# Approximate Solution of $K(p,q)$ and $K(p,q,1)$ Equations with Time Fractional Derivatives

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***Abstract*—The coupled Kamal transform and Adomian polynomial are used in this article to solve nonlinear fractional partial differential equations. The technique was developed to obtain an approximate solution to the fractional order $K(p,q)$: ${}^cD_t^\alpha u + (u^p)_x + (u^q)_{xxx} = 0$ and $K(p,q,1)$: ${}^cD_t^\alpha u + (u^p)_x + (u^q)_{xxx} + u_{5x} = 0$ equations with initial conditions. The solutions were calculated in the form of an infinite series that converges rapidly to the exact solution when $\alpha = 1$. The results, as well as the graphs that were presented demonstrate the method's reliability and efficiency.**

***Index Terms*—Solitary wave $K(p,q)$ and $K(p,q,1)$ equations, nonlinear fractional order differential equations, Kamal transform, Adomian polynomial.**

## I. Introduction

The study of fractional order differential equations is extremely essential and has been a mathematical topic with many real-world applications. It serves as a great tool for describing dynamical behavior, memory, and hereditary traits in relation to physical systems and processes. Fractional differential equations give more accurate models for real-world problems than integer-order derivatives because of their usefulness in describing some physical issues in the simulation of complex applied science phenomena [1], [10], [11], [13], [16], [29].

The fundamental applications of fractional differential equations prompted this study to consider the nonlinear time fractional order solitary waves, which are expressed as follows:

$$ {}^cD_t^\alpha u + (u^p)_x + (u^q)_{xxx} = 0, \qquad p,\ q > 1. \quad (1)$$

And

$$ {}^cD_t^\alpha u + (u^p)_x + (u^q)_{xxx} + u_{5x} = 0, \qquad p,\ q > 1. \quad (2)$$

Equations 1 and 2 presented above are called $K(p,q)$ and $K(p,q,1)$, which are the classes of solitary waves with compact supports. The first term, expressed in Caputo sense, is the generalized evolution term; the second is the nonlinear term; and the third and fourth terms are the dispersion terms that cause the wave form to spread [20], [21]. The $K(p,q)$ equation, which was first studied by Rosenan and Hyman [20], belongs to the highly nonlinear Kortwge-deVries (Kdv) family, which has applications in the investigation of nonlinear dispersion in pattern generation in liquid drops, and the obtained solitary wave solution with compact support is called compactons. This compacton has the remarkable property that the solitary wave solution reemerges after colliding with other compact solitons [20], [22], [30], [24]. The $K(p,q,1)$ equation is an extension of $K(p,q)$ equation, which was conducted as a result of further studies of compactons. Hence, equation 2 is regarded as the higher order Kdv equation [3], [24]. Obtaining the approximate solutions of fractional order differential equations for the solitary wave solution of $K(p,q)$ and $K(p,q)$ have been studied using a number of approaches, including: homotopy perturbation method [8], [15], [26], [28], reduced differential transform method [2], [17], Adomian decomposition method [24], [31], variational principle [30], lie symmetry analysis [33], homotopy analysis method [12], decomposition method [27]. In an effort to present analytical and approximative solutions to the nonlinear dispersive equations with fractional time derivatives, this research extends the application of the coupled Kamal transform and Adomian polynomial to solve nonlinear $K(p,q)$ and $K(p,q,1)$ equations, which are in the Caputo sense. The Kamal transform [7] was proposed primarily to solve linear differential equations. However, it is worth mentioning that when coupling the Adomian polynomial with the Kamal transform, nonlinear differential equations were successfully solved. Hence, one of the remarkable features is that coupling the Kamal transform with the Adomian polynomial is efficient in handling both linear and nonlinear fractional order differential equations, which are of the Caputo sense [13].

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## II. Definitions

This section provides a basic definition of fractional differentiation and integration.

### A. Definition 1

The Caputo fractional derivative of order $\alpha$, denoted by ${}^cD^\alpha$, is defined by [8], [5], [13], [17].

$$ {}^cD^\alpha f(x) = J^{n-\alpha}D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^n(\tau)}{(x-\tau)^{\alpha+1-n}}d\tau. \quad (3)$$

where $n-1 < \alpha \le n$, $n \in \mathbf{N}$, $x > 0$, $\alpha > 0$, and $f \in \mathbf{C}_{-1}^n$.

### B. Definition

The fractional integral operator of order $\alpha$, called Riemann-Liouville, is defined by [6], [8], [9], [17].

$$ J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt. \quad (4)$$

where $\alpha \ge 0$, $f \in \mathbf{C}_l$, $l \ge -1$, $x > 0$, and $J^0 f(x) = f(x)$. The properties of Reimann Liouville operator $J^\alpha$ for

$f \in \mathbf{C}_l$, $l \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$ is given by [1], [10], [12], [23].

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,

(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

(3) $J^\alpha x^\gamma = \Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1) x^{\alpha+\gamma}$.

### C. Definition

Kamal transform is denoted by the operator $K(\cdot)$ which is defined as [7], [13], [14], [18]:

$$K[f(t)] = \int_0^\infty f(t) e^{-t/v} dt = G(v). \quad t \geq 0, \quad k_1 \leq v \leq k_2. \tag{5}$$

where $v$ is used to factor $t$ in the argument of the function $f$.

**Theorem 1.** Assuming that $K[f(t)] = G(v)$ and that $G(v)$ is a Kamal transform of the function $f(t)$. Then,

1. $K[f'(t)] = \frac{1}{v} G(v) - f(0)$,
2. $K[f''(t)] = \frac{1}{v^2} G(v) - \frac{1}{v} f(0) - f'(0)$,
3. $K[f^n(t)] = \frac{1}{v^n} G(v) - \sum_{k=0}^{n-1} v^{k-n+1} f^k(0)$.

The proof of **Theorem 1** was achieved by the principle of mathematical induction in [18], [19].

**Theorem 2.** Assuming the $n$th order of Caputo fractional derivative of order $\alpha$ is given, then the Kamal transform is given as

$$K\left[{}^c D_x^\alpha f(x)\right] = v^{-\alpha} G(v) - \sum_{k=0}^{n-1} v^{k-\alpha+1} f^k(0). \tag{6}$$

where $n-1 < \alpha < n, \quad n \in N$.

The proof of **Theorem 2** can be found in [13] as well.

## III. The Coupled Kamal transform and Adomian Polynomial for Fractional order differential equations

A general Caputo fractional order differential equation is defined as [13].

$$^c D_x^\alpha u(x,t) + Lu(x,t) + Mu(x,t) = h(x,t). \tag{7}$$

where $t > 0$, $0 < \alpha \leq 1$, $Lu(x,t)$, $Mu(x,t)$ , and $h(x,t)$ are the linear operator, nonlinear operator and the source term respectively. The term ${}^c D_x^\alpha u(x,t)$ is the Caputo fractional derivative of order $\alpha$.

The Kamal transform of equation (7) gives;

$$K\left[{}^c D_x^\alpha u(x,t)\right] + K\left[Lu(x,t) + Mu(x,t)\right] = K\left[h(x,t)\right]. \tag{8}$$

Simplifying equation (8) gives;

$$v^{-\alpha} K\left[u(x,t)\right] - \sum_{k=0}^{n=1} v^{k-\alpha+1} u^k(0)$$
$$+ K\left[Lu(x,t) + Mu(x,t)\right] = K\left[h(x,t)\right]. \tag{9}$$

Further simplification of equation (9) and applying the inverse Kamal transform gives;

$$u(x,t) = K^{-1}\left[v^\alpha \left\{\sum_{k=0}^{n=1} v^{k-\alpha+1} u^k(0) + K\left[h(x,t)\right]\right\}\right]$$
$$+ K^{-1}\left[v^\alpha K\left\{Lu(x,t) + Mu(x,t)\right\}\right] \tag{10}$$

Let

$$u_0(x,t) = K^{-1}\left[v^\alpha \left\{\sum_{k=0}^{n=1} v^{k-\alpha+1} u^k(0) + K\left[h(x,t)\right]\right\}\right] \tag{11}$$

where $u_0(x,t)$ is obtained by simplifying the given initial condition and the source term. Then, the recursive relation is given as

$$u_{n+1}(x,t) = K^{-1}\left[v^\alpha K\left\{Lu(x,t) + A_n\right\}\right]. \tag{12}$$

The nonlinear function in equation (8) is expressed as

$$Mu(x,t) = \sum_{n=0}^\infty A_n, \tag{13}$$

where $A_n$ denote the nonlinear term which was decomposed by Adomian polynomial formula [18], [19].

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n}\left[M\left(\sum_{i=0}^\infty \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, \cdots \tag{14}$$

$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^N u_n(x,t). \tag{15}$$

The mean absolute error can be calculated by

$$\bar{X} = \frac{\sum_{i=1}^n \alpha_i}{N}.$$

where $\bar{X}$ is the mean absolute error, $\sum_{i=1}^n \alpha_i$ is the summation of the absolute difference between the exact and approximate solution at a particular point, and $N$ is the number of points considered within the domain in each of the problems considered.

## IV. Applications

### A. Consider the K(p, q) equation for $p = q = 2$ [1], [26].

$$^c D_t^\alpha u + (u^2)_x + (u^2)_{xxx} = 0, \qquad 0 < \alpha \leq 1, \qquad t > 0. \tag{16}$$

with initial condition

$$u(x,0) = \frac{4k}{3} \sin^2\left(\frac{x}{4}\right). \tag{17}$$

The Kamal transform of equation (16) gives;

$$K\left[{}^c D_t^\alpha u\right] = -K\left[(u^2)_x + (u^2)_{xxx}\right]. \tag{18}$$

Simplifying equation (18) gives;

$$\frac{u(x,v)}{v^\alpha} - \sum_{k=0}^{0} v^{k-\alpha+1} u^k(x,0) = -K\left[(u^2)_x + (u^2)_{xxx}\right] \tag{19}$$

Then, equation (19) will give:

$$u(x,v) = v^\alpha\left(v^{-\alpha+1} u(x,0)\right) - v^\alpha K\left[(u^2)_x + (u^2)_{xxx}\right]. \tag{20}$$

By substituting the initial conditions given in equation (17) into equation (20). Then,

$$u(x,v) = \frac{4k}{3} \sin^2\left(\frac{x}{4}\right) v - v^\alpha K\left[(u^2)_x + (u^2)_{xxx}\right]. \tag{21}$$

Applying the inverse Kamal transform on equation (21) give;

$$u(x,t) = K^{-1}\left[\frac{4k}{3}\sin^2\left(\frac{x}{4}\right)v\right] - K^{-1}\left[v^\alpha K\left[(u^2)_x + (u^2)_{xxx}\right]\right]. \quad (22)$$

Then, equation (22) give;

$$u(x,t) = \frac{4k}{3}\sin^2\left(\frac{x}{4}\right) - K^{-1}\left[v^\alpha K\left[(u^2)_x + (u^2)_{xxx}\right]\right]. \quad (23)$$

From equation (23), let

$$u_0(x,t) = \frac{4k}{3}\sin^2\left(\frac{x}{4}\right), \quad (24)$$

Then,

$$u_{n+1}(x,t) = -K^{-1}\left[v^\alpha K\left[(A_n)_x + (B_n)_{xxx}\right]\right]. \quad (25)$$

Put $n = 0$ in equation (25) such that,

$$u_1(x,t) = -k^{-1}\left[v^\alpha K\left[(A_0)_x + (B_0)_{xxx}\right]\right]. \quad (26)$$

where $A_0$ and $B_0$ are the nonlinear terms which will be decomposed by Adomian polynomial formula given in equation (14) and can be express as

$$A_0 = u_0^2, \qquad B_0 = u_0^2. \quad (27)$$

Simplifying equation (26) so as to have

$$u_1(x,t) = -K^{-1}\left[v^\alpha K\left[\frac{2}{3}k^2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\right]\right]. \quad (28)$$

Equation (28) becomes;

$$u_1(x,t) = -K^{-1}\left[\frac{2}{3}k^2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)v^{\alpha+1}\right]. \quad (29)$$

Thus,

$$u_1(x,t) = -\frac{2}{3}k^2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}. \quad (30)$$

If $n = 1$ in equation (25) then,

$$u_2(x,t) = -K^{-1}\left[v^\alpha K\left[(A_1)_x + (B_1)_{xxx}\right]\right] \quad (31)$$

where

$$A_1 = 2u_0u_1, \qquad B_1 = 2u_0u_1. \quad (32)$$

Simplifying equation (31) gives;

$$u_2(x,t) = -K^{-1}\left[\frac{-1}{6}k^3\left(2\cos^2\left(\frac{x}{4}\right)-1\right)v^{2\alpha+1}\right] \quad (33)$$

Therefore,

$$u_2(x,t) = \frac{1}{6}k^3\left(2\cos^2\left(\frac{x}{4}\right)-1\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (34)$$

Putting $n = 2$ in equation (25) to give,

$$u_3(x,t) = -K^{-1}\left[v^\alpha K\left[(A_2)_x + (B_2)_{xxx}\right]\right], \quad (35)$$

where

$$A_2 = 2u_0u_2 + u_1^2, \qquad B_1 = 2u_0u_2 + u_1^2. \quad (36)$$

Simplifying equation (35) gives;

$$u_3(x,t) = -K^{-1}\left[\frac{-1}{6}k^4\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)v^{2\alpha+1}\right], \quad (37)$$

Therefore,

$$u_3(x,t) = \frac{1}{6}k^4\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (38)$$

Thus, the series solution becomes;

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \quad (39)$$

Then,

$$u(x,t) = \frac{4k}{3}\sin^2\left(\frac{x}{4}\right) - \frac{2}{3}k^2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{6}k^3\left(2\cos^2\left(\frac{x}{4}\right)-1\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{1}{6}k^4\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (40)$$

In order to check for the classical solution, put $\alpha = 1$ in equation (40), then the close solution becomes;

$$u(x,t) = \frac{4k}{3}sin^2\left(\frac{x-kt}{4}\right). \quad (41)$$

*B. Consider the K(p, q) equation for $p = q = 3$ [1], [26].*

$$^cD_t^\alpha u + (u^3)_x + (u^3)_{xxx} = 0, \quad 0 < \alpha \le 1, \quad t > 0. \quad (42)$$

with initial condition

$$u(x,0) = \frac{\sqrt{6}}{2}k^{\frac{1}{2}}\sin\left(\frac{x}{3}\right). \quad (43)$$

The Kamal transform of equation (42) gives;

$$K\left[D_t^\alpha u\right] = -K\left[(u^3)_x + (u^3)_{xxx}\right]. \quad (44)$$

Simplifying equation (44) gives;

$$\frac{u(x,v)}{v^\alpha} - \sum_{k=0}^{0} v^{k-\alpha+1}u^k(x,0) = -K\left[(u^3)_x + (u^3)_{xxx}\right] \quad (45)$$

Then, equation (45) will give:

$$u(x,v) = v^\alpha\left(v^{-\alpha+1}u(x,0)\right) - v^\alpha K\left[(u^3)_x + (u^3)_{xxx}\right]. \quad (46)$$

By substituting the initial conditions given in equation (43) into equation (46). Then,

$$u(x,v) = \frac{\sqrt{6}}{2}k^{\frac{1}{2}}\sin\left(\frac{x}{3}\right)v - v^\alpha K\left[(u^3)_x + (u^3)_{xxx}\right]. \quad (47)$$

Applying the inverse Kamal transform on equation (47) give;

$$u(x,t) = K^{-1}\left[\frac{\sqrt{6}}{2}k^{\frac{1}{2}}\sin\left(\frac{x}{3}\right)v\right] - K^{-1}\left[v^\alpha K\left[(u^3)_x + (u^3)_{xxx}\right]\right]. \quad (48)$$

Then, equation (48) give;

$$u(x,t) = \frac{\sqrt{6}}{2}k^{\frac{1}{2}}\sin\left(\frac{x}{3}\right) - K^{-1}\left[v^\alpha K\left[(u^3)_x + (u^3)_{xxx}\right]\right]. \quad (49)$$

From equation (49), let

$$u_0(x,t) = \frac{\sqrt{6}}{2}k^{\frac{1}{2}}\sin\left(\frac{x}{3}\right), \quad (50)$$

Then,

$$u_{n+1}(x,t) = -k^{-1}\left[v^{\alpha}K\left[(A_n)_x + (B_n)_{xxx}\right]\right].. \quad (51)$$

Given that $n = 0$ in equation (51) then,

$$u_1(x,t) = -K^{-1}\left[v^{\alpha}K\left[(A_0)_x + (B_0)_{xxx}\right]\right]. \quad (52)$$

where $A_0$ and $B_0$ are the nonlinear terms given as

$$A_0 = u_0^3, \qquad B_0 = u_0^3.$$

Simplifying equation (52) gives

$$u_1(x,t) = -K^{-1}\left[v^{\alpha}K\left[\frac{\sqrt{6}}{6}k^{\frac{3}{2}}\cos\left(\frac{x}{3}\right)\right]\right]. \quad (53)$$

Equation (53) becomes;

$$u_1(x,t) = -K^{-1}\left[\frac{\sqrt{6}}{6}k^{\frac{3}{2}}\cos\left(\frac{x}{3}\right)v^{\alpha+1}\right]. \quad (54)$$

Thus,

$$u_1(x,t) = -\frac{\sqrt{6}}{6}k^{\frac{3}{2}}\cos\left(\frac{x}{3}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \quad (55)$$

moreso, putting $n = 1$ in equation (51) gives;

$$u_2(x,t) = -K^{-1}\left[v^{\alpha}K\left[(A_1)_x + (B_1)_{xxx}\right]\right], \quad (56)$$

Simplifying equation (56) gives;

$$u_2(x,t) = -K^{-1}\left[\frac{\sqrt{6}}{18}k^{\frac{5}{2}}\sin\left(\frac{x}{3}\right)v^{2\alpha+1}\right], \quad (57)$$

Therefore,

$$u_2(x,t) = -\frac{\sqrt{6}}{18}k^{\frac{5}{2}}\sin\left(\frac{x}{3}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \quad (58)$$

Thus,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (59)$$

Then,

$$u(x,t) = \frac{\sqrt{6}}{2}k^{\frac{1}{2}}\sin\left(\frac{x}{3}\right) - \frac{\sqrt{6}}{6}k^{\frac{3}{2}}\cos\left(\frac{x}{3}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\sqrt{6}}{18}k^{\frac{5}{2}}\sin\left(\frac{x}{3}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad (60)$$

The classical solution is checked when $\alpha = 1$ in equation (60), then the close solution becomes;

$$u(x,t) = \frac{\sqrt{6}}{2}k^{\frac{1}{2}}\cos\left(\frac{x-kt}{3}\right). \quad (61)$$

*C. Consider the K(2, 2, 1) equation for $p = q = 2$ [8], [25], [27].*

$$^{c}D_t^{\alpha}u + (u^2)_x - (u^2)_{xxx} + u_{5x} = 0, \quad 0 < \alpha \le 1, \quad t > 0. \quad (62)$$

with initial condition

$$u(x,0) = \frac{(16k-1)}{12}\cosh^2\left(\frac{x}{4}\right). \quad (63)$$

The Kamal transform of equation (62) gives;

$$K\left[D_t^{\alpha}u\right] = K\left[(u^2)_{xxx} - (u^2)_x - u_{5x}\right]. \quad (64)$$

Simplifying equation (64) gives;

$$\frac{u(x,v)}{v^{\alpha}} - \sum_{k=0}^{0} v^{k-\alpha+1}u^k(x,0) = K\left[(u^2)_{xxx} - (u^2)_x - u_{5x}\right]. \quad (65)$$

Then, equation (65) will give:

$$u(x,v) = v^{\alpha}\left(v^{-\alpha+1}u(x,0)\right) + v^{\alpha}K\left[(u^2)_{xxx} - (u^2)_x - u_{5x}\right]. \quad (66)$$

By substituting the initial conditions given in equation (63) into equation (66). Then,

$$u(x,v) = \frac{16k-1}{12}\cosh^2\left(\frac{x}{4}\right)v + v^{\alpha}K\left[(u^2)_{xxx} - (u^2)_x - u_{5x}\right]. \quad (67)$$

Applying the inverse Kamal transform on equation (67) give;

$$u(x,t) = K^{-1}\left[\frac{16k-1}{12}\cosh^2\left(\frac{x}{4}\right)v\right] + K^{-1}\left[v^{\alpha}K\left[(u^2)_{xxx} - (u^2)_x - u_{5x}\right]\right]. \quad (68)$$

Then, equation (68) give;

$$u(x,t) = \frac{16k-1}{12}\cosh^2\left(\frac{x}{4}\right) + K^{-1}\left[v^{\alpha}K\left[(u^2)_{xxx} - (u^2)_x - u_{5x}\right]\right]. \quad (69)$$

From equation (69), let

$$u_0(x,t) = \frac{16k-1}{12}\cosh^2\left(\frac{x}{4}\right), \quad (70)$$

Then,

$$u_{n+1}(x,t) = K^{-1}\left[v^{\alpha}K\left[(A_n)_{xxx} - (B_n)_x - (u_n)_{5x}\right]\right]. \quad (71)$$

Assigning $n = 0$ in equation (71) then,

$$u_1(x,t) = K^{-1}\left[v^{\alpha}K\left[(A_0)_{xxx} - (B_0)_x - (u_0)_{5x}\right]\right]. \quad (72)$$

where $A_0$ and $B_0$ are the nonlinear terms.
Simplifying equation (72) gives

$$u_1(x,t) = -K^{-1}\left[v^{\alpha}K\left[\frac{(16k^2-k)}{24}\cosh\left(\frac{x}{4}\right)\sinh\left(\frac{x}{4}\right)\right]\right]. \quad (73)$$

Equation (73) becomes;

$$u_1(x,t) = -K^{-1}\left[\frac{(16k^2-k)}{24}\cosh\left(\frac{x}{4}\right)\sinh\left(\frac{x}{4}\right)v^{\alpha+1}\right]. \quad (74)$$

Thus,

$$u_1(x,t) = -\frac{(16k^2-k)}{24}\cosh\left(\frac{x}{4}\right)\sinh\left(\frac{x}{4}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}. \quad (75)$$

Similarly, if $n = 1$ in equation (71) then,

$$u_2(x,t) = -K^{-1}\left[v^{\alpha}K\left[(A_1)_{xxx} - (B_1)_x - (u_1)_{5x}\right]\right], \quad (76)$$

Simplifying equation (76) gives;

$$u_2(x,t) = K^{-1}\left[\frac{(16k^3-k^2)}{96}\left(2\cosh^2\left(\frac{x}{4}\right) - 1\right)v^{2\alpha+1}\right], \quad (77)$$

Therefore,

$$u_2(x,t) = \frac{(16k^3 - k^2)}{96}\left(2\cosh^2\left(\frac{x}{4}\right) - 1\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{78}$$

Thus,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{79}$$

Then,

$$u(x,t) = \frac{16k-1}{12}\cosh^2\left(\frac{x}{4}\right) - \frac{(16k^2-k)}{24}\cosh\left(\frac{x}{4}\right)\sinh\left(\frac{x}{4}\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(16k^3-k^2)}{96}\left(2\cosh^2\left(\frac{x}{4}\right) - 1\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{80}$$

When $\alpha = 1$ in equation (80) the classical solution is obtained , then the close solution is given as;

$$u(x,t) = \frac{(16k-1)}{12}\cosh^2\left(\frac{kt-x}{4}\right). \tag{81}$$

*D. Consider the K(3, 3, 1) equation for $p = q = 3$ [8], [25], [27].*

$${}^cD_t^\alpha u + (u^3)_x - (u^3)_{xxx} + u_{5x} = 0, \quad 0 < \alpha \le 1, \quad t > 0. \tag{82}$$

with initial condition

$$u(x,0) = \sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{x}{3}\right). \tag{83}$$

The Kamal transform of equation (82) gives;

$$K\left[D_t^\alpha u\right] = K\left[(u^3)_{xxx} - (u^3)_x - u_{5x}\right]. \tag{84}$$

Simplifying equation (84) gives;

$$\frac{u(x,v)}{v^\alpha} - \sum_{k=0}^{0} v^{k-\alpha+1}u^k(x,0) = K\left[(u^3)_{xxx} - (u^3)_x - u_{5x}\right]. \tag{85}$$

Then, equation (85) will give:

$$u(x,v) = v^\alpha\left(v^{-\alpha+1}u(x,0)\right) + v^\alpha K\left[(u^3)_{xxx} - (u^3)_x - u_{5x}\right]. \tag{86}$$

By substituting the initial conditions given in equation (83) into equation (86). Then,

$$u(x,v) = \sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{x}{3}\right)v + v^\alpha K\left[(u^3)_{xxx} - (u^3)_x - u_{5x}\right]. \tag{87}$$

Applying the inverse Kamal transform on equation (87) give;

$$u(x,t) = K^{-1}\left[\sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{x}{3}\right)v\right] + K^{-1}\left[v^\alpha K\left[(u^3)_{xxx} - (u^3)_x - u_{5x}\right]\right]. \tag{88}$$

Then, equation (88) give;

$$u(x,t) = \sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{x}{3}\right) + K^{-1}\left[v^\alpha K\left[(u^3)_{xxx} - (u^3)_x - u_{5x}\right]\right]. \tag{89}$$

From equation (89), let

$$u_0(x,t) = \sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{x}{3}\right), \tag{90}$$

Then,

$$u_{n+1}(x,t) = K^{-1}\left[v^\alpha K\left[(A_n)_{xxx} - (B_n)_x - (u_n)_{5x}\right]\right], \tag{91}$$

setting $n = 0$ in equation (91) then,

$$u_1(x,t) = K^{-1}\left[v^\alpha K\left[(A_0)_{xxx} - (B_0)_x - (u_0)_{5x}\right]\right]. \tag{92}$$

where $A_0$ and $B_0$ are the nonlinear terms
Simplifying equation (92) gives

$$u_1(x,t) = -K^{-1}\left[v^\alpha K\left[\frac{1}{54}\sqrt{486k-6}\sinh\left(\frac{x}{3}\right)k\right]\right]. \tag{93}$$

Equation (93) becomes;

$$u_1(x,t) = -K^{-1}\left[\frac{1}{54}\sqrt{486k-6}\sinh\left(\frac{x}{3}\right)kv^{\alpha+1}\right]. \tag{94}$$

Thus,

$$u_1(x,t) = -\frac{1}{54}\frac{\sqrt{486k-6}\sinh\left(\frac{x}{3}\right)kt^\alpha}{\Gamma(\alpha+1)}. \tag{95}$$

Also when $n = 1$ in equation (91) then,

$$u_2(x,t) = -K^{-1}\left[v^\alpha K\left[(A_1)_{xxx} - (B_1)_x - (u_1)_{5x}\right]\right], \tag{96}$$

Simplifying equation (96) gives;

$$u_2(x,t) = K^{-1}\left[\frac{1}{162}\sqrt{486k-6}\cosh\left(\frac{x}{3}\right)k^2v^{2\alpha+1}\right], \tag{97}$$

Therefore,

$$u_2(x,t) = \frac{1}{162}\frac{\sqrt{486k-6}\cosh\left(\frac{x}{3}\right)k^2t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{98}$$

Thus,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{99}$$

Then,

$$u(x,t) = \sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{x}{3}\right) - \frac{1}{54}\frac{\sqrt{486k-6}\sinh\left(\frac{x}{3}\right)kt^\alpha}{\Gamma(\alpha+1)} + \frac{1}{162}\frac{\sqrt{486k-6}\cosh\left(\frac{x}{3}\right)k^2t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{100}$$

The classical solution is obtained when $\alpha = 1$ in equation (100) and the close solution becomes;

$$u(x,t) = \sqrt{\frac{(81k-1)}{54}}\cosh\left(\frac{kt-x}{3}\right). \tag{101}$$

## V. RESULTS

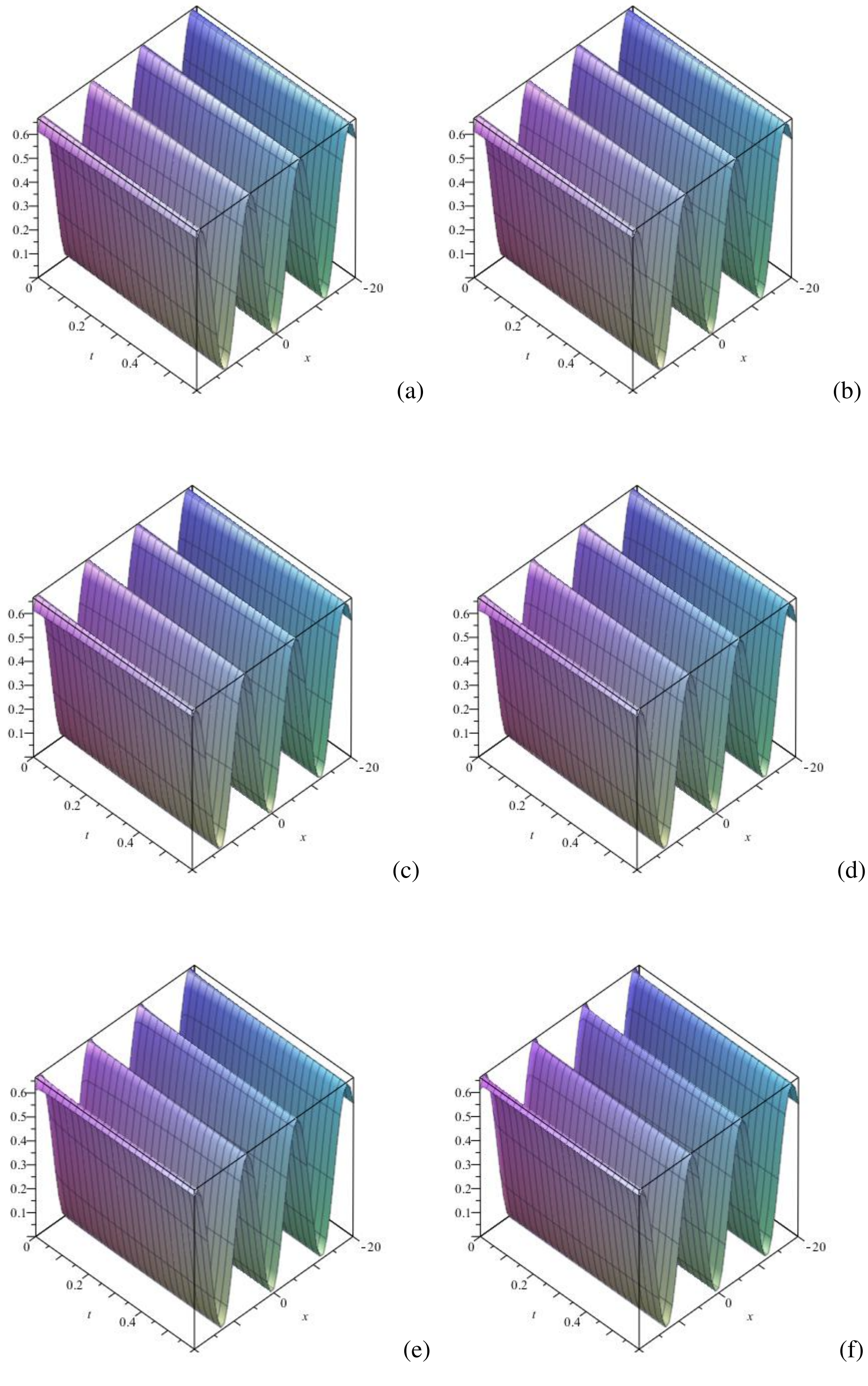


Fig. 1. The soliton solution of equation (16) when $k = 0.5$, (a) plot of $\alpha = 1.0$, (b) plot of $\alpha = 0.9$, (c) plot for $\alpha = 0.7$, (d) plot for $\alpha = 0.5$, (e) plot for $\alpha = 0.3$, (f) plot for $\alpha = 0.1$.

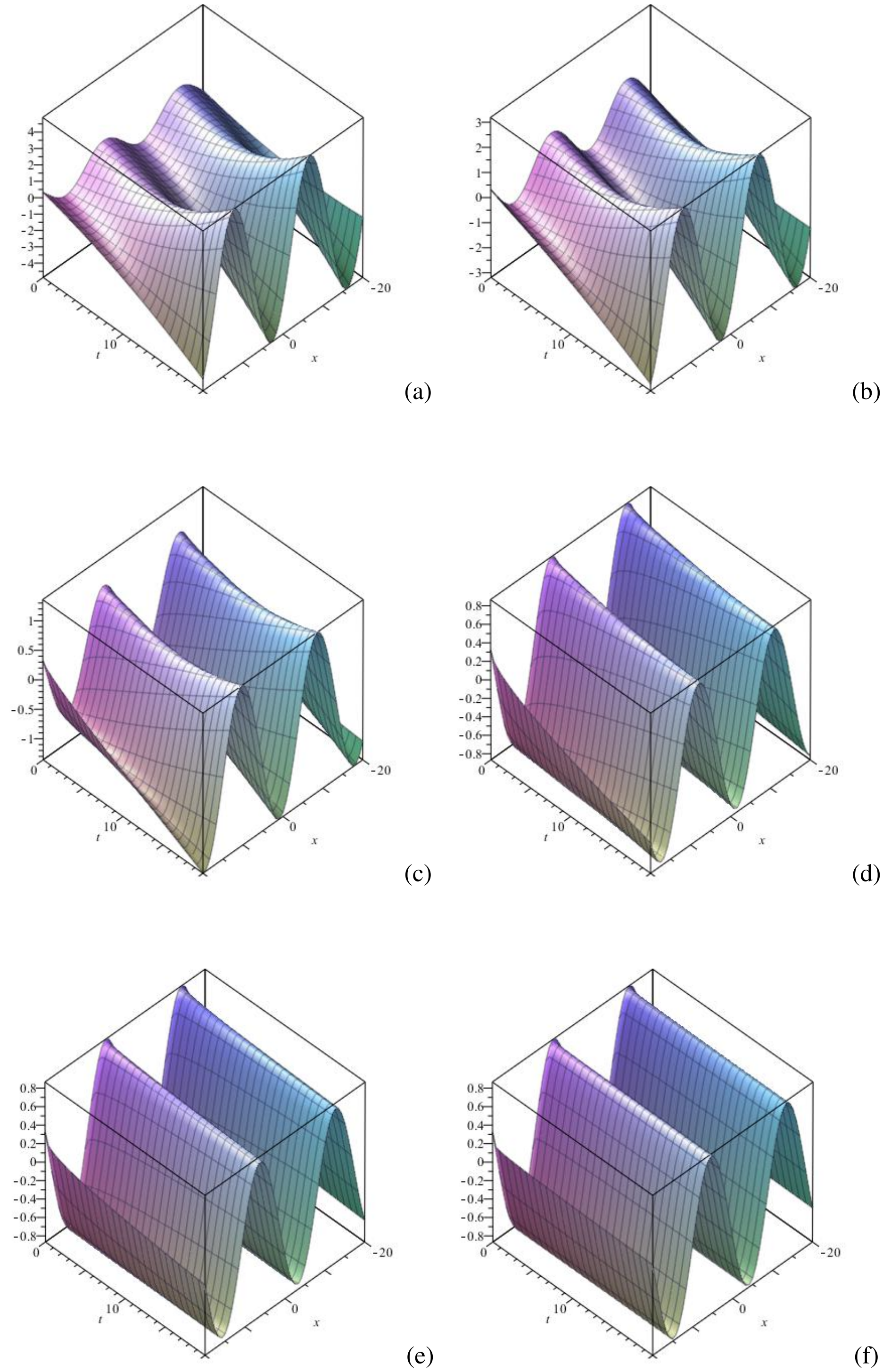


Fig. 2. The soliton solution of equation (42) when $k = 0.5$, (a) plot of $\alpha = 1.0$, (b) plot of $\alpha = 0.9$, (c) plot for $\alpha = 0.7$, (d) plot for $\alpha = 0.5$, (e) plot for $\alpha = 0.3$, (f) plot for $\alpha = 0.1$.

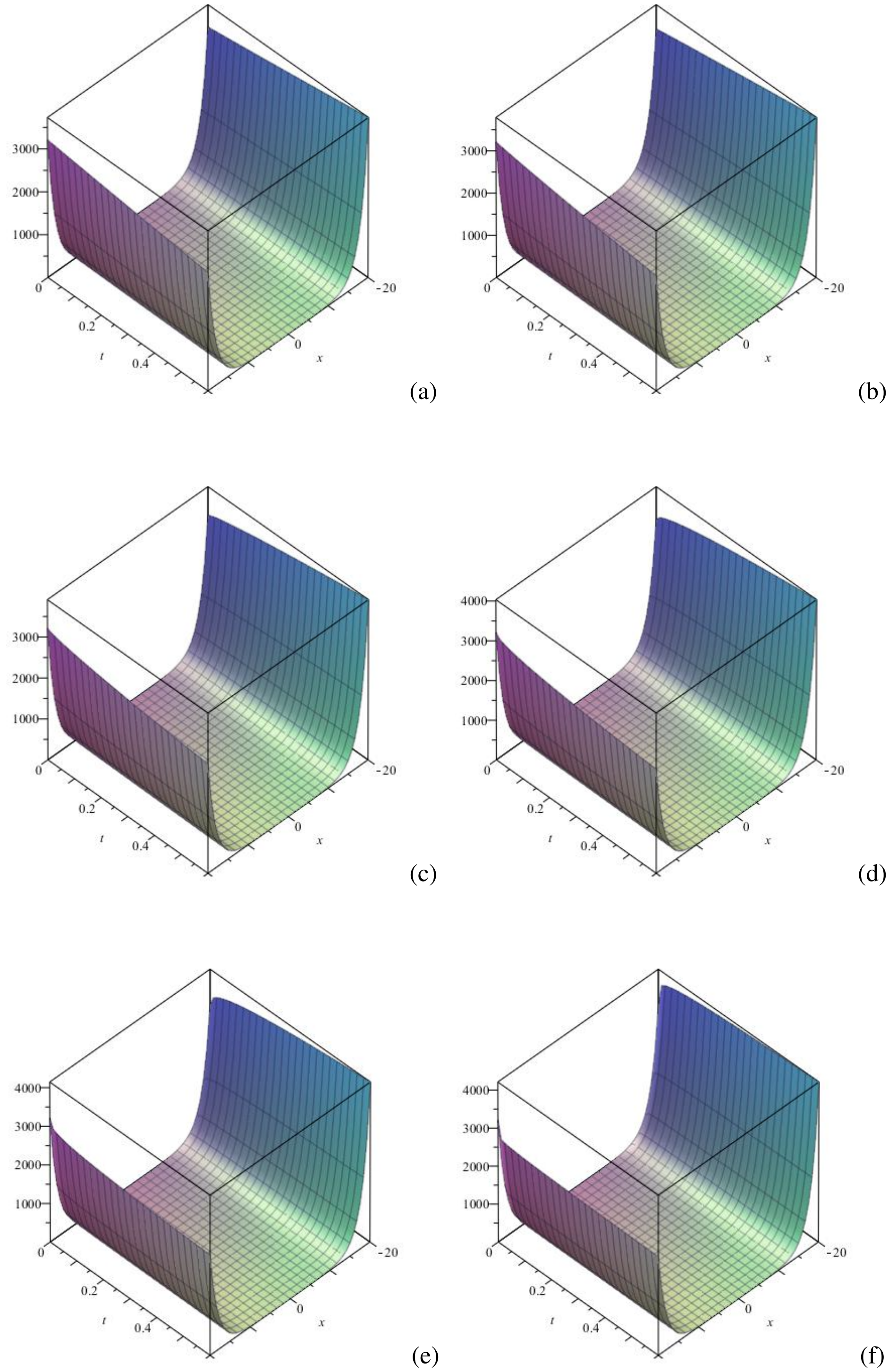


Fig. 3. The soliton solution of equation (62) when $k = 0.5$, (a) plot of $\alpha = 1.0$, (b) plot of $\alpha = 0.9$, (c) plot for $\alpha = 0.7$, (d) plot for $\alpha = 0.5$, (e) plot for $\alpha = 0.3$, (f) plot for $\alpha = 0.1$.

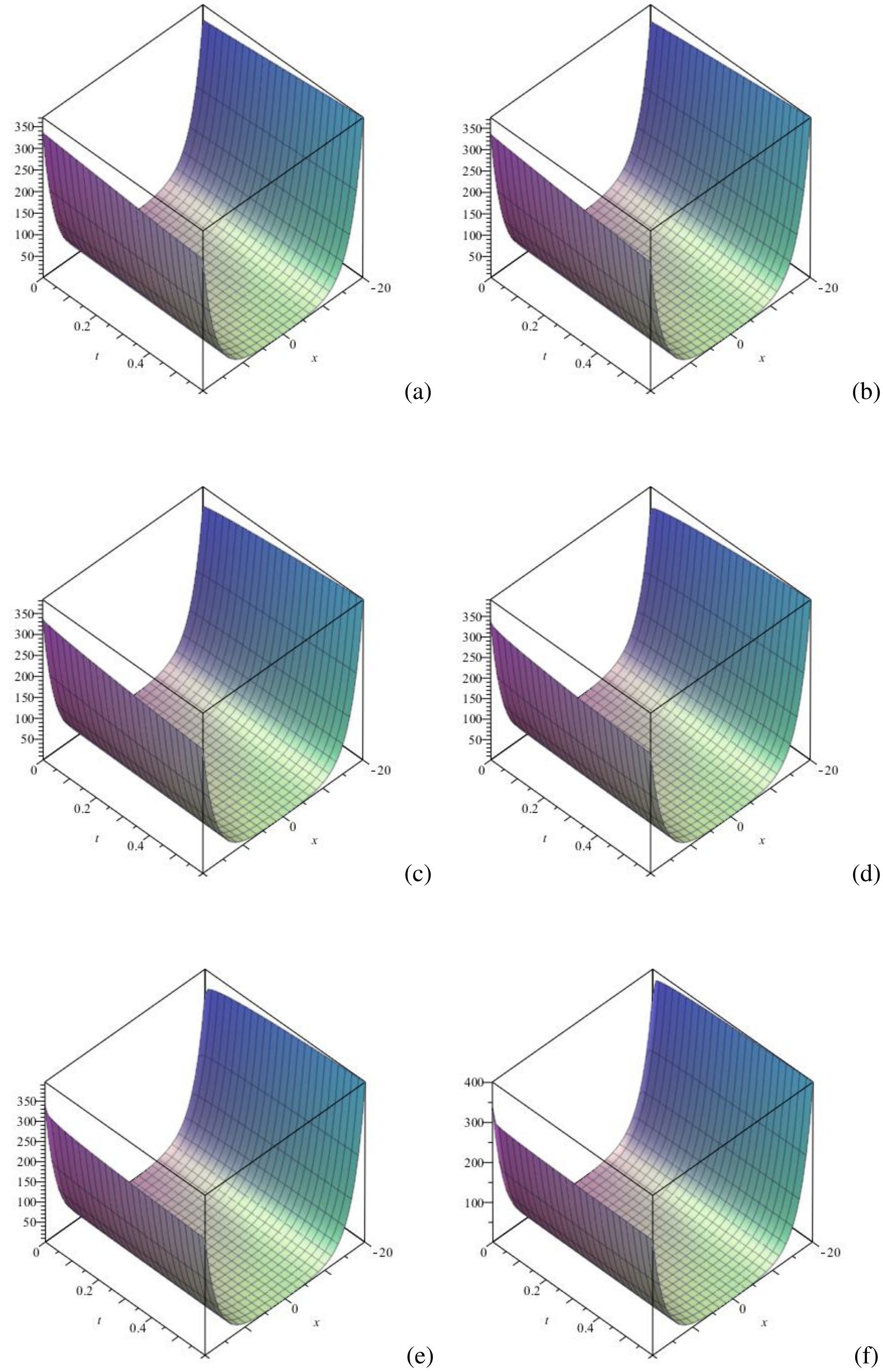


Fig. 4. The soliton solution of equation (95) when $k = 0.5$, (a) plot of $\alpha = 1.0$, (b) plot of $\alpha = 0.9$, (c) plot for $\alpha = 0.7$, (d) plot for $\alpha = 0.5$, (e) plot for $\alpha = 0.3$, (f) plot for $\alpha = 0.1$.

TABLE I
THE COUPLED KAMAL TRANSFORM AND ADOMIAN POLYNOMIAL SOLUTION OF DIFFERENTIAL FRACTIONAL-ORDER $\alpha$ OF $K(2,2)$ EQUATION GIVEN IN PROBLEM 1

| $(x,t)$ | u(x, t) at $\alpha = 0.5$ | u(x, t) at $\alpha = 0.7$ | u(x, t) at $\alpha = 1.0$ | Exact solution | Absolute error at $\alpha = 1.0$ |
|---|---|---|---|---|---|
| (0.1, 0.1) | 0.0010373 | 0.0001778 | 0.0001050 | 0.00010417 | 0.0000012 |
| (0.2, 0.2) | 0.0017171 | 0.0004949 | 0.00042638 | 0.00041656 | 0.0000098 |
| (0.3, 0.3) | 0.0024597 | 0.0010324 | 0.00096910 | 0.00093706 | 0.0000320 |
| (0.4, 0.4) | 0.0034097 | 0.0018381 | 0.0017380 | 0.0016653 | 0.0000727 |
| (0.5, 0.5) | 0.004649 | 0.0029448 | 0.0027382 | 0.0026007 | 0.0001375 |
| (0.6, 0.6) | 0.006229 | 0.0043758 | 0.0039715 | 0.0037430 | 0.0002285 |
| (0.7, 0.7) | 0.008183 | 0.006145 | 0.0054378 | 0.0050911 | 0.0003464 |
| (0.8, 0.8) | 0.010534 | 0.008263 | 0.0071413 | 0.0066443 | 0.0004973 |
| (0.9, 0.9) | 0.013303 | 0.010734 | 0.0090765 | 0.0084012 | 0.0006755 |
| (1.0, 1.0) | 0.016506 | 0.013563 | 0.011241 | 0.010362 | 0.0008790 |

TABLE II
THE COUPLED KAMAL TRANSFORM AND ADOMIAN POLYNOMIAL SOLUTION OF DIFFERENTIAL FRACTIONAL-ORDER $\alpha$ OF $K(3,3)$ EQUATION GIVEN IN PROBLEM 2

| $(x,t)$ | u(x, t) at $\alpha = 0.5$ | u(x, t) at $\alpha = 0.7$ | u(x, t) at $\alpha = 1.0$ | Exact solution | Absolute error at $\alpha = 1.0$ |
|---|---|---|---|---|---|
| (0.1, 0.1) | -0.022693 | -0.002840 | 0.014433 | 0.014433 | 0.00000 |
| (0.2, 0.2) | -0.015302 | 0.006185 | 0.028859 | 0.028864 | 0.000005 |
| (0.3, 0.3) | -0.003021 | 0.018056 | 0.043267 | 0.043284 | 0.000017 |
| (0.4, 0.4) | 0.01175 | 0.031518 | 0.057656 | 0.057691 | 0.000035 |
| (0.5, 0.5) | 0.02811 | 0.046030 | 0.072010 | 0.072091 | 0.000081 |
| (0.6, 0.6) | 0.04554 | 0.06129 | 0.086321 | 0.086460 | 0.000139 |
| (0.7, 0.7) | 0.06377 | 0.07712 | 0.10059 | 0.10081 | 0.000222 |
| (0.8, 0.8) | 0.08262 | 0.09341 | 0.11480 | 0.11514 | 0.00034 |
| (0.9, 0.9) | 0.10193 | 0.11003 | 0.12895 | 0.12942 | 0.00047 |
| (1.0, 1.0) | 0.12158 | 0.12691 | 0.14303 | 0.14369 | 0.00064 |

TABLE III
THE COUPLED KAMAL TRANSFORM AND ADOMIAN POLYNOMIAL SOLUTION OF DIFFERENTIAL FRACTIONAL-ORDER $\alpha$ OF $K(3,3,1)$ EQUATION GIVEN IN PROBLEM 3

| $(x,t)$ | u(x, t) at $\alpha = 0.5$ | u(x, t) at $\alpha = 0.7$ | u(x, t) at $\alpha = 1.0$ | Exact solution | Absolute error at $\alpha = 1.0$ |
|---|---|---|---|---|---|
| (0.1, 0.1) | 0.58421 | 0.58347 | 0.58341 | 0.58345 | 0.000045886 |
| (0.2, 0.2) | -0.58482 | 0.58378 | 0.58376 | 0.58368 | 0.00007452 |
| (0.3, 0.3) | -0.58535 | 0.58415 | 0.58414 | 0.58415 | 0.00001381 |
| (0.4, 0.4) | 0.58612 | 0.58480 | 0.58478 | 0.58485 | 0.0000707 |
| (0.5, 0.5) | 0.58719 | 0.58574 | 0.58563 | 0.58566 | 0.0000287 |
| (0.6, 0.6) | 0.58859 | 0.58689 | 0.58668 | 0.58660 | 0.0000758 |
| (0.7, 0.7) | 0.59037 | 0.58854 | 0.58792 | 0.58776 | 0.0001629 |
| (0.8, 0.8) | 0.59256 | 0.59041 | 0.58937 | 0.58916 | 0.0002037 |
| (0.9, 0.9) | 0.59508 | 0.59260 | 0.59092 | 0.59068 | 0.0002402 |
| (1.0, 1.0) | 0. 59823 | 0.59528 | 0.59283 | 0.59249 | 0.0003404 |

TABLE IV
THE COUPLED KAMAL TRANSFORM AND ADOMIAN POLYNOMIAL SOLUTION OF DIFFERENTIAL FRACTIONAL-ORDER $\alpha$ OF $K(3,3,1)$ EQUATION GIVEN IN PROBLEM 4

| $(x,t)$ | u(x, t) at $\alpha = 0.5$ | u(x, t) at $\alpha = 0.7$ | u(x, t) at $\alpha = 1.0$ | Exact solution | Absolute error at $\alpha = 1.0$ |
|---|---|---|---|---|---|
| (0.1, 0.1) | 0.85647 | 0.85536 | 0.85543 | 0.85536 | 0.00007475 |
| (0.2, 0.2) | 0.85712 | 0.85579 | 0.85574 | 0.85578 | 0.0000420 |
| (0.3, 0.3) | 0.85790 | 0.85634 | 0.85635 | 0.85638 | 0.0000336 |
| (0.4, 0.4) | 0.85888 | 0.85719 | 0.85718 | 0.85715 | 0.0000350 |
| (0.5, 0.5) | 0.86016 | 0.85835 | 0.85825 | 0.85826 | 0.000014 |
| (0.6, 0.6) | 0.86194 | 0.85992 | 0.85962 | 0.85955 | 0.000070 |
| (0.7, 0.7) | 0.86402 | 0.86178 | 0.86112 | 0.86109 | 0.000027 |
| (0.8, 0.8) | 0.86677 | 0.86418 | 0.86300 | 0.86288 | 0.000117 |
| (0.9, 0.9) | 0.86990 | 0.86690 | 0.86501 | 0.86493 | 0.000083 |
| (1.0, 1.0) | 0. 87374 | 0.87019 | 0.86741 | 0.86716 | 0.000249 |

The mean absolute errors for Figures I-IV were calculated as 0.00028799, 0.00019491, 0.00012566, and 0.00007454, respectively.

## VI. DISCUSSION OF RESULTS

A new scheme for solving nonlinear $K(p,q)$ and $K(p,q,1)$ equations has been derived from the basic principles of the Kamal transform and the Adomian polynomial. Four applications were considered to validate the efficiency of the scheme, and the results were obtained in series form in equations (40), (60), (80), and (100) for applications (4.1), (4.2), (4.3), and (4.4), respectively. At $\alpha = 1$, which is the classical form of the fractional problems considered, the series solutions converge to the exact solution, which indicates the reliability of the method used. Figures $1(a-f)$ showed that varying the fractional order $\alpha$ does not have effect on the shape of the soliton considered in equation (16); however, figures $2(a-f)$, which depicts a $3D$ graph of equation (42), revealed that there is a corresponding change in the shape of the soliton as $\alpha$ changes from $0.1$ to $1.0$, which showed the hidden effect of $\alpha$ in this soliton and also gave credence to the non-local feature of fractional calculus. Figures $3(a-f)$ also present results of equation (82), which revealed that the v-shape of the soliton was maintained for different $\alpha$; however, the value decreased with reduced value of $\alpha$ from $1.0$ through $0.9$, $0.7$, $0.5$, $0.3$, to $0.1$, while figures $4(a-f)$ also behave in the same manner as $3(a-f)$.

In addition, Tables I, II, III, and IV show the numerical results of $u(x,t)$ at different values of $\alpha$ (fractional order) for the four problems considered in sections A, B, C, and D, respectively. There are six columns in each table; column six depicts the errors obtained when the approximate solution is compared with the closed form solution of each problem considered, as each problem has an exact solution at $\alpha = 1$. These errors for all four problems are very negligible, demonstrating the efficiency of the method used. In order to account for all the errors within the domain, the mean absolute errors were calculated for each problem at classical order $\alpha = 1$, and they are $0.00028799$, $0.00019491$, $0.00012566$, and $0.00007454$, respectively, for problems 1 to 4. These errors are very small and also demonstrate the cumulative efficiency of the method across the points of the problem. Columns two and three also depict the solutions at $\alpha = 0.5$ and $0.7$, and these show the change in the values of $u(x,t)$ from fractional order to classical order. This phenomenon also supports the existence of a hidden effect of $\alpha$ in the fractional order differential equations.

## VII. CONCLUSION

The nonlinear dispersive $K(p,q)$ and $K(p,q,1)$ equations with Caputo time fractional derivatives were successfully solved with the formulated coupled Kamal transform and Adomian polynomial scheme. The approximate series solutions obtained converge to the exact solution at $\alpha = 1$ for all the problems considered. 3D graphs were plotted to demonstrate the effect of fractional order $\alpha$ as well as the physical behavior of each soliton considered, and therefore, the results obtained showed that the coupled Kamal transform and Adomian polynomial are useful mathematical tools for solving any nonlinear time-fractional differential equations.

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