	3.30e-11	2.22e-12	8.99e-14	3.43e-15	1.05e-16
0.5	1.93e-10	4.57e-13	6.22e-14	2.59e-15	7.82e-17
0.6	2.87e-11	1.42e-12	5.53e-14	2.12e-15	6.733-17
0.7	2.98e-11	1.08e-12	4.71e-14	1.80e-15	6.84e-17
0.8	1.94e-11	1.03e-12	4.19e-14	1.61e-15	8.12e-17
0.9	1.26e-13	9.69e-13	3.92e-14	1.54e-15	1.06e-16
1	2.94e-10	7.39e-13	3.01e-14	1.54e-15	1.45e-16

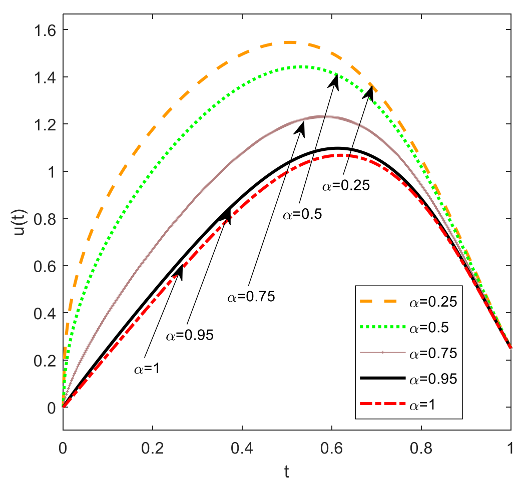
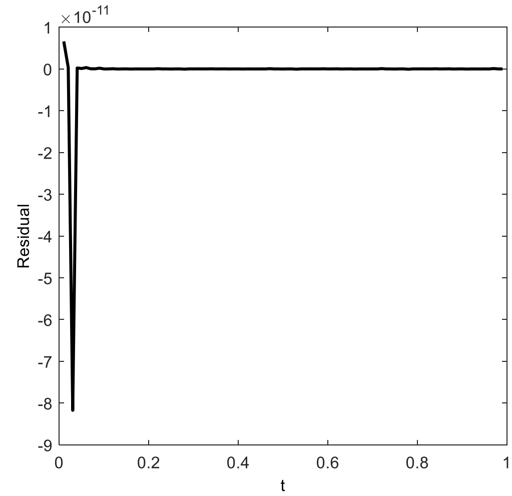
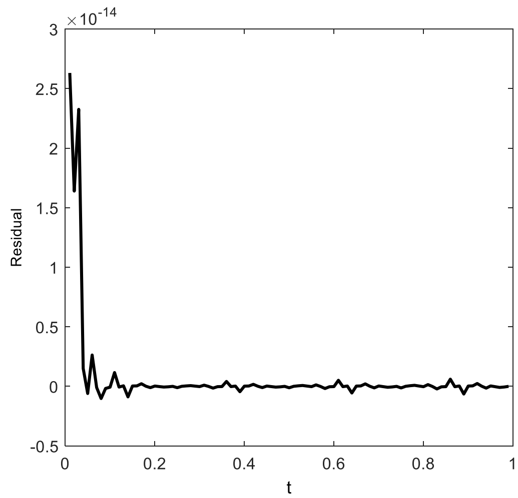


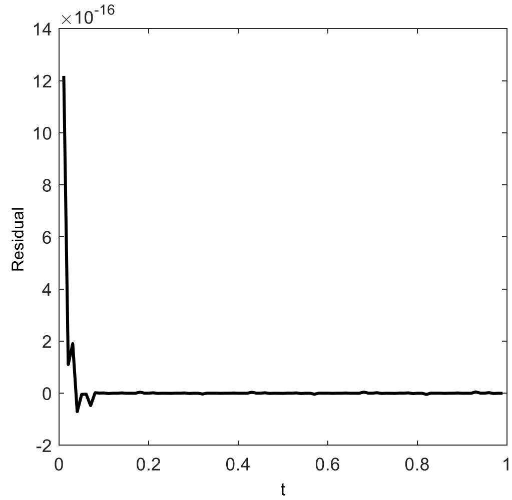
Fig. 3. Comparison of approximate solution with $M = 7$, $k = 5$, and iteration=73 of Example 3 for different α.



(a) $k = 6$



(b) $k = 7$



(c) $k = 8$

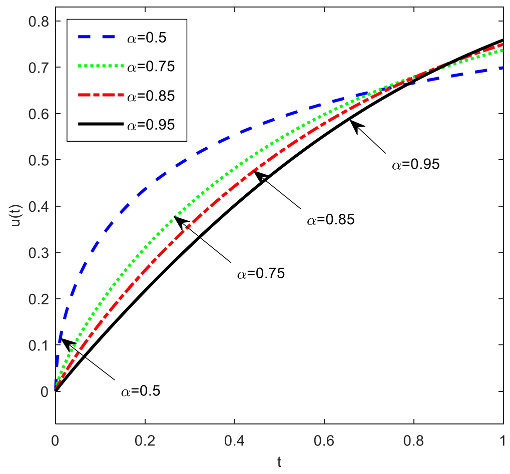
Fig. 4. Residual error of Example 3 with various values of k at $M = 7$, $\alpha = 0.5$ and $iteration = 54$.

Example 4. Consider the nonlinear FDE as defined in [11]

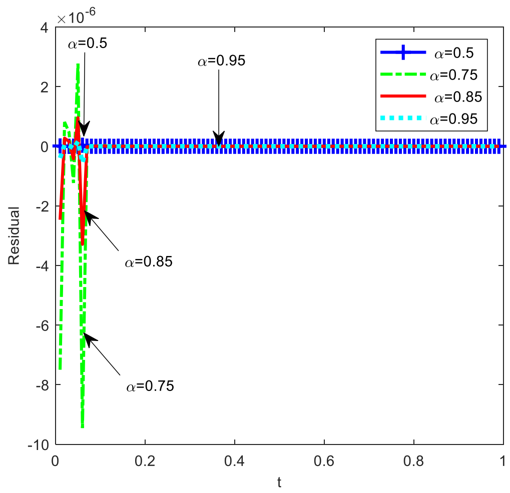
$$D^\alpha u(t) + u^2(t) - 1 = 0 \quad (27)$$

with $u(0) = 0, , 0 < \alpha \leq 1$. Let $u_{k,M}(t)$ represents the approximate solution of LW parameters M and k. The exact solution of (27) is $u(t) = \frac{e^{2t}-1}{e^{2t}+1}$ for $\alpha = 1$. The problem (27) has already been solved in Modified Homotopy Perturbation Method (MHPM) [31], Enhanced Homotopy Perturbation

Method (EHPM) [32], Bernstein Polynomials Collocation Method (BPCM)[20], Haar wavelet Operational Matrix Method (HWOMM) [6], where they have used different techniques. To verify the effect of α in the present scheme, we display the approximate solutions for various α in Figure 5 (a), with their residual errors is plotted in Figure 5 (b).



(a)



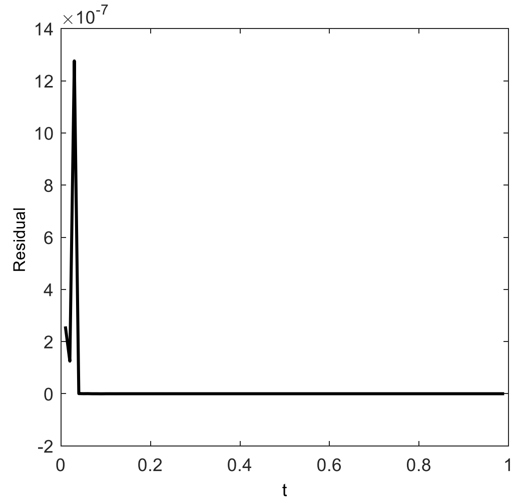
(b)

Fig. 5. For parameter $k = M = 5$, (a) approximate solution (b) residual error of Example 4 at $iteration = 35$

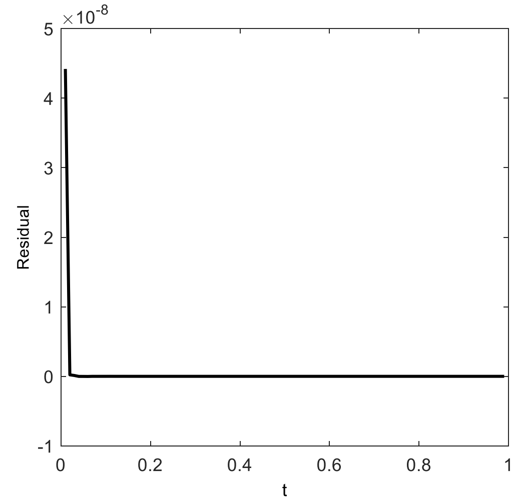
The difference in solution between different values of k is tabulated in Table VI. For $\alpha = 0.75$ the residual error is plotted in Figure 6 for different values of k.

TABLE VI
DIFFERENCES OF APPROXIMATE SOLUTION $(u_{k,M} - u_{k-1,M})$ AT $M = 6, \alpha = 0.75$, AND ITERATION=20 FOR EXAMPLE 4.

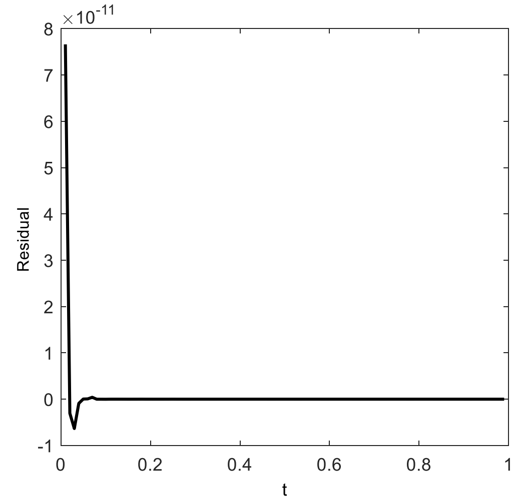
t	$k = 5$	$k = 6$	$k = 7$	$k = 8$
0.1	-7.05e-07	-1.53e-07	-2.63e-08	-4.61e-09
0.2	-6.65e-07	-1.14e-07	-1.99e-08	-3.52e-09
0.3	-5.25e-07	-9.14e-08	-1.62e-08	-2.84e-09
0.4	-4.27e-07	-7.48e-08	-1.32e-08	-2.33e-09
0.5	-3.52e-07	-6.18e-08	-1.09e-08	-1.92e-09
0.6	-2.92e-07	-5.14e-08	-9.07e-09	-1.60e-09
0.7	-2.45e-07	-4.31e-08	-7.60e-09	-1.34e-09
0.8	-2.06e-07	-3.08e-08	-5.44e-09	-9.61e-10
0.9	-1.49e-07	-2.63e-08	-4.64e-09	-8.20e-10



(a) $k = 6$



(b) $k = 7$



(c) $k = 8$

Fig. 6. Residual error of Example 4 with various values of k at $M = 6, \alpha = 0.75$ and $iteration = 20$.

For various values of parameter M, k of LW, the absolute error is tabulated in Table VII. For $\alpha = 0.75, 0.9,$ the approximate solution is listed in Table VIII-IX and compared with [31], [32], [6], [11], [20]. The given tables demonstrate that the achieved results are more accurate as compared to existing methods. Table X shows the absolute error for $k = 7, M = 6$ and compared with those provided in [29], [28] where it is observed that the achieved results are more accurate as compared to [29], [28].

Example 5. Consider the nonlinear FDE [14], [33], [12]

$$D^{\alpha}u(t) = 1 - 2u^2(0.5t), 0 < \alpha \leq 1 \quad (28)$$

with

$$u(0) = 0.$$

The exact solution of (28) is $u(t) = sin(t)$ for $\alpha = 1$. Table XI presents the approximate solution for $k = 2$, $M = 9$, which is compared with [33]. The data given in this table illustrates that the achieved results are more accurate as compared to other existing methods. For $0 < \alpha < 1$, since (28) lacks an exact solution, the reliability of the proposed scheme is assessed using the error estimation formula in (20). The results, depicted in Figure 7 show the residual error for different k with $\alpha = 0.25$, $M = 5$. It is clear from the figure that as LW parameter k increases the residual error decreases.

TABLE VII
ABSOLUTE ERROR OF EXAMPLE 4 WITH VARIOUS VALUES OF PARAMETER M, k OF LW AND ITERATION $= 23$.

t	$M = 6$				$k = 5$			
	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$M = 7$	$M = 8$	$M = 9$	$M = 10$
0.1	1.41e-11	3.13e-13	5.72e-15	8.43e-17	7.25e-16	6.54e-17	1.82e-19	1.33e-20
0.2	3.39e-11	5.99e-13	9.05e-15	1.38e-16	1.10e-15	1.10e-16	2.60-19	1.96e-20
0.3	4.04e-11	6.33e-13	1.00e-14	1.57e-16	1.04e-15	1.05e-16	2.12e-19	1.61e-20
0.4	3.19e-11	5.39e-13	8.72e-15	1.35e-16	6.47e-16	6.09e-17	4.28e-20	3.53e-21
0.5	2.28e-11	3.54e-13	5.52e-15	8.61e-17	2.68e-16	1.33e-17	2.64e-20	5.44e-21
0.6	1.31e-11	1.45e-13	1.72e-15	3.02e-17	2.95e-16	2.40e-17	1.28e-19	9.25e-21
0.7	1.84e-12	9.17e-14	1.10e-15	1.39e-17	5.21e-16	4.86e-17	1.08e-19	8.19e-21
0.8	1.28e-11	1.68e-13	2.91e-15	4.78e-17	4.91e-16	4.72e-17	7.33e-20	5.65e-21
0.9	1.67e-11	2.51e-13	3.77e-15	6.01e-17	3.99e-16	3.64e-17	1.50e-20	1.30e-21
1	1.60e-11	2.48e-13	3.87e-15	6.04e-17	9.51e-17	2.22e-17	1.74e-20	9.82e-22

TABLE VIII
COMPARISON OF APPROXIMATE SOLUTION WITH DIFFERENT METHODS AT $M = 5, \alpha = 0.75$, AND $ITERATION = 16$ FOR EXAMPLE 4.

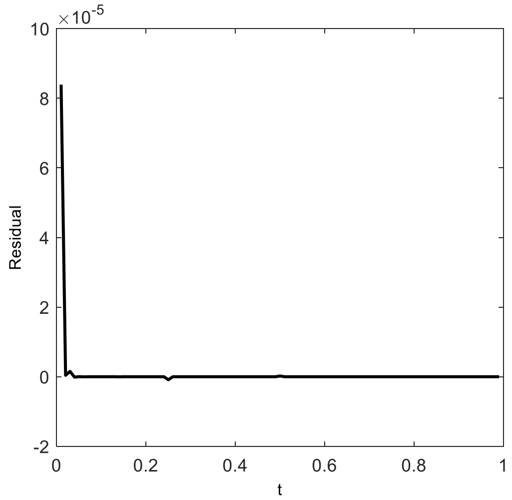
t	MHPM [31]	EHPM [32]	HWOMM [6]	PLSM [11]	BPCM [20]	Present Method	
						$k = 8$	$k = 9$
0.2	0.3138	0.3214	0.3095	0.307	0.309975526145	**0.30997528**50	**0.30997528**42
0.4	0.4929	0.5077	0.4814	0.4819	0.481631848538	**0.48163169**14	**0.48163169**08
0.6	0.5974	0.6259	0.5977	0.5969	0.597782779119	**0.59778267**15	**0.597782671**1
0.8	0.6604	0.7028	0.6788	0.6783	0.678849572343	**0.67884949**62	**0.67884949**55
1	0.7183	0.7542	0.7368	0.7365	0.736836860238	**0.736836670**4	**0.736836670**2

TABLE IX
COMPARISON OF APPROXIMATE SOLUTION OBTAINED BY DIFFERENT METHODS AT $M = 5, \alpha = 0.9$, AND $ITERATION = 13$ FOR EXAMPLE 4.

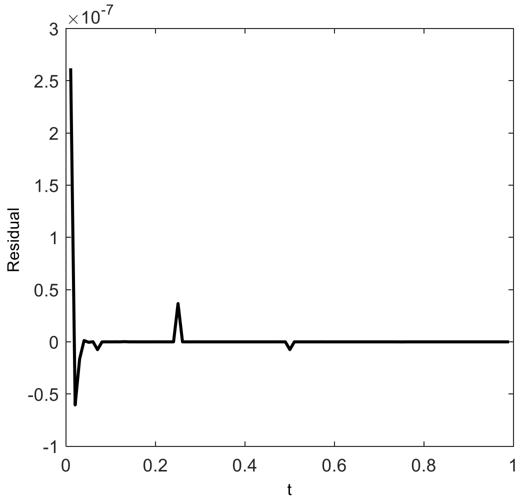
t	MHPM [31]	EHPM [32]	PLSM [11]	BPCM [20]	Present Method	
					$k = 8$	$k = 9$
0	0	0	0	0	0	0
0.2	0.2391	0.2647	0.2369	0.238789150071	**0.238789136**91	**0.238789136**85
0.4	0.4229	0.4591	0.4211	0.422583099027	**0.4225830884**75	**0.4225830884**27
0.6	0.5653	0.6031	0.5651	0.566171571136	**0.566171156**30	**0.566171156**298
0.8	0.674	0.7068	0.6738	0.674627004671	**0.6746269981**67	**0.6746269981**4
1	0.7569	0.7806	0.7541	0.754589017236	**0.7545888085**69	**0.7545888085**74

TABLE XI
COMPARISON OF APPROXIMATE SOLUTION FOR EXAMPLE 5 WITH $\alpha = 1,\ k = 2,\ M = 9$ AND ITERATION = 6

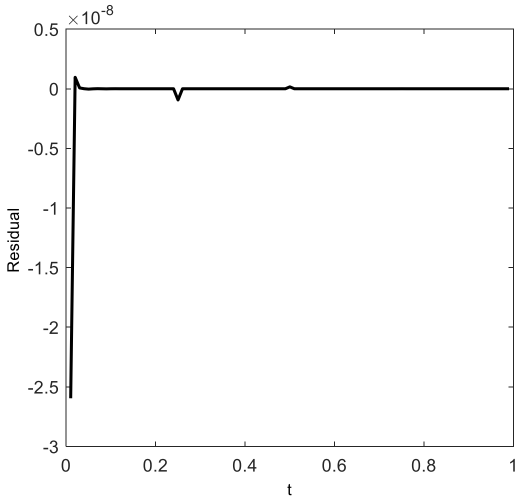
t	HPM [33]	LWM [33]	Approximate Solution	Exact Solution
0	0	0	0	0
0.125	0.124674731	0.124674731	**0.12467473338522**61	**0.12467473338522**77
0.25	0.247403957	0.247403957	**0.24740395925452**14	**0.24740395925452**29
0.375	0.366272528	0.366272527	**0.36627252908604**61	**0.36627252908604**76
0.5	0.479425533	0.479425537	**0.479425538604203**8	**0.479425538604203**0
0.625	0.585097232	0.585097271	**0.58509727294044**58	**0.58509727294044**62
0.75	0.681638554	0.681638759	**0.681638760023330**5	**0.681638760023334**2
0.875	0.767542679	0.767543501	**0.76754350223602**36	**0.76754350223602**71
1	0.841468253	0.841470984	**0.84147098480789**85	**0.84147098480789**65



(a) $k = 7$



(b) $k = 8$



(c) $k = 9$

Fig.7. Residual error of Example 5 with different values of k at $M = 5, \alpha = 0.25$ and $iteration = 20$.

The absolute errors are tabulated in Table XII and also compared with LWM [33], CWM [12], and Jacobi collocation method (JCM) [14] for different values of t with step length 0.125. It is evident from the given table that the achieved results are more accurate as compared to the existing technique.

TABLE X
COMPARISON OF ABSOLUTE ERROR OF EXAMPLE 4 WITH DIFFERENT METHODS.

t	IRKHSM [29]	GBCM [28]	Present Method $M = 6, k = 7$, $iter = 23$
0.1	9.05e-06	2.51e-08	8.43e-17
0.2	1.72e-05	4.64e-08	1.38e-16
0.3	2.38e-05	5.60e-08	1.57e-16
0.4	2.85e-05	1.08e-08	1.35e-16
0.5	3.11e-05	1.46e-07	8.61e-17
0.6	3.17e-05	2.42e-07	3.02e-17
0.7	3.07e-05	1.17e-06	1.39e-17
0.8	2.81e-05	1.57e-06	4.78e-17
0.9	2.32e-05	1.16e-06	6.01e-17
1	11.19e-05	1.04e-06	6.04e-17

TABLE XII
COMPARISON OF ABSOLUTE ERROR WITH DIFFERENT METHODS FOR EXAMPLE 5 AT $M = 9, k = 2, ITERATION = 6$.

t	LWM [33]	CWM [12]	JCM [14]	Present Method
0.125	2.38522e-9	1.70708e-12	2.04656e-12	1.61e-15
0.25	2.25452e-9	2.47323e-12	1.61024e-12	1.54e-15
0.375	2.08604e-9	9.36426e-11	3.47906e-11	1.42e-15
0.5	1.60420e-9	1.78774e-11	5.31130e-11	7.49e-16
0.625	1.94046e-9	1.88843e-11	2.06581e-11	3.93e-15
0.75	1.02333e-9	3.04277e-12	2.47129e-11	3.67e-15
0.875	1.23602e-9	1.64433e-12	1.59291e-11	3.42e-15
1	8.07896e-10	4.51086e-10	3.85050e-10	2.04e-15

VI. CONCLUSION

The article successfully implements an effective new numerical technique to solve various types of nonlinear fractional order initial values problems. The primary advantage of this methodology resides in its adeptness at transforming nonlinear equations to linear equations. The efficacy of the proposed approach has been assessed through diverse numerical illustrations, elucidating a comparative analysis of the solution and their errors in tables and figures. Our findings show that our numerical solution offers a high degree of accuracy as compared to existing methods. Increasing the value of the LW parameter or increasing the number of iterations enhances solution precision. Furthermore, we've included error estimations for the proposed scheme using the residual function, and the effectiveness of this error estimation is demonstrated through numerical examples. It's worth nothing that this article focuses on solutions for nonlinear problems, but the proposed methodology can also be applied to solve linear problems.

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