On Certain Coupled Fixed Point Theorems Via C
Star Class Functions in $C^*$-Algebra Valued Fuzzy
Soft Metric Spaces With Applications

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Abstract—The discussion of this paper is to aim to examine
application of the notion of $C^*$-algebra valued fuzzy soft metric
to homotopy theory using common coupled fixed point results
from $C_*$-class functions. We also tried to provide an illustration
of our major findings. The results attained expand upon and
apply to many of the findings in the literature.

Index Terms—$C_*$-class function, $\omega$-compatible mapping,
$C^*$-algebra valued fuzzy soft metric and coupled fixed points.

I. INTRODUCTION

NUMEROUS real-world issues deal with ambiguous
data and cannot be adequately described in classical
mathematics. Fuzzy set theory, developed by Zadeh [1], and
the theory of soft sets, developed by Molodstov [2], are two
types of mathematical tools that can be used to deal with
uncertainties and help with difficulties in a variety of fields.
Thangaraj Beaula et al. defined fuzzy soft metric space in
terms of fuzzy soft points in the cited work [3], and they
supported various claims. However, numerous authors have
established a great deal of findings regarding fuzzy soft sets
and fuzzy soft metric spaces (see [4] -[6]).

A concept of $C^*$-algebra valued metric space was pre-
sented in 2006 by Ma et al. in [7], and certain fixed and
coupled fixed point solutions for mapping under contraction
conditions in these spaces were established. This line of
inquiry was pursued in (see [8]-[14]). Recently, R.P.Agarwal
et al. introduced the idea of $C^*$-algebra valued fuzzy soft
metric spaces and demonstrated some associated fixed point
solutions on this space (see. [15]-[19]).

The purpose of this article is to establish two pairs of
$\omega$-compatible mappings meeting generalised contractive
requirements as unique common coupled fixed point theorems
using $C_*$-class functions in the context of $C^*$-algebra valued
fuzzy soft metric spaces. Additionally, we may provide
pertinent examples and applications for homotopy.

II. PRELIMINARIES

In this section, we review several fundamental notations
and definitions.

Definition II.1:([15]) Assume that $C \subseteq \Theta$ and $\hat{\Theta}$ are
the absolute fuzzy soft set and $\psi_0(\alpha) = \hat{l}$ for all $\alpha \in \Theta$,
respectively. Let the $C^*$-algebra be represented by $\hat{C}$. The
mapping $d_{C^*}: \Theta \times \Theta \rightarrow \hat{C}$ satisfying the given constraints is
known as the $C^*$-algebra valued fuzzy soft metric utilising
fuzzy soft points.

(i) $\tilde{0}_{\hat{C}} \leq d_{C^*}(\psi_{\alpha}, \psi_{\beta})$ for all $\psi_{\alpha}, \psi_{\beta} \in \Theta$,
(ii) $d_{C^*}(\psi_{\alpha}, \psi_{\beta}) = 0_{\hat{C}} \iff \psi_{\alpha} = \psi_{\beta}$,
(iii) $d_{C^*}(\psi_{\alpha}, \psi_{\beta}) = d_{C^*}(\psi_{\beta}, \psi_{\alpha})$,
(iv) $d_{C^*}(\psi_{\alpha}, \psi_{\beta}) \leq d_{C^*}(\psi_{\alpha}, \psi_{\gamma}) + d_{C^*}(\psi_{\gamma}, \psi_{\beta})$
$\forall \psi_{\alpha}, \psi_{\beta}, \psi_{\gamma} \in \Theta$.

The $C^*$-algebra valued fuzzy soft metric space is made up
of the fuzzy soft set $\Theta$ and the fuzzy soft metric $d_{C^*}$. It is
represented by the symbol $(\Theta, C^*, d_{C^*})$.

Remark II.1: ([15]) It is clear that fuzzy soft metric spaces
with $C^*$-algebra valued fuzzy soft metrics generalise the idea
of fuzzy soft metric spaces by substituting the set of fuzzy
soft real numbers with $C^*$. The idea of a fuzzy soft metric
space with $C^*$-algebra values is similar to the definition of
real metric spaces if we assume that $C^*_+ = \mathcal{R}$.

Example II.1:([15]) If $C$ and $\Theta$ are subsets of $\mathcal{R}$, then $\hat{\Theta}$
is an absolute fuzzy soft set, where $\theta(\alpha) = \hat{l}$ for every $\alpha \in
\Theta$, and $\hat{C}$ is defined as $\mathcal{M}_2(\mathcal{R}(C^*))$.

Define $d_{C^*}: \Theta \times \Theta \rightarrow \hat{C}$ by $d_{C^*}(\psi_{\alpha}, \psi_{\beta}) = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}$,
where $\kappa = \inf\{\|\mu_{\psi_{\alpha}}(t) - \mu_{\psi_{\beta}}(t)\|/t \in C\}$ and
$\psi_{\alpha}, \psi_{\beta} \in \hat{\Theta}$. Then, by the completeness of $\mathcal{R}(C^*)$,
$(\hat{\Theta}, C^*, d_{C^*})$ is a complete $C^*$-algebra valued fuzzy soft metric
distance and $d_{C^*}$ is a $C^*$-algebra valued fuzzy soft metric.

Definition II.2:([15]) Assume that $(\hat{\Theta}, C^*, d_{C^*})$ is a
$C^*$-algebra valued fuzzy soft metric space. According to $\hat{C}$
a sequence $\{\psi_{\alpha_k}\}$ in $\hat{\Theta}$ is defined as:

(1) $C^*$-algebra valued fuzzy soft Cauchy sequence if, for
each $\tilde{0}_{\hat{C}} \prec \tilde{\delta}$, there exist $\tilde{0}_{\hat{C}} \prec \tilde{\delta}$ and a positive
integer $N = N(\tilde{\epsilon})$ such that $\|d_{C^*}(\psi_{\alpha_k}, \psi_{\alpha_l})\| \prec 0_{\hat{C}}$ implies
that $\|\mu_{\psi_{\alpha_k}}(t) - \mu_{\psi_{\alpha_l}}(s)\| \prec 0_{\hat{C}}$ whenever $k, l \geq N$.
That is $\|d_{C^*}(\psi_{\alpha_k}, \psi_{\alpha_l})\| \rightarrow 0_{\hat{C}}$ as $k, l \rightarrow \infty$.

(2) $C^*$-algebra valued fuzzy soft convergent to a point $\psi_{\alpha_0} \in \hat{\Theta}$ if, for each $\tilde{0}_{\hat{C}} \prec \tilde{\delta}$ and a positive
integer $N = N(\tilde{\epsilon})$ such that $\|d_{C^*}(\psi_{\alpha_k}, \psi_{\alpha_l})\| \prec 0_{\hat{C}}$ implies
$\|\mu_{\psi_{\alpha_k}}(t) - \mu_{\psi_{\alpha_l}}(t)\| \prec 0_{\hat{C}}$ whenever $k, l \geq N$. It is usually denoted as
$\lim_{k \rightarrow \infty} \psi_{\alpha_k} = \psi_{\alpha_0}$.

(3) It is referred to as being complete when a $C^*$-algebra
valued fuzzy soft metric space $(\hat{\Theta}, C^*, d_{C^*})$ is present.
If each Cauchy sequence in $\Theta$ converges to a fuzzy
soft point in $\Theta$.

Lemma II.1:([15]) Let $\hat{C}$ be a $C^*$-algebra with the identity

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element $I_C$ and $\tilde{\theta}$ be a positive element of $C$. If $\lambda \in \bar{C}$ is such that $||\lambda|| < 1$ then for $p < q$, we have

$$\lim_{q \to \infty} \sum_{k=p}^{q} (\lambda_k)^p \tilde{\theta}(\lambda_k) \in I_C(\tilde{\theta}(\bar{\lambda})^2) \left( \frac{||\lambda||^p}{1-||\lambda||} \right).$$

**Defn II.2:** (151) Suppose that $C$ is a unital $\mathcal{C}^*$-algebra with unit $1$. Let $\bar{\lambda} \in \bar{C}$ with $||\bar{\lambda}|| < 1$ then $I - \bar{\lambda}$ is invertible and $||\bar{\lambda}(I - \bar{\lambda})^{-1}|| < 1$.

**Defn II.3:** Suppose that $\bar{\lambda}, \bar{\lambda} \in \bar{C}$ with $\bar{\lambda} \leq \bar{\lambda}$ and $\bar{\lambda} \bar{\lambda} = \bar{\lambda} \bar{\lambda}$ then $\bar{\lambda} \bar{\lambda} \geq 0$.

**Defn II.4:** Let $C'$ be a $\mathcal{C}^*$-algebra, if $\bar{\lambda}, \bar{\lambda} \in C'$ with $\bar{\lambda} \geq \bar{\lambda} \geq 0$ and $I - \bar{\lambda} \in C_+$ is an invertible operator, then $(I - \bar{\lambda})^{-1}$ is an invertible operator, where $\bar{\lambda} \in C_+ \cap C'$.

Notice that in $\mathcal{C}^*$-algebra, if $0 \leq \bar{\lambda}, \bar{\lambda}$, one can’t conclude that $0 \leq \bar{\lambda} \bar{\lambda}$. Indeed, consider the $\mathcal{C}^*$-algebra $M_2(\mathbb{R}(\mathcal{C}))$ and set

$$\tilde{\kappa} = \begin{bmatrix} \psi_{a1}(a) & \psi_{a2}(a) \\ \psi_{b1}(a) & \psi_{b2}(a) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

and

$$\tilde{\lambda} = \begin{bmatrix} \psi_{a1}(c) & \psi_{a2}(c) \\ \psi_{b1}(d) & \psi_{b2}(d) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}$$

then clearly $\tilde{\kappa} \geq 0$ and $\tilde{\lambda} \geq 0$ but $\tilde{\lambda} \tilde{\lambda}$ is not.

For more properties of a $\mathcal{C}^*$-algebra valued fuzzy soft metric and $\mathcal{C}^*$-algebra we refer the reader to ([15], [20]).

### III. MAIN RESULTS

For $\mathcal{C}_*$-class functions in $\mathcal{C}^*$-algebra valued fuzzy soft metric spaces, we will demonstrate various coupled fixed point theorems in this section.

**Defn III.1:** Let $(\theta, C, d_\times)$ be a $\mathcal{C}^*$-algebra valued fuzzy soft metric space. Let $S : \theta \times \theta \to \theta$ be a mapping. Then an element $(\psi_{a1}, \psi_{a2}) \in \theta \times \theta$ is called coupled fixed point of $S$ if $S(\psi_{a1}, \psi_{a2}) = \psi_{a1}$ and $S(\psi_{a2}, \psi_{a1}) = \psi_{a2}$.

**Defn III.2:** Let $\theta$ be absolute fuzzy soft set and $S : \theta \times \theta \to \theta$ and $f : \theta \to \theta$ be two mappings. An element $(\psi_{a1}, \psi_{a2}) \in \theta \times \theta$ is called

- (i) a coupled coincidence point of $S$ and $f$ if $f(\psi_{a1}) = S(\psi_{a1}, \psi_{a2})$ and $f(\psi_{a2}) = S(\psi_{a2}, \psi_{a1})$
- (ii) a common coupled fixed point of $S$ and $f$ if $\psi_{a1} = f(\psi_{a1}) = S(\psi_{a1}, \psi_{a2})$ and $\psi_{a2} = f(\psi_{a2}) = S(\psi_{a2}, \psi_{a1})$.

**Defn III.3:** Let $\theta$ be absolute fuzzy soft set and $S : \theta \times \theta \to \theta$ and $f : \theta \to \theta$. Then the sequence $(f)$ is said to be $\omega$-compatible if $f(S(\psi_{a1}, \psi_{a2})) = S(f(\psi_{a1}), f(\psi_{a2}))$ and $f(S(\psi_{a2}, \psi_{a1})) = S(f(\psi_{a2}), f(\psi_{a1}))$.

**Defn III.4:** Let $C$ be a unital $\mathcal{C}^*$-algebra. Then a continuous function $\Gamma : C_+ \times C_+ \to C_+$ is called a $C_*$-class function if for all $A, B \in C_+$,

(a) $\Gamma(\bar{A}, \bar{B}) \leq \bar{A}$;
(b) $\Gamma(\bar{A}, \bar{B}) = \bar{A} \Rightarrow \bar{A} = \bar{0}$ or $\bar{B} = \bar{0}$.

We denote $C_+$ as the family of all $\mathcal{C}_*$-class functions.

**Defn III.5:** A function $\eta : C_+ \to C_+$ is called an altering distance function if the following properties are satisfied:

- (a) $\eta$ is nondecreasing and continuous,
- (b) $\eta(A) = 0$ if and only if $A = 0$.

The family of all altering distance functions is denoted by $\Omega$.

**Theorem III.1:** Assume that $\mathcal{C}^*$-algebra valued fuzzy soft metric space $(\theta, C, d_\times)$ and suppose two mappings $S : \theta \times \theta \to \theta$ and $f : \theta \to \theta$ be satisfying

$$\eta\left( d_\times(\psi_{a1}, \psi_{a2}), S(\psi_{a1}, \psi_{a2}) \right) \leq \Gamma\left( \eta\left( \tilde{\kappa} d_\times(\psi_{a1}, \psi_{a2}), \theta(\tilde{\kappa} d_\times(\psi_{a1}, \psi_{a2})) \right) \right)$$

for all $\psi_{a1}, \psi_{a2} \in \theta$, where $\tilde{\kappa} \in C$ with $||\tilde{\kappa}|| < 1$ and $\eta, \theta \in \Omega$ and $\Gamma \in C_*$.

(i) $S(\theta \times \theta) \subseteq f(\theta)$,
(ii) $(S, f)$ is $\omega$-compatible pairs,
(iii) $f(\theta)$ is complete $\mathcal{C}^*$-algebra valued fuzzy soft metric space $\theta$.

Then, in $\tilde{\kappa}, S$ and $f$ have a unique common coupled fixed point.

**Proof:** Let $\psi_{a1}, \psi_{a2} \in \theta$. From (i) we can construct the sequences $(\psi_{a1})_{n=1}^\infty$, $(\psi_{a2})_{n=1}^\infty$ and $(\xi_{a1})_{n=1}^\infty$, $(\xi_{a2})_{n=1}^\infty$ such that

$$S(\psi_{a1}, \psi_{a2}) = \xi_{a1}, S(\psi_{a2}, \psi_{a1}) = \xi_{a2}$$

for $n = 0, 1, 2, \ldots$

Observe that in $\mathcal{C}^*$-algebra, if $\tilde{\kappa}, \tilde{\lambda} \in C_+$ and $\tilde{\kappa} \leq \tilde{\lambda}$, then for any $\tilde{x} \in C_+$ both $\tilde{x} \tilde{\kappa} \tilde{x}$ and $\tilde{x}^* \tilde{\lambda} \tilde{x}$ are positive. We conveniently refer to the element $d_\times(\xi_{a1}, \xi_{a2})$ in $C$ as $Q$.

From (1), we get

$$\eta\left( d_\times(\xi_{a1}, \xi_{a2}) \right) \leq \eta\left( d_\times(\psi_{a1}, \psi_{a2}), \psi_{a1}) \right) \leq \Gamma\left( \eta\left( \tilde{\kappa} d_\times(\psi_{a1}, \psi_{a2}), \theta(\tilde{\kappa} d_\times(\psi_{a1}, \psi_{a2})) \right) \right)$$

By the definition of $\eta$, we have

$$d_\times(\xi_{a1}, \xi_{a2}) \leq \tilde{\kappa} d_\times(\xi_{a1}, \xi_{a2}) \leq (\tilde{\kappa})^2 d_\times(\xi_{a1}, \xi_{a2}) \leq \cdots \leq (\tilde{\kappa})^n d_\times(\xi_{a1}, \xi_{a2}) \leq (\tilde{\kappa})^n Q \tilde{\kappa}^n.$$
→ 0 as m → ∞.

As a result, \{ξn\} is a Cauchy sequence in \(\tilde{C}\) with regard to \(\tilde{d}\). We can also demonstrate that \{\tilde{ξn}\} is a Cauchy sequence with regard to \(\tilde{C}\). Let’s say \(f(\Theta)\) the complete subspace of \((\Theta, \tilde{C}, \tilde{d})\). Then the sequences \{ξn\} and \{\tilde{ξn}\} are convergent to \(ξ'\) and \(\tilde{ξ}^*\) respectively in \(f(\Theta)\). Thus, there exist \(\psi''\) and \(\tilde{\psi}''\) in \(f(\Theta)\) such that

\[
\lim_{n \to \infty} \xi_n = ξ' \quad \text{and} \quad \lim_{n \to \infty} \tilde{ξ}_n = \tilde{ξ}' \Rightarrow ξ' = \tilde{θ}(ξ, ξ').
\]

Now, we claim that \(S(ξ', ξ') = ξ'\) and \(S(\tilde{ξ}', \tilde{ξ}') = \tilde{ξ}'\).

From (1) and using the triangular inequality

\[
\tilde{d}(S(\tilde{ξ}', \tilde{ξ}'), ξ') \leq \tilde{d}(\tilde{ξ}', \tilde{ξ}') + \tilde{d}(\tilde{ξ}', ξ') \leq \tilde{d}(\tilde{ξ}', ξ') \Rightarrow \tilde{d}(\tilde{ξ}', ξ') = 0.
\]

By the definition of \(\eta\), we have

\[
\eta \left( \tilde{d}(S(\tilde{ξ}', \tilde{ξ}'), ξ') \right) = \lim_{n \to \infty} \eta \left( \tilde{d}(S(\tilde{ξ}', \tilde{ξ}'), ξ') \right) = 0.
\]

Similarly, we prove \(S(ξ', ξ') = ξ'\). Therefore, it follows \(S(ξ, ξ') = ξ'\) and \(S(\tilde{ξ}', ξ') = \tilde{ξ}'\).

\begin{align*}
\tilde{d}(S(ξ', ξ'), ξ') & \leq \tilde{d}(S(ξ', ξ'), ξ') + \tilde{d}(S(ξ', ξ'), ξ') \\
& \leq \tilde{d}(S(ξ', ξ'), ξ') \Rightarrow \tilde{d}(S(ξ', ξ'), ξ') = 0.
\end{align*}

\begin{align*}
\tilde{d}(S(ξ, ξ'), ξ') & \leq \tilde{d}(S(ξ, ξ'), ξ') + \tilde{d}(S(ξ, ξ'), ξ') \\
& \leq \tilde{d}(S(ξ, ξ'), ξ') \Rightarrow \tilde{d}(S(ξ, ξ'), ξ') = 0.
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\tilde{d}(S(ξ', ξ'), ξ') & \leq \tilde{d}(S(ξ', ξ'), ξ') + \tilde{d}(S(ξ', ξ'), ξ') \\
& \leq \tilde{d}(S(ξ', ξ'), ξ') \Rightarrow \tilde{d}(S(ξ', ξ'), ξ') = 0.
\end{align*}

Therefore, we have \(\tilde{d}(S(ξ', ξ'), ξ') = 0\) implies \(S(ξ', ξ', \tilde{ξ}') = 0\) implies \(S(ξ', \tilde{ξ}', \tilde{ξ}') = ξ'\).

Let \(S(ξ', \tilde{ξ}', \tilde{ξ}') = ξ'\). Then \(\tilde{c}(\tilde{ξ}', \tilde{ξ}') = \tilde{ξ}'\). Therefore, it follows \(\tilde{c}(\tilde{ξ}', \tilde{ξ}') = ξ'\) and \(\tilde{c}(\tilde{ξ}', \tilde{ξ}') = \tilde{ξ}'\).

\begin{align*}
\tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') & \leq \tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') + \tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') \\
& \leq \tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') \Rightarrow \tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') = 0.
\end{align*}

By the definition of \(\eta\), which further induces that

\[
\eta \left( \tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') \right) = \lim_{n \to \infty} \eta \left( \tilde{d}(\tilde{c}(\tilde{ξ}', \tilde{ξ}'), ξ') \right) = 0.
\]

It follows from the fact \(||\tilde{d}|| < 1\) that \(\tilde{d}(\tilde{ξ}', \tilde{ξ}') = 0\), thus \(\tilde{ξ}' = \tilde{ξ}'\). Which means that \(S\) and \(f\) have a unique fixed point in \(\Theta\).

**Corollary III.1:** Let \((\tilde{θ}, \tilde{C}, \tilde{d})\) be a complete \(C\)-algebra valued fuzzy soft metric space. Suppose \(S: \Theta \times \Theta \to \Theta\) satisfies

\[
\eta \left( \tilde{d}(S(\psi, \phi), \tilde{c}(\psi, \phi)) \right) \leq \Gamma \left( \tilde{d}(S(\psi, \phi), \tilde{c}(\psi, \phi)) \right),
\]

for all \(\psi, \phi, \tilde{c}\) and \(\theta\) in \(\Theta\) and \(\Gamma\) in \(\tilde{C}\). Then \(S\) has a unique fixed point in \(\Theta\).

**Example III.1:** Let \(\Theta = \{\alpha_1, \alpha_2, \alpha_3\}, U = \{x, y, z, w\}\) and \(C\) and \(D\) are two subset of \(\Theta\) with \(C = \{\alpha_1, \alpha_2, \alpha_3\}, D = \{\alpha_1, \alpha_2, \alpha_3\}\). Define fuzzy soft set as,

\[
\begin{align*}
\langle \psi, C \rangle &= \left\{ \begin{array}{ll}
\alpha_1 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}, \\
\alpha_2 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}, \\
\alpha_3 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}.
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\langle \phi, D \rangle &= \left\{ \begin{array}{ll}
\alpha_1 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}, \\
\alpha_2 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}, \\
\alpha_3 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}.
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\tilde{c}_1 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}, \\
\tilde{c}_2 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}, \\
\tilde{c}_3 &= \{0.9, 0.7, 0.5, 0.3, 0.1\}.
\end{align*}
\]
and $FSC(\Phi_0) = \{\psi_0, \psi_2, \psi_3, \phi_0, \phi_2\}$, let $\Theta$ be a absolute fuzzy soft set, that is $\Theta(\alpha) = 1$ for all $\alpha \in \Theta$, and $\mathcal{C} = M_2(\mathcal{R}(\mathcal{C}))$, be a $C^*$-algebra.

Define $d_{c^*}(\psi_0, \psi_2)$ by $d_{c^*}(\psi_0, \psi_2) = \left\{ \inf \{\psi_0(x) - \psi_2(y) / x \in U \} \right\}$, then obviously $(\Theta, \mathcal{C}, d_{c^*})$ is a complete $C^*$-algebra valued fuzzy soft metric space. We define $S: \Theta \times \Theta \to \Theta$ by $S(\psi_0, \psi_2)(x) = \frac{\psi_0(x) + \psi_2(x)}{2^\alpha + 2^\beta + 3^\gamma}$, $\eta \to \Theta$ by $\eta \to \Theta$, $\eta \in C_+$ such that $\eta(\hat{\psi}) = \hat{\psi}$ and $\hat{\psi} = \hat{\psi}$ for all $\hat{\psi} \in C_+$. Theorem 3.11 and $\Gamma : C_+ \times C_+ \to C_+$ by $\Gamma(\hat{\psi}, \hat{\psi}) = \hat{\psi} - \hat{\psi}$ for all $\hat{\psi} \in C_+$. Then, obviously, $S(\Theta \times \Theta) \subseteq f(\Theta)$ and $\{S, f\}$ is $\omega$-compatible pair.

Observe that $d_{c^*}(\psi_0, \psi_2) = \frac{2^\alpha + 2^\beta + 3^\gamma}{5}$, $\eta \to \Theta$ by $\eta \to \Theta$, $\eta \in C_+$ such that $\eta(\hat{\psi}) = \hat{\psi}$ and $\hat{\psi} = \hat{\psi}$ for all $\hat{\psi} \in C_+$. Then, obviously, $S(\Theta \times \Theta) \subseteq f(\Theta)$ and $\{S, f\}$ is $\omega$-compatible pair.

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Letting \( \tilde{p} \) such that
\[
|\tilde{p}| = \frac{\{H_{\alpha}(w_{\alpha}, w_{\alpha})\}}{\{H_{\alpha}(w_{\alpha}, w_{\alpha})\}} + 1.
\]
Similarly, we can prove,
\[
||d_{c'}(\mathcal{H}(w_{\alpha}, w_{\alpha}, \alpha)\mathcal{H}(w_{\alpha}, w_{\alpha}, \alpha))|| \leq r + ||d_{c'}(w_{\alpha}, w_{\alpha})||.
\]
Thus for each fixed \( s_{\alpha} \in (s_{\alpha} - \epsilon, s_{\alpha} + \epsilon) \),
\[
\mathcal{H}_{\alpha}(s_{\alpha}) : B_{d_{c'}}(\mathcal{H}(w_{\alpha}, w_{\alpha}, \alpha)) \to B_{d_{c'}}(\mathcal{H}(w_{\alpha}, w_{\alpha}, \alpha)).
\]
Then all conditions of Theorem IV are satisfied. Thus \( \mathcal{H}_{\alpha}(s_{\alpha}) \) has a coupled fixed point in \( \Delta' \). But this must be in \( \Delta' \) since \( \tau_0 \) holds. Thus, \( s_{\alpha} \in \Theta \) for any \( s_{\alpha} \in (s_{\alpha} - \epsilon, s_{\alpha} + \epsilon) \). Hence \( (s_{\alpha} - \epsilon, s_{\alpha} + \epsilon) \subset \Theta \). Clearly \( \Theta \) is open in \( [0, 1] \). For the reverse implication, we use the same strategy.

V. CONCLUSION

This paper finishes various applications to homotopy theory via coupled fixed point theorems for \( C_* \)-class functions in the setting up of \( C^* \)-algebra valued fuzzy soft metric spaces.

Significance Statement

This study proposed a framework for establishing fixed point results in \( C^* \)-algebra valued fuzzy soft metric spaces using generalised contractions of \( C_* \)-class functions. The findings of this study will help to broaden the generalisation of various contractions in \( C^* \)-algebra valued fuzzy soft metric spaces and other metric spaces, facilitating their use in homotopy theory. As a result, a novel framework for fuzzy soft metric spaces with \( C^* \)-algebra values can be established.

REFERENCES


