# Regular Semigroups Characterized in Terms of Bipolar Fuzzy Bi-Interior-Ideals

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*Abstract*—**The concpets of bipolar fuzzy sets is an extension of fuzzy sets which was by Zhang in 1994. In this paper, we give the definitions of bipolar fuzzy bi-interior ideals and we display some properties of bipolar fuzzy bi-interior ideals. We discuss the connection between bi-interior ideals and the characteristics of bipolar fuzzy bi-interior ideals. The results reveal the beneficial application of the characterization of regular semigroups in terms of bipolar fuzzy ideals are very useful for applications. Moreover, we discussed the properties of bipolar fuzzy prime bi-interior ideals in semigroups.**

*Index Terms*—**Bipolar fuzzy sets, bi-interior ideals, prime bi-interior ideals, semiprime bi-interior ideals, regular**

## I. Introduction

IN HUMAN life everyday, there is no exception to this rule in any field of expertise. Which mathematics is not always effective. As mathematical models for dealing with uncertainty and variability, methods such as fuzzy set theory, vague set theory, interval mathematics, etc., have been developed. Zadeh [1] proposed fuzzy logic, which is an extension of classical logic used to manage problems an uncertainty. Today, several further fuzzy set extensions have been proposed, such as interval valued sets, bipolar fuzzy sets, and others. The concept was applied in many areas, such as robotics, computer science, medical science, theoretical physics, control engineering, information science, measure theory, logic, set theory, topology, etc. In 1979, Kuroki [2] used knowledge of a fuzzy set in semigroup theory and various kinds of ideals in semigroups and characterized them. In 1994, Zhang [3] introduced the notion of bipolar fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. It is used in decision-making problems, organization problems, economic problems, evaluation, risk management, environmental and social impact assessments. Thus, the concept of bipolar fuzzy sets is more relevant in mathematics [4], [5], [6], [7], [8]. Later in 2000, Lee [9] used the term bipolar valued fuzzy sets and applied it to algebraic structures. In 2011 Kim et al, [10] studied properties of bipolar fuzzy ideals in semigroups. In 2021 T. Gaketem and P. Khamrot [11] discussed bipolar fuzzy weakly interior ideals in semigroups. In the 19th century [12], Dedekind and E. Noether initiated the concepts of ideals and the theory of algebraic numbers for associative rings. The class of quasi-ideals in semigroups was studied by Stienfeld, which is a generalization of one-sided ideals, whereas the bi-ideals are a generalization of quasi-ideals [13]. In 2018, MK. Rao [14] introduced and studied the definition and properties of the bi-interior ideals in the semigroups.

In this paper, we give the concepts of bipolar fuzzy interior ideals. We provide properties of bipolar fuzzy bi-interior ideals. The regular semigroups are characterized in terms of bipolar fuzzy bi-interior ideals. Finally, we define and study the properties of bipolar fuzzy prime bi-interior ideals in semigroups.

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## II. Preliminaries

In this section, some basic definitions are given as the follows. By a *subsemigroup* of a semigroup $F$ we mean a non-empty subset $M$ of $F$ such that $M^2 \subseteq M$, and by a *left (right) ideal* of $F$ we mean a non-empty subset $M$ of $F$ such that $FM \subseteq M(MF \subseteq M)$. By a *two-sided ideal* or simply an ideal, we mean a non-empty subset of a semigroup $F$ that is both a left and a right ideal of $F$. A non-empty subset $M$ of $F$ is called a *quasi-ideal* of $F$ if $MF \cap FM \subseteq M$. A subsemigroup $M$ of $F$ is called a *bi-ideal* of $F$ if $MFM \subseteq M$. A subsemigroup $M$ of a semigroup $F$ is called an interior ideal of $F$ if $FMF \subseteq M$. A subsemigroup $M$ of a semigroup $F$ is said to be a *bi-interior ideal* of $F$ if $M$ is a subsemigroup of $F$ and $FMF \cap MFM \subseteq M$.[12]. We note here that the properties are hold:

(1) Every left ideal is a bi-interior ideal of $F$.
(2) Every right ideal is a bi-interior ideal of $F$.
(3) Every ideal is a bi-interior ideal of $F$.
(4) Every quasi ideal is a bi-interior ideal of $F$.
(5) The arbitrary intersection of the bi-interior of $F$ is also the bi-interior ideal of $F$.
(6) If $M$ is a bi-interior ideal of $F$ then $MF$ and $FM$ are bi-interior ideals of $F$ [12].

For any $\nu_i \in [0,1]$ where $i \in \mathcal{K}$, define

$$\bigvee_{i\in\mathcal{K}} \nu_i := \sup_{i\in\mathcal{K}}\{\nu_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{K}} \nu_i := \inf_{i\in\mathcal{K}}\{\nu_i\}.$$

We note here that for any $\nu, \xi \in [0,1]$, we have

$$\nu \vee \xi = \max\{\nu, \xi\} \quad \text{and} \quad \nu \wedge \xi = \min\{\nu, \xi\}.$$

A fuzzy set $\omega$ of a non-empty set $F$ is a function from $F$ into the closed interval $[0,1]$, i.e, $\omega : F \to [0,1]$.

**Definition 2.1.** *A* bipolar fuzzy set *(shortly, BF set) $\omega$ on $F$ is an object having the form*

$$\omega := \{(r, \omega^p(r), \omega^n(r)) \mid r \in F\},$$

*where $\omega^p : F \to [0,1]$ and $\omega^n : F \to [-1,0]$.*

**Remark 2.2.** *For the sake of simplicity we shall use the symbol $\omega = (F; \omega^p, \omega^n)$ for the BF set $\omega = \{(r, \omega^p(r), \omega^n(r)) \mid r \in F\}$.*

**Definition 2.3.** *Let $M$ be a non-empty set of a semigroup $F$. A* positive characteristic function and a negative characteristic function *are respectively defined as*

$$\chi_M^p : F \to [0,1], u \mapsto \chi_M^p(r) := \begin{cases} 1 & r \in M \\ 0 & r \notin M \end{cases}$$

*and*

$$\chi_M^n : F \to [-1,0], u \mapsto \chi_M^n(r) := \begin{cases} -1 & r \in M \\ 0 & r \notin M. \end{cases}$$

**Remark 2.4.** *For the sake of simplicity we shall use the symbol $\chi_M = (F; \chi_M^p, \chi_M^n)$ for the BF set $\chi_M = \{(h, \chi_M^p(r), \chi_M^n(r)) \mid r \in F\}$.*

In this case of $M = F$ defined $\chi_F^p(r) = 1$ and $\chi_F^n(r) = -1$.

The products $\omega^p \circ \psi^p$ and $\omega^n \circ \psi^n$ were defined as follows: For $r \in F$

$$(\omega^p \circ \psi^p)(r) = \begin{cases} \bigvee\limits_{(k,o)\in A_r} \{\omega^p(k) \wedge \psi^p(o)\} & \text{if} \quad A_r \neq \emptyset \\ 0 & \text{if} \quad A_r = \emptyset \end{cases}$$

and

$$(\omega^n \circ \psi^n)(r) = \begin{cases} \bigwedge\limits_{(k,o)\in A_r} \{\omega^n(k) \vee \psi^n(o)\} & \text{if} \quad A_r \neq \emptyset \\ 0 & \text{if} \quad A_r = \emptyset, \end{cases}$$

where $A_r := \{(k,o) \in F \times F \mid r = ko\}$.

**Definition 2.5.** *[10] A BF set $\omega = (F; \omega^p, \omega^n)$ on a semigroup $F$ is called*

(1) *a* BF subsemigroup *on $F$ if $\omega^p(r_1 r_2) \geq \omega^p(r_1) \wedge \omega^p(r_2)$ and $\omega^n(r_1 r_2) \leq \omega^n(r_1) \vee \omega^n(r_2)$ for all $r_1, r_2 \in F$.*
(2) *a* BF left (right) ideal *on $F$ if $\omega^p(r_1 r_2) \geq \omega^p(r_2)$ ($\omega^p(r_1 r_2) \geq \omega^p(r_1)$) and $\omega^n(r_1 r_2) \leq \omega^n(r_2)$ ($\omega^n(r_1 r_2) \leq \omega^n(r_1)$) for all $r_1, r_2 \in F$.*
(3) *a* BF bi-ideal *on $F$ if $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup, $\omega^p(r_1 r_2 r_3) \geq \omega^p(r_1) \wedge \omega^p(r_3)$ and $\omega^n(r_1 r_2 r_3) \leq \omega^n(r_1) \vee \omega^n(r_3)$ for all $r_1, r_2, r_3 \in F$.*
(4) *a* BF interior ideal *on $F$ if $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup, $\omega^p(r_1 r_2 r_3) \geq \omega^p(r_2)$ and $\omega^n(r_1 r_2 r_3) \leq \omega^n(r_2)$ for all $r_1, r_2, r_3 \in F$.*
(5) *a* BF quasi ideal *on $F$ if $\omega^p(r) \geq (\omega^p \circ \chi_F^p)(r) \wedge (\chi_F^p \circ \omega^p)(r)$ and $\omega^n(r) \leq (\omega^n \circ \chi_F^n)(r) \vee (\chi_F^n \circ \omega^n)(r)$ for all $r \in F$.*

**Theorem 2.6.** *[10] Let $\omega = (F; \omega^p, \omega^n)$ be a BF ideal on semigroup $F$. Then the following statements hold:*

(1) *$\omega = (F; \omega^p, \omega^n)$ is a BF bi-ideal of $F$.*
(2) *$\omega = (F; \omega^p, \omega^n)$ is a BF interior ideal of $F$.*
(3) *$\omega = (F; \omega^p, \omega^n)$ is a BF quasi-ideal of $F$.*

**Theorem 2.7.** *[10] Let $M$ be a non-empty subset of a semigroup $F$ and $\chi_M = (\chi_M^p, \chi_M^n)$ be the characteristic BF set of $M$. Then $M$ is a subsemigroup of a semigroup $F$ if and only if $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF subsemigroup of a semigroup $F$.*

## III. Bipolar fuzzy bi-interior ideals in semigroups

In this section, we give the concepts of bipolar fuzzy bi-interior ideals and ideals in semigroups. Also, we study the important properties for reference in the next part.

**Definition 3.1.** *A BF subsemigroup $\omega = (F; \omega^p, \omega^n)$ on a semigroup $F$ is called a BF bi-interior ideal of $F$ if it satisfies the following conditions: $\chi_S^p \circ \omega^p \circ \chi_S^p \cap \omega^p \circ \chi_S^p \circ \omega^p \subseteq \omega^p$ and $\chi_S^n \circ \omega^n \circ \chi_S^n \cup \omega^n \circ \chi_S^n \circ \omega^n \supseteq \omega^n$*

**Example 3.2.** *Let $\mathbb{Q}$ be the set of all rational numbers and $F = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Q} \right\}$. Then $F$ is a semigroup under addition of matrix.*

*Define $\omega^p : K \to [0,1]$ by*

$$\omega^p(e) = \begin{cases} 1 & \text{if} \quad e \in F, \\ 0 & \text{if} \quad e \notin F. \end{cases}$$

*and $\omega^n : K \to [-1,0]$ by*

$$\omega^n(e) = \begin{cases} -1 & \text{if} \quad e \in F, \\ 0 & \text{if} \quad e \notin F. \end{cases}$$

*Then $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$.*

The next Theorems study BF ideals in semigroup are BF bi-interior ideals of semigroups

**Theorem 3.3.** *Every BF left ideal of a semigroup $F$ is a BF bi-interior ideal of $F$.*

*Proof:* Let $\omega = (F; \omega^p, \omega^n)$ be a BF left ideal of a semigroup $F$. Let $x \in F$. Then

$$\begin{aligned}(\chi_F^p \circ \omega^p)(x) &= \bigvee_{x=yz} \{\chi_F^p(y) \wedge \omega^p(z)\} \\ &= \bigvee_{x=yz} \{\omega^p(z)\} \\ &\subseteq \bigvee_{x=yz} \{\omega^p(yz)\} \\ &= \bigvee_{x=yz} \{\omega^p(x)\} \\ &= \omega^p(x).\end{aligned}$$

We have,

$$\begin{aligned}(\omega^p \circ \chi_F^p \circ \omega^p)(x) &= \bigvee_{x=abc} \{\omega^p(a) \wedge (\chi_F^p \circ \omega^p)(bc)\} \\ &\leq \bigvee_{x=abc} \{\omega^p(a) \wedge \omega^p(bc)\} \\ &= \omega^p(x).\end{aligned}$$

Now,

$$\begin{aligned}&(\chi_F^p \circ \omega^p \circ \chi_F^p \wedge \omega^p \circ \chi_F^p \circ \omega^p)(x) \\ &= (\chi_F^p \circ \omega^p \circ \chi_F^p)(x) \wedge (\omega^p \circ \chi_F^p \circ \omega^p)(x) \\ &\leq (\chi_F^p \circ \omega^p \circ \chi_F^p)(x) \wedge \omega^p(x) \leq \omega^p(x).\end{aligned}$$

Therefore, $\chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega^p \circ \chi_F^p \circ \omega^p \subseteq \omega^p$. And

$$\begin{aligned}(\chi_F^n \circ \omega^n)(x) &= \bigwedge_{x=yz} \{\chi_F^n(y) \vee \omega^n(z)\} \\ &= \bigwedge_{x=yz} \{\omega^n(z)\} \\ &\supseteq \bigwedge_{x=yz} \{\omega^n(yz)\} \\ &= \bigwedge_{x=yz} \{\omega^n(x)\} \\ &= \omega^n(x).\end{aligned}$$

We have,

$$\begin{aligned}(\omega^n \circ \chi_F^n \circ \omega^n)(x) &= \bigwedge_{x=abc} \{\omega^n(a) \vee (\chi_F^n \circ \omega^n)(bc)\} \\ &\leq \bigwedge_{x=abc} \{\omega^p(a) \vee \omega^n(bc)\} \\ &= \omega^n(x).\end{aligned}$$

Now,

$$\begin{aligned}&(\chi_F^n \circ \omega^n \circ \chi_F^n \vee \omega^n \circ \chi_F^n \circ \omega^n)(x) \\ &= (\chi_F^n \circ \omega^n \circ \chi_F^n)(x) \vee (\omega^n \circ \chi_F^n \circ \omega^n)(x) \\ &\leq (\chi_F^n \circ \omega^n \circ \chi_F^n)(x) \vee \omega^n(x) \leq \omega^n(x).\end{aligned}$$

Therefore, $\chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega^n \circ \chi_F^n \circ \omega^n \supseteq \omega^n$.
Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$. ■

**Theorem 3.4.** *Every BF right ideal of a semigroup $F$ is a BF bi-interior ideal of $F$.*

*Proof:* It follows Theorem 3.3. ■

**Corollary 3.5.** *Every BF ideal of a semigroup $F$ is a BF bi-interior ideal of $F$.*

**Theorem 3.6.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF quasi ideal of semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$.*

*Proof:* Assume that $\omega = (F; \omega^p, \omega^n)$ is a BF quasi-ideal of $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of $F$. Thus, by Corollary 3.5, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$. ■

We know that every BF ideal is a BF bi-interior ideal then the following theorem holds.

**Theorem 3.7.** *If $\omega = (F; \omega^p, \omega^n)$ and $\tau = (F; \tau^p, \tau^n)$ are BF right ideals and a BF left ideal of a semigroup $F$ respectively. Then $\omega \cap \tau = (\omega^p \wedge \tau^p, \omega^n \vee \tau^n)$ is a BF bi-interior ideal of $F$.*

*Proof:* Assume that $\omega = (F; \omega^p, \omega^n)$ and $\tau = (F; \tau^p, \tau^n)$ are BF right ideals and BF left ideal of $F$ respectively. Then by Theorems 3.3 and 3.4 we have $\omega = (F; \omega^p, \omega^n)$ and $\tau = (F; \tau^p, \tau^n)$ are BF bi-interior ideal of $F$. By Theorem 3.22 we have $\omega \cap \tau = (\omega^p \wedge \tau^p, \omega^n \vee \tau^n)$ is a BF bi-interior ideal of $F$. ■

The following are tools the converse of a BF bi-interior ideals are BF ideals on semigroups.

**Definition 3.8.** *A semigroup $F$ is called regular if for all $r \in F$, there exists $k \in F$ such that $r = rkr$.*

**Theorem 3.9.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF set on regualr semigroup $F$. Then every BF bi-ideal is a BF ideal of $F$.*

*Proof:* Suppose that $\omega = (F; \omega^p, \omega^n)$ is a BF bi-ideal of $F$ and $x, y \in F$. Since $F$ is regular, we have $xy \in (xFx)F \subseteq xFx$. Thus, there exists $k \in F$ such that $xy = xkx$. So $\omega^p(xy) = \omega^p(xkx) \geq \omega^p(x) \wedge \omega^p(x) = \omega^p(x)$. And $\omega^n(xy) = \omega^n(xkx) \leq \omega^n(x) \vee \omega^n(x) = \omega^n(x)$. Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF right ideal of $F$. Similarly, we can show that $\omega^p(xy) \geq \omega^p(y)$ and $\omega^n(xy) \leq \omega^n(y)$. Thus, $\omega = (F; \omega^p, \omega^n)$ is a BF left ideal of $F$. Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of $F$. ■

**Theorem 3.10.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF set on regualr semigroup $F$. Then every BF interior ideal is a BF ideal of $F$.*

*Proof:* Suppose that $\omega = (F; \omega^p, \omega^n)$ is a BF interior ideal of $F$ and let $u, v \in F$. Since $F$ is regular, there exists $x \in F$ such that $u = uxu$. Thus,

$$\omega^p(uv) = \omega^p((uxu)v) = \omega^p((ux)uv) \geq \omega^p(u)$$

and

$$\omega^n(uv) = \omega^n((uxu)v) = \omega^n((ux)uv) \leq \omega^n(u).$$

Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF right ideal of $F$. Similarly, we can show that $\omega = (F; \omega^p, \omega^n)$ is a BF left ideal of $F$. Thus, $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of $F$. ■

**Theorem 3.11.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF quasi ideal of a regular semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of a semigroup $F$.*

*Proof:* Assume that $\omega = (F; \omega^p, \omega^n)$ is a BF quasi-ideal of $F$ and let $x, y \in F$. Then

$$\begin{aligned}\omega^p(xy) &\geq (\omega^p \circ \chi_F^p)(xy) \wedge (\chi_F^p \circ \omega^p)(xy) \\ &= \bigvee_{xy=ab} \{\omega^p(a) \wedge \chi_F^p(b)\} \wedge \\ &\quad \bigvee_{xy=ij} \{\chi_F^p(i) \wedge \omega^p(j)\} \\ &\geq \omega^p(x) \wedge \chi_F^p(y) \wedge \chi_F^p(x) \wedge \omega^p(y) \\ &= (\omega^p(x) \wedge 1) \wedge (1 \wedge \omega^p(y)) \\ &= \omega^p(x) \wedge \omega^p(y).\end{aligned}$$

Thus, $\omega^p(xy) \geq \omega^p(x) \wedge \omega^p(y)$. And

$$\begin{aligned}\omega^n(xy) &\leq (\omega^n \circ \chi_F^n)(xy) \vee (\chi_F^n \circ \omega^n)(xy) \\ &= \bigwedge_{xy=ab} \{\omega^n(a) \wedge \chi_F^n(b)\} \vee \\ &\quad \bigwedge_{xy=ij} \{\chi_F^n(i) \vee \omega^n(j)\} \\ &\leq \omega^n(x) \vee \chi_F^n(y) \vee \chi_F^n(x) \vee \omega^n(y) \\ &= (\omega^n(x) \vee -1) \vee (-1 \vee \omega^n(y)) \\ &= \omega^n(x) \vee \omega^n(y).\end{aligned}$$

Thus, $\omega^n(xy) \leq \omega^n(x) \vee \omega^n(y)$.
Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$.

Let $x, y, z \in F$. Then

$$\begin{aligned}\omega^p(xyz) &\geq (\omega^p \circ \chi_F^p)(xyz) \wedge (\chi_F^p \circ \omega^p)(xyz) \\ &= \bigvee_{xyz=ab} \{\omega^p(a) \wedge \chi_F^p(b)\} \wedge \\ &\quad \bigvee_{xyz=ij} \{\chi_F^p(i) \wedge \omega^p(j)\} \\ &\geq \omega^p(x) \wedge \chi_F^p(yz) \wedge \chi_F^p(xy) \wedge \omega^p(z) \\ &= (\omega^p(x) \wedge 1) \wedge (1 \wedge \omega^p(z)) \\ &= \omega^p(x) \wedge \omega^p(z).\end{aligned}$$

Thus, $\omega^p(xyz) \geq \omega^p(x) \wedge \omega^p(z)$. And

$$\begin{aligned}\omega^n(xyz) &\leq (\omega^n \circ \chi_F^n)(xyz) \vee (\chi_F^n \circ \omega^n)(xyz) \\ &= \bigwedge_{xyz=ab} \{\omega^n(a) \vee \chi_F^n(b)\} \vee \\ &\quad \bigwedge_{xyz=ij} \{\chi_F^n(i) \vee \omega^n(j)\} \\ &\leq \omega^n(x) \vee \chi_F^n(yz) \vee \chi_F^n(xy) \vee \omega^n(z) \\ &= (\omega^n(x) \vee -1) \vee (-1 \vee \omega^n(z)) \\ &= \omega^n(x) \vee \omega^n(z).\end{aligned}$$

Thus, $\omega^n(xyz) \leq \omega^n(x) \vee \omega^n(z)$. Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-ideal of $F$. By Theorem 3.9, $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of $F$. ■

**Theorem 3.12.** *Let $F$ be a regular semigroup. Then $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$ if and only if $\omega = (F; \omega^p, \omega^n)$ is a BF quasi ideal of $F$.*

*Proof:* Let $\omega = (F; \omega^p, \omega^n)$ be a BF bi-interior ideal of $F$ and $x \in F$. Then $(\chi_F^p \circ \omega^p \circ \chi_F^p)(x) \wedge (\omega^p \circ \chi_F^p \circ \omega^p)(x) \leq \omega^p(x)$ and $(\chi_F^n \circ \omega^n \circ \chi_F^n)(x) \vee (\omega^n \circ \chi_F^n \circ \omega^n)(x) \geq \omega^n(x)$. Assume that $(\chi_F^p \circ \omega^p)(x) > \omega^p(x)$ and $(\chi_F^m \circ \omega^n)(x) < \omega^n(x)$. Since $F$ is regular, there exists $y \in F$ such that $x = xyx$. Then

$$\begin{aligned}(\omega^p \circ \chi_F^p \circ \omega^p)(x) &= \bigvee_{x=xyx}\{\omega^p(xy) \wedge (\chi_F^p \circ \omega^p)(x)\} \\ &\leq \bigvee_{x=xyx}\{\omega^p(x) \wedge \omega^p(x)\} \\ &= \omega^p(x).\end{aligned}$$

Thus, $(\omega^p \circ \chi_F^p \circ \omega^p)(x) \leq \omega^p(x)$. And

$$\begin{aligned}(\omega^n \circ \chi_F^n \circ \omega^n)(x) &= \bigwedge_{x=xyx}\{\omega^n(xy) \vee (\chi_F^n \circ \omega^n)(x)\} \\ &\geq \bigwedge_{x=xyx}\{\omega^n(x) \vee \omega^n(x)\} \\ &= \omega^n(x).\end{aligned}$$

Thus, $(\omega^n \circ \chi_F^n \circ \omega^n)(x) \geq \omega^n(x)$. Which is a contradiction. Therefore, $\omega = (F; \omega^p, \omega^n)$ is a BF quasi ideal of $F$. For the converse, assume that $\omega = (F; \omega^p, \omega^n)$ is a BF quasi-ideal. Then by Theorem 3.11, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$. ■

**Theorem 3.13.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF bi-interior ideal of a regular semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of a semigroup $F$.*

*Proof:* Suppose that $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$. Then by Theorem 3.12, $\omega = (F; \omega^p, \omega^n)$ is a BF quasi ideal of $F$. Thus, by Theorem 3.11, $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of $F$. Hence, the theorem is complete. ■

**Theorem 3.14.** *If $\omega = (F; \omega^p, \omega^n)$ be a BF bi-ideal of a regular semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of a semigroup $F$.*

*Proof:* Suppose that $\omega = (F; \omega^p, \omega^n)$ is a BF bi-ideal of $F$. Then by Theorem 3.9, $\omega = (F; \omega^p, \omega^n)$ is a BF ideal of $F$. Thus, by Corollary 3.5, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior of $F$. ■

**Theorem 3.15.** *If $\omega = (F; \omega^p, \omega^n)$ be a BF interior ideal of a regular semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of a semigroup $F$.*

*Proof:* Suppose that $\omega = (F; \omega^p, \omega^n)$ is a BF interior ideal of $F$. Then by Theorem 3.10, $\omega = (F; \omega^p, \omega^n)$ is a ideal of $F$. Thus, by Corollary 3.5, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior of $F$. ■

**Theorem 3.16.** *Let $M$ be a non-empty subset of a semigroup $F$ and $\chi_M = (F; \chi_M^p, \chi_M^n)$ be the characteristic BF set of $M$. Then $M$ is a bi-interior ideal of a semigroup $F$ if and only if $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of a semigroup $F$.*

*Proof:* Suppose $M$ is a bi-interior ideal of $F$. Then $M$ is a subsemigroup of $F$. Thus, by Theorem 2.7, $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF subsemigroup of $F$. Let $x \in F$. Since $M$ is a bi-interior ideal of $F$, we have $FMF \cap MFM \subseteq M$. Thus,

$$\begin{aligned}&(\chi_F^p \circ \chi_M^p \circ \chi_F^p \wedge \chi_M^p \circ \chi_F^p \circ \chi_M^p)(x) \\ &= (\chi_F^p \circ \chi_M^p \circ \chi_F^p)(x) \wedge (\chi_M^p \circ \chi_F^p \circ \chi_M^p)(x) \\ &= \chi_{FMF}^p(x) \wedge \chi_{MFM}^p(x) = \chi_{FMF\cap MFM}^p(x) \leq \chi_M^p(x).\end{aligned}$$

and

$$\begin{aligned}&(\chi_F^n \circ \chi_M^n \circ \chi_F^n \vee \chi_M^n \circ \chi_F^n \circ \chi_M^n)(x) \\ &= (\chi_F^n \circ \chi_M^n \circ \chi_F^n)(x) \vee (\chi_M^n \circ \chi_F^n \circ \chi_M^n)(x) \\ &= \chi_{FMF}^n(x) \vee \chi_{MFM}^n(x) = \chi_{FMF\cup MFM}^n(x) \geq \chi_M^n(x).\end{aligned}$$

Therefore, $(\chi_F^p \circ \chi_M^p \circ \chi_F^p \wedge \chi_M^p \circ \chi_F^p \circ \chi_M^p)(x) \leq \chi_M^p(x)$ and $(\chi_F^n \circ \chi_M^n \circ \chi_F^n \vee \chi_M^n \circ \chi_F^n \circ \chi_M^n)(x) \geq \chi_M^n(x)$. Hence $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$.

Conversely, suppose that $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$. Then $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF subsemigroup of $F$. Thus, by Theorem 2.7, $M$ is a subsemigroup of $F$. Let $x \in FMF \cap MFM$. By assumption, $(\chi_F^p \circ \chi_M^p \circ \chi_F^p \wedge \chi_M^p \circ \chi_F^p \circ \chi_M^p)(x) \leq \chi_M^p(x)$ and $(\chi_F^n \circ \chi_M^n \circ \chi_F^n \vee \chi_M^n \circ \chi_F^n \circ \chi_M^n)(x) \geq \chi_M^n(x)$. Thus, $x \in M$. Therefore, $FMF \cap MFM \subseteq M$. Hence, $M$ is a bi-interior ideal of $F$. ■

The *support* of $\omega = (F; \omega^p, \omega^n)$ instead of $\mathrm{supp}(\omega) = \{x \in F \mid \omega^p(x) \neq 0 \,\text{and}\, \omega^n(x) \neq 0\}$.

**Theorem 3.17.** *[10] Let $\omega = (F; \omega^p, \omega^n)$ be a BF subset of a semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$ if and only if $\mathrm{supp}(\omega)$ is a subsemigroup of $F$.*

This Theorem is a study of the $\mathrm{supp}(\omega)$ is a bi-interior ideal on semigroups.

**Theorem 3.18.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF subset of a semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$ if and only if $\mathrm{supp}(\omega)$ is a bi-interior ideal of $F$.*

*Proof:* Let $\omega = (F; \omega^p, \omega^n)$ be a BF bi-interior ideal of $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$. Thus, by Theorem 3.17, $\mathrm{supp}(\omega)$ is a subsemigroup of $F$. Let $m \in F$. Then $(\chi_F^p \circ \omega^p \circ \chi_F^p \wedge \omega^p \circ \chi_F^p \circ \omega^p)(x) \neq 0$ and $(\chi_F^n \circ \omega^n \circ \chi_F^n \vee \omega^n \circ \chi_F^n \circ \omega^n)(x) \neq 0$ for all $x \in F$. Thus, there exists $a_1, a_2, a_3 \in F$ such that $x = ma_1m = a_2ma_3$, $\omega^p(a_1) \neq 0$, $\omega^p(a_2) \neq 0$, $\omega^p(a_3) \neq 0$, $\omega^p(x) \neq 0$ and $\omega^n(a_1) \neq 0$, $\omega^n(a_2) \neq 0$, $\omega^n(a_3) \neq 0$, $\omega^n(x) \neq 0$. It implies that $x, a_1, a_2, a_3 \in \omega$. Thus, $(\chi_\omega^p \circ \omega^p \circ \chi_\omega^p \wedge \omega^p \circ \chi_\omega^p \circ \omega^p)(x) \neq 0$ and $(\chi_F^n \circ \omega^n \circ \chi_\omega^n \vee \omega^n \circ \chi_\omega^n \circ \omega^n)(x) \neq 0$. Hence, $\chi_M = (F; \chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$. By Theorem 3.16, $\mathrm{supp}(\omega)$ is a bi-interior ideal of $F$.

Conversely, let $\mathrm{supp}(\omega)$ be a bi-interior ideal of $F$. Then $\mathrm{supp}(\omega)$ is a subsemigroup of $F$. Thus, by Theorem 3.17, $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$.

Since $(\chi_\omega^p \circ \omega^p \circ \chi_\omega^p \wedge \omega^p \circ \chi_\omega^p \circ \omega^p)(x) \neq 0$ and $(\chi_F^n \circ \omega^n \circ \chi_\omega^n \vee \omega^n \circ \chi_\omega^n \circ \omega^n)(x) \neq 0$ for all $x \in F$. Thus, there exists $a_1, a_2, a_3 \in F$ such that $x = ma_1m = a_2ma_3$, $\omega^p(a_1) \neq 0$, $\omega^p(a_2) \neq 0$, $\omega^p(a_3) \neq 0$, $\omega^p(x) \neq 0$ and $\omega^n(a_1) \neq 0$, $\omega^n(a_2) \neq 0$, $\omega^n(a_3) \neq 0$, $\omega^n(x) \neq 0$. Hence, $(\chi_F^p \circ \omega^p \circ \chi_F^p \wedge \omega^p \circ \chi_F^p \circ \omega^p)(x) \neq 0$ and $(\chi_F^n \circ \omega^n \circ \chi_F^n \vee \omega^n \circ \chi_F^n \circ \omega^n)(x) \neq 0$ for all $x \in F$. Therefore, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$. ■

The following we discuss BF level set are BF bi-interior ideals in semigroups.

**Definition 3.19.** *Let $\omega = (F; \omega^p, \omega^n)$ be a BF set in a non-empty set $X$. Define $U(\omega; t, m) = \{x \in X | t \leq \omega^p(x), \omega^n(x) \geq m\}$ where $t \in [0, 1]$ and $m \in [-1, 0]$ is called the BF level set of $\omega$.*

**Theorem 3.20.** *[10] Let $F$ be a semigroup and $\omega = (F; \omega^p, \omega^n)$ be a non-empty BF set of $F$. Then the BF set $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$ if and only if the BF level set $U(\omega; t, m)$ is a subsemigroup of $F$ for every $t \in [0, 1]$ and $m \in [-1, 0]$, where $U(\omega; t, m) \neq \emptyset$.*

**Theorem 3.21.** *Let $F$ be a semigroup and $\omega = (F; \omega^p, \omega^n)$ be a non-empty BF set of $F$. A BF set $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of $F$ if and only if the BF level set $U(\omega; t, m)$ is a bi-interior ideal of $F$ for every $t \in [0, 1]$ and $m \in [-1, 0]$, where $U(\omega; t, m) \neq \emptyset$.*

*Proof:* Assume that $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of a semigroup $F$. Then $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$. By Theorem 3.20, $U(\omega; t, m)$ is a subsemigorup of $F$. Let $x \in FU(\omega; t, m)F \cap U(\omega; t, m)FU(\omega; t, m)$. Then $x = bau = cde$ where $b, u, d \in F$ and $a, c, e \in U(\omega; t, m)$. Then $t \leq (\chi_F^p \circ \omega \circ \chi_F^p)(x)$ and $t \leq (\omega^p \circ \chi_F^p \circ \omega)(x)$ implies that $t \leq \omega^p(x)$ and $(\chi_F^n \circ \omega^n \circ \chi_F^n)(x) \geq m$ and $(\omega^n \circ \chi_F^n \circ \omega^n)(x) \geq m$ implies that $\omega^n(x) \geq m$. Then $x \in U(\omega; t, m)$. Therefore, $U(\omega; t, m)$ is a bi-interior ideal of $F$.

Conversely suppose that $U(\omega; t, m)$ is a bi-interior ideal of the semigroup $F$, for all $t \in Im(\omega^p)$ and $m \in Im(\omega^n)$. Then $U(\omega; t, m)$ is a subsemigroup of $F$. By Theorem 3.20, $\omega = (F; \omega^p, \omega^n)$ is a BF subsemigroup of $F$. Let $x, y \in F$. Then $\omega^p(x) = t_1$, $\omega^p(y) = t_2$, $\omega^n(x) = m_1$, $\omega^n(y) = m_2$, where $t_1 \geq t_2$ and $m_1 \leq m_2$. Then $x, y \in U(\omega; t, m)$. Thus, $FU(\omega; t, m)F \cap U(\omega; t, m)FU(\omega; t, m) \subseteq U(\omega; t, m)$, for all $t \in Im(\omega^p)$ and $m \in Im(\omega^n)$. Suppose $t = \min\{Im(\omega^p)\}$ and $m = \max\{Im(\omega^n)\}$. Then $(\omega; t, m)F \cap U(\omega; t, m)FU(\omega; t, m) \subseteq U(\omega; t, m)$. Therefore, $\chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega \circ \chi_F^p \circ \omega^p \subseteq \omega^p$ and $\chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega \circ \chi_F^n \circ \omega^n \supseteq \omega^n$. Hence, $\omega = (F; \omega^p, \omega^n)$ is a BF bi-interior ideal of a semigroup $F$. ■

This Theorem is a study of the intersection of a BF bi-interior ideal on semigroups.

**Theorem 3.22.** *If $\omega = (F; \omega^p, \omega^n)$ and $\tau = (F; \tau^p, \tau^n)$ are BF bi-interior ideals of a semigroup $F$, then $\omega \cap \tau = (\omega^p \wedge \tau^p, \omega^n \vee \tau^n)$ is a BF bi-interior ideal of $F$.*

*Proof:* Let $\omega = (F; \omega^p, \omega^n)$ and $\tau = (F; \tau^p, \tau^n)$ be BF bi-interior ideals of $F$. Then $\omega = (F; \omega^p, \omega^n)$ and $\tau = (F; \tau^p, \tau^n)$ are BF subsemigroup of $F$. Thus, $\omega \cap \tau$ is a BF subsemigroup of $F$. Let $x \in F$

$$
\begin{aligned}
(\chi_F^p \circ (\omega^p \wedge \tau^p))(x) &= \bigvee_{x=ab}\{\chi_F^p(a) \wedge (\omega^p \wedge \tau^p)(b)\} \\
&= \bigvee_{x=ab}\{\chi_F^p(a) \wedge \omega^p(b) \wedge \tau^p(b)\} \\
&= \bigvee_{x=ab}\{\{\chi_F^p(a) \wedge \omega^p(b)\} \wedge \\
&\quad \{\chi_F^p(a) \wedge \tau^p(b)\}\} \\
&= \bigvee_{x=ab}\{\chi_F^p(a) \wedge \omega^p(b)\} \wedge \\
&\quad \bigvee_{x=ab}\{\chi_F^p(a) \wedge \tau^p(b)\} \\
&= (\chi_F^p \circ \omega^p)(x) \wedge (\chi_F^p \circ \tau^p)(x) \\
&= (\chi_F^p \circ \omega^p \wedge \chi_F^p \circ \tau^p)(x).
\end{aligned}
$$

Therefore, $\chi_F^p \circ (\omega^p \wedge \tau^p) = \chi_F^p \circ \omega^p \wedge \chi_F^p \circ \tau^p$.

$(\omega^p \wedge \tau^p \circ \chi_F^p \circ \omega^p \wedge \tau^p)(x)$
$= \bigvee_{x=abc}\{(\omega^p \wedge \tau^p)(a) \wedge (\chi_F^p \circ \omega^p \wedge \tau^p)(bc)\}$
$= \bigvee_{x=abc}\{(\omega^p \wedge \tau^p)(a) \wedge \{(\chi_F^p \circ \omega^p \wedge \chi_S^p \circ \tau^p)(bc)\}\}$
$= \bigvee_{x=abc}\{(\omega^p \wedge \tau^p)(a) \wedge \{(\chi_F^p \circ \omega^p)(bc) \wedge (\chi_F^p \circ \tilde{\tau^p})(bc)\}\}$
$= \bigvee_{x=abc}\{\{\omega(a) \wedge (\chi_F^p \circ \omega^p)(bc)\} \wedge \{\tau^p(a) \wedge (\chi_F^p \circ \tau^p)(bc)\}\}$
$= (\omega^p \circ \chi_F^p \circ \omega^p)(x) \wedge (\tau^p \circ \chi_F^p \circ \tau^p)(x)$
$= (\omega^p \circ \chi_F^p \circ \omega^p \wedge \tau^p \circ \chi_F^p \circ \tau^p)(x)$.

Therefore, $(\omega^p \cap \tau^p \circ \chi_F^p \circ \omega^p \cap \tau^p) =$
$(\omega^p \circ \chi_F^p \circ \omega^p \cap \tau^p \circ \chi_F^p \circ \tau^p)$.
Similarly, $\chi_F^p \circ \omega^p \cap \tau^p \circ \chi_F^p = (\chi_F^p \circ \omega^p \circ \chi_F^p) \cap (\chi_F^p \circ \tau^p \circ \chi_S^p)$.
Then

$(\chi_F^p \circ \omega^p \wedge \tau^p \circ \chi_F^p)(x) \wedge (\omega^p \wedge \tau^p \circ \chi_F^p \circ \omega^p \wedge \tau^p)(x)$
$= (\chi_F^p \circ \omega^p \circ \chi_F^p)(x) \wedge (\omega^p \circ \chi_F^p \circ \omega^p)(x) \wedge (\chi_F^p \circ \tau^p \circ \chi_F^p)(x) \wedge (\tau^p \circ \chi_F^p \circ \tau^p)(x)$
$\leq (\omega^p \wedge \tau^p)(x)$.

Therefore, $(\chi_F^p \circ \omega^p \cap \tau^p \circ \chi_F^p) \cap (\omega^p \cap \tau^p \circ \chi_F^p \circ \omega^p \cap \tau^p) \subseteq \omega^p \cap \tau^p$.
And

$$
\begin{aligned}
(\chi_F^n \circ (\omega^n \vee \tau^n))(x) &= \bigwedge_{x=ab}\{\chi_F^n(a) \vee (\omega^n \wedge \tau^n)(b)\} \\
&= \bigwedge_{x=ab}\{\chi_F^n(a) \vee \omega^n(b) \wedge \tau^n(b)\} \\
&= \bigwedge_{x=ab}\{\{\chi_F^n(a) \vee \omega^n(b)\} \wedge \\
&\quad \{\chi_F^n(a) \vee \tau^n(b)\}\} \\
&= \bigwedge_{x=ab}\{\chi_F^n(a) \vee \omega^p(b)\} \vee \\
&\quad \bigwedge_{x=ab}\{\chi_F^n(a) \vee \tau^n(b)\} \\
&= (\chi_F^n \circ \omega^n)(x) \vee (\chi_F^n \circ \tau^n)(x) \\
&= (\chi_F^n \circ \omega^n \vee \chi_F^n \circ \tau^n)(x).
\end{aligned}
$$

Therefore, $\chi_F^n \circ (\omega^n \vee \tau^n) = \chi_F^n \circ \omega^n \vee \chi_F^n \circ \tau^n$.

$(\omega^n \vee \tau^n \circ \chi_F^n \circ \omega^n \vee \tau^n)(x)$
$= \bigwedge_{x=abc}\{(\omega^n \vee \tau^n)(a) \vee (\chi_F^n \circ \omega^n \wedge \tau^n)(bc)\}$
$= \bigwedge_{x=abc}\{(\omega^n \vee \tau^n)(a) \vee \{(\chi_F^n \circ \omega^n \vee \chi_S^n \circ \tau^p)(bc)\}\}$
$= \bigwedge_{x=abc}\{(\omega^n \vee \tau^n)(a)$
$\vee\{(\chi_F^n \circ \omega^n)(bc) \vee (\chi_F^n \circ \tilde{\tau^n})(bc)\}\}$
$= \bigwedge_{x=abc}\{\{\omega(a) \vee (\chi_F^p \circ \omega^n)(bc)\} \vee \{\tau^n(a) \wedge (\chi_F^n \circ \tau^n)(bc)\}\}$
$= (\omega^n \circ \chi_F^n \circ \omega^n)(x) \vee (\tau^n \circ \chi_F^n \circ \tau^n)(x)$
$= (\omega^n \circ \chi_F^p \circ \omega^p \vee \tau^p \circ \chi_F^p \circ \tau^p)(x)$.

Hence, $(\omega^n \cap \tau^n \circ \chi_F^p \circ \omega^p \cap \tau^p) = (\omega^p \circ \chi_F^n \circ \omega^n \cap \tau^n \circ \chi_F^n \circ \tau^n)$.
Similarly, $\chi_F^n \circ \omega^n \cap \tau^n \circ \chi_F^n = (\chi_F^n \circ \omega^n \circ \chi_F^n) \cap (\chi_F^n \circ \tau^n \circ \chi_S^n)$.
Then

$(\chi_F^n \circ \omega^n \vee \tau^n \circ \chi_F^n)(x) \vee (\omega^n \vee \tau^n \circ \chi_F^n \circ \omega^n \vee \tau^n)(x)$
$= (\chi_F^n \circ \omega^n \circ \chi_F^n)(x) \wedge (\omega^n \circ \chi_F^n \circ \omega^n)(x) \wedge$
$(\chi_F^n \circ \tau^n \circ \chi_F^n)(x) \vee (\tau^n \circ \chi_F^p \circ \tau^n)(x) \geq (\omega^n \vee \tau^n)(x)$.

Therefore, $(\chi_F^n \circ \omega^n \cap \tau^n \circ \chi_F^n) \cap (\omega^n \cap \tau^n \circ \chi_F^n \circ \omega^n \cap \tau^n) \supseteq \omega^n \cap \tau^n$. Hence, $\omega \cap \tau$ is a BF bi-interior ideal of $F$. ■

## IV. Characterization regalur semigroup in terms bipolar fuzzy bi-interior ideals.

The following theorems are tools in the characterization regular semigorup in terms of BF bi-interior ideals on semigroups.

**Theorem 4.1.** *[10] For non-empty subsets $G$ and $H$ of a semigroup $F$, we have*
(1) $\chi_G \circ \chi_H = \chi_{GH}$,
(2) $\chi_G \wedge \chi_H = \chi_{G \cap H}$.

**Theorem 4.2.** *[12] Let $F$ be a semigroup. Then $F$ is a regular semigroup if and only if $M = FMF \cap FMF$, for every bi-interior ideal $M$ of $F$.*

The following theorems are the characterization of regular semigroup in terms of BF bi-interior ideals in semigroups.

**Theorem 4.3.** *Let $F$ be a semigroup. Then $F$ is a regular if and only if $\omega^p = \chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega^p \circ \chi_F^p \circ \omega^p$ and $\omega^n =$*

$\chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega^n \circ \chi_F^n \circ \omega^n$, *for every BF bi-interior ideal of a semigroup* $F$.

*Proof:* Let $\omega = (F; \omega^p, \omega^n)$ be a BF bi-interior ideal of the regular semigroup $F$ and let $r \in F$. Since $F$ is regular, there exists $k \in F$ such that $r = rkr$. Thus,

$$\begin{aligned}(\omega^p \circ \chi_F^p \circ \omega^p)(r) &= \bigvee_{r=rkr} \{\omega^p(r) \wedge (\chi_F^p \circ \omega^p)(kr)\} \\ &= \bigvee_{r=rkr} \{\omega^p(r) \wedge \bigvee_{kr=yz} \{\chi_F^p(y) \wedge \omega^p(z)\}\} \\ &= \bigvee_{r=rkr} \{\omega^p(r) \wedge \bigvee_{kr=yz} \{1 \wedge \omega^p(z)\}\} \\ &= \bigvee_{r=rkr} \{\omega^p(r) \wedge \bigvee_{kr=yz} \{\omega^p(z)\}\} \\ &\geq \bigvee_{r=rkr} \{\omega^p(r) \wedge \omega^p(r)\} = \omega^p(r).\end{aligned}$$

And

$$\begin{aligned}(\omega^n \circ \chi_F^n \circ \omega^n)(r) &= \bigwedge_{r=rkr} \{\omega^n(r) \vee (\chi_F^n \circ \omega^n)(kr)\} \\ &= \bigwedge_{r=rkr} \{\omega^n(r) \vee \bigwedge_{kr=yz} \{\chi_F^n(y) \vee \omega^n(z)\}\} \\ &= \bigwedge_{r=rkr} \{\omega^n(r) \vee \bigwedge_{kr=yz} \{-1 \vee \omega^n(z)\}\} \\ &= \bigwedge_{r=rkr} \{\omega^n(r) \vee \bigvee_{kr=yz} \{\omega^n(z)\}\} \\ &\leq \bigwedge_{r=rkr} \{\omega^n(r) \vee \omega^n(r)\} = \omega^n(r).\end{aligned}$$

Hence, $(\omega^p \circ \chi_F^p \circ \omega^p)(r) \geq \omega^p(r)$ and $(\omega^n \circ \chi_F^n \circ \omega^n)(r) \leq \omega^n(r)$. Therefore, $\omega^p = \chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega^p \circ \chi_F^p \circ \omega^p$ and $\omega^n = \chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega^n \circ \chi_F^n \circ \omega^n$.

Conversely suppose that $\omega^p = \chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega^p \circ \chi_F^p \circ \omega^p$ and $\omega^n = \chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega^n \circ \chi_F^n \circ \omega^n$ and $M$ is a bi-interior ideal of a semigroup $F$. Then $FMF \cap MFM \subseteq M$.
Let $r \in M$. Then by Theorem 3.16, $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$. Thus, by Theorem 4.1,

$$\begin{aligned}\chi_M^p(r) &= \chi_F^p \circ \chi_M^p \circ \chi_F^p(r) \wedge \chi_M^p \circ \chi_F^p \circ \chi_M^p(x) \\ &= \chi_{FMF}^p(r) \wedge \chi_{MFM}^p(r) = \chi_{FMF \cap MFM}^p(r).\end{aligned}$$

And

$$\begin{aligned}\chi_M^n(r) &= \chi_F^n \circ \chi_M^n \circ \chi_F^n(r) \vee \chi_M^n \circ \chi_F^n \circ \chi_M^n(x) \\ &= \chi_{FMF}^n(r) \vee \chi_{MFM}^n(r) = \chi_{FMF \cap MFM}^n(r).\end{aligned}$$

Hence, $M \subseteq FMF \cap MFM$. Therefore, $M = FMF \cap MFM$. By Theorem 4.2, $F$ is a regular semigroup. ■

**Theorem 4.4.** *Let* $F$ *be a semigroup. Then* $F$ *is regular if and only if* $\omega^p \cap \tau^p \subseteq \tau^p \circ \omega^p \circ \tau^p \cap \omega^p \circ \tau^p \circ \omega^p$ *and* $\omega^n \cup \tau^n \supseteq \tau^n \circ \omega^n \circ \tau^n \cup \omega^n \circ \tau^n \circ \omega^n$, *for every BF bi-interior ideal* $\omega = (F; \omega^p, \omega^n)$ *and every BF ideal* $\tau = (F; \tau^p, \tau^n)$ *of a semigroup* $F$.

*Proof:* Supposet that $\omega = (F; \omega^p, \omega^n)$ be a BF bi-interior ideal, $\tau = (F; \tau^p, \tau^n)$ be a BF ideal of a regular semigroup $F$ and let $r \in F$. Then there exists $k \in F$ such that $r = rkr$. Since $\tau = (F; \tau^p, \tau^n)$ is a BF ideal of $F$ we have $\tau = (F; \tau^p, \tau^n)$ is a BF interior ideal of $F$. Thus,

$$\begin{aligned}(\omega^p \circ \tau^p \circ \omega^p)(r) &= \bigvee_{r=rkr} \{(\omega^p \circ \tau^p)(rk) \wedge \omega^p(r)\} \\ &= \bigvee_{r=rkr} \{ \bigvee_{rk=rkrk} \{\omega^p(r) \wedge \tau^p(krk)\} \\ &\wedge \omega^p(r)\} \\ &\geq \{\omega^p(r) \wedge \tau^p(r)\} \wedge \omega^p(r) \\ &= \omega^p(r) \wedge \tau^p(r) \\ &= (\omega^p \wedge \tau^p)(r).\end{aligned}$$

And,

$$\begin{aligned}(\tau^p \circ \omega^p)(r) &= \bigvee_{r=rkr} \{\tau^p(rk) \wedge \omega^p(r)\} \\ &\geq \{\tau^p(r) \wedge \omega^p(r)\} \\ &= \omega^p(r) \wedge \tau^p(r) \\ &= (\omega^p \wedge \tau^p)(r).\end{aligned}$$

Hence, $\omega^p \cap \tau^p \subseteq \tau^p \circ \omega^p \circ \tau^p \cap \omega^p \circ \tau^p \circ \omega^p$. Similarly,

$$\begin{aligned}(\omega^n \circ \tau^n \circ \omega^n)(r) &= \bigwedge_{r=rkr} \{(\omega^n \circ \tau^n)(rk) \vee \omega^n(r)\} \\ &= \bigwedge_{r=rkr} \{ \bigwedge_{rk=rkrk} \{\omega^n(r) \vee \tau^n(krk)\} \\ &\vee \omega^n(r)\} \\ &\leq \{\omega^n(r) \vee \tau^n(r)\} \vee \omega^n(r) \\ &= \omega^n(r) \vee \tau^n(r) \\ &= (\omega^n \wedge \tau^n)(r).\end{aligned}$$

And,

$$\begin{aligned}(\tau^n \circ \omega^n)(r) &= \bigwedge_{r=rkr} \{\tau^n(rk) \vee \omega^n(r)\} \\ &\leq \{\tau^n(r) \vee \omega^n(r)\} \\ &= \omega^n(r) \vee \tau^n(r) \\ &= (\omega^n \vee \tau^n)(r).\end{aligned}$$

Hence, $\omega^n \cup \tau^n \subseteq \tau^n \circ \omega^n \circ \tau^n \cup \omega^n \circ \tau^n \circ \omega^n$.

Conversely suppose that $\omega^p \cap \tau^p \subseteq \tau^p \circ \omega^p \circ \tau^p \cap \omega^p \circ \tau^p \circ \omega^p$ and $\omega^n \cup \tau^n \supseteq \tau^n \circ \omega^n \circ \tau^n \cup \omega^n \circ \tau^n \circ \omega^n$ and let $\omega = (F; \omega^p, \omega^n)$ be a BF bi-interior ideal. We have $\omega^p \cap \chi_F^p \subseteq \chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega^p \circ \chi_F^p \circ \omega^p$ and $\omega^n \cup \chi_F^u \supseteq \chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega^n \circ \chi_F^n \circ \omega^n$ implies that $\omega^p \subseteq \chi_F^p \circ \omega^p \circ \chi_F^p \cap \omega^p \circ \chi_F^p \circ \omega^p$ and $\omega^n \supseteq \chi_F^n \circ \omega^n \circ \chi_F^n \cup \omega^n \circ \chi_F^n \circ \omega^n$.
By Theorem 4.3, $F$ is a regular semigroup. ■

## V. Bipolar fuzzy Prime bi-interior ideals in semigroups

In this section, we introduce the notion of bipolar fuzzy prime, semiprime bi-interior ideals of semigroups and we study the properties of bipolar fuzzy prime, semiprime bi-interior ideals and relations between them.

**Definition 5.1.** *A bi-interior ideal* $M$ *of a semigroup* $F$ *is called a*

(1) prime bi-inetior ideal *of* $F$ *if* $M_1 M_2 \subseteq M$ *implies* $M_1 \subseteq M$ *or* $M_2 \subseteq M$.
(2) semiprime bi-inetior ideal *of* $F$ *if* $M^2 \subseteq M$ *implies* $M \subseteq M$.

**Definition 5.2.** *A BF bi-interior ideal $\omega = (F; \omega^p, \omega^n)$ of a semigroup $F$ is a*

(1) prime bi-interior ideal *of $F$ if $\omega^p(rk) \leq \omega^p(r) \vee \omega^p(k)$ and $\omega^p(rk) \geq \omega^p(r) \wedge \omega^p(k)$ for all $r, k \in F$.*
(2) semiprime bi-interior ideal *of $F$ if $\omega^p(r^2) \leq \omega^p(r)$ and $\omega^p(r^2) \geq \omega^p(r)$ for all $r \in F$.*

**Theorem 5.3.** *Let $M$ be a non-empty subset of a semigroup $F$. Then the following statement holds*

(1) *$M$ is a prime bi-interior ideal of $F$ if and only if $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF prime bi-interior ideal of $F$,*
(2) *$M$ is a semiprime bi-interior ideal of $F$ if and only if $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF semiprime bi-interior ideal of $F$.*

*Proof:*

(1) Suppose that $M$ is a prime bi-interior ideal of $F$ and let $x, y \in F$. By assumption, $M$ is bi-interior ideal of $F$. Thus, by Theorem 3.16 $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$.
If $rk \in M$, then $r \in M$ or $k \in M$.
Thus, $\chi_M^p(r) \vee \chi_M^p(k) = 1 \geq \chi_M^p(rk)$ and $\chi_M^n(r) \wedge \chi_M^n(k) = 0 \leq \chi_M^n(rk)$.
If $rk \notin A$, then $\chi_M^p(rk) = 0 \leq \chi_M^p(r) \vee \chi_M^p(k)$ and $\chi_M^n(rk) = 1 \geq \chi_M^n(r) \wedge \chi_M^n(k)$.
Thus, $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF prime bi-interior ideal of $F$.
Conversely, suppose that $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF prime bi-interior ideal of $F$ and let $x, y \in F$. By assumption, $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$. Thus, by Theorem 3.16, $M$ is a bi-interior ideal of $F$. If $rk \in M$, then $\chi_M^p(rk) = 1$ and $\chi_M^n(rk) = 0$. By assumption, $\chi_M^p(rk) \leq \chi_M^p(r) \vee \chi_M^p(k)$ and $\chi_M^n(rk) \geq \chi_M^n(r) \wedge \chi_M^n(k)$. Thus, $\chi_M^p(r) \vee \chi_M^p(k) = 1$ and $\chi_M^p(r) \wedge \chi_M^p(k) = 0$ so $r \in M$ or $k \in M$. Hence $M$ is a prime bi-interior ideal of $F$.

(2) Suppose that $M$ is a semiprime bi-interior ideal of $F$ and let $r \in F$. By assumption, $M$ is a bi-interior ideal of $F$. Thus, by Theorem 3.16 $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF bi-interior ideal of $F$.
If $r^2 \in M$, then $r \in M$ Thus, $\chi_M^p(r) = \chi_M^p(r^2) = 1$ and $\chi_M^n(r) = \chi_M^n(r^2) = 0$. Hence, $\chi_M^p(r^2) \leq \chi_M^p(r)$ and $\chi_M^n(r^2) \geq \chi_M^n(r)$.
If $r^2 \notin M$, then $\chi_M^p(r^2) = 0 \leq \chi_M^p(r)$ and $\chi_M^n(r^2) = 1 \geq \chi_M^n(r)$.
Thus, $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF semiprime bi-interior ideal of $F$.
Conversely, suppose that $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF semiprime bi-interior ideal of $F$ and let $r \in F$. By assumption, $\chi_M = (\chi_M^p, \chi_M^n)$ is a BF bi-interior ideal. Thus, by Theorem 3.16, $M$ is a bi-interior ideal of $F$. If $r^2 \in M$, then $\chi_M^p(r^2) = 1$ and $\chi_M^n(r^2) = 0$. By assumption $\chi_M^p(r^2) \leq \chi_M^p(r)$ and $\chi_n^p(r^2) \geq \chi_M^n(r)$. Thus, $\chi_M^p(r) = 1$ and $\chi_M^n(r) = 0$ so $r \in M$. Hence $M$ is a semiprime bi-interior ideal of $F$.

■

## VI. Conclusion

In this article, we propose the concept of bipolar fuzzy bi-interior ideals in semigroups. We study properties of bipolar fuzzy bi-interior ideals. The relation between bipolar fuzzy bi-interior ideals and bi-interior ideals is proved. In the further, we study bipolar fuzzy bi-quasi ideals in semigroups.

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