On Certain Coupled Fixed Point Theorems Via C Star Class Functions in C^*-Algebra Valued Fuzzy Soft Metric Spaces With Applications

C.Ushabhavani, G.Upender Reddy, and B.Srinuvasa Rao

***Abstract*—The discussion of this paper is to aim to examine application of the notion of C^*-algebra valued fuzzy soft metric to homotopy theory using common coupled fixed point results from C_*-class functions. We also tried to provide an illustration of our major findings. The results attained expand upon and apply to many of the findings in the literature.**

***Index Terms*—C_*-class function, ω-compatible mapping, C^*-algebra valued fuzzy soft metric and coupled fixed points.**

I. Introduction

NUMEROUS real-world issues deal with ambiguous data and cannot be adequately described in classical mathematics. Fuzzy set theory, developed by Zadeh [1], and the theory of soft sets, developed by Molodstov [2], are two types of mathematical tools that can be used to deal with uncertainties and help with difficulties in a variety of fields. Thangaraj Beaula *et al.* defined fuzzy soft metric space in terms of fuzzy soft points in the cited work [3], and they supported various claims. However, numerous authors have established a great deal of findings regarding fuzzy soft sets and fuzzy soft metric spaces (see [4] -[6]).

A concept of C^*- algebra valued metric space was presented in 2006 by Ma *et al.* in [7], and certain fixed and coupled fixed point solutions for mapping under contraction conditions in these spaces were established. This line of inquiry was pursued in (see [8]-[14]). Recently, R.P.Agarwal *et al.* introduced the idea of C^*-algebra valued fuzzy soft metric spaces and demonstrated some associated fixed point solutions on this space (see. [15]-[19]).

The purpose of this article is to establish two pairs of ω-compatible mappings meeting generalised contractive requirements as unique common coupled fixed point theorems using C_*-class functions in the context of C^*-algebra valued fuzzy soft metric spaces. Additionally, we may provide pertinent examples and applications for homotopy.

II. Preliminaries

In this section, we review several fundamental notations and definitions.

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Definition II.1:([15]) Assume that $C \subseteq \Theta$ and $\tilde{\Theta}$ are the absolute fuzzy soft set and $\psi_\Theta(\alpha) = \tilde{1}$ for all $\alpha \in \Theta$, respectively. Let the C^*-algebra be represented by $\tilde{C}$. The mapping $\tilde{d_{c^*}}: \tilde{\Theta} \times \tilde{\Theta} \to \tilde{C}$ satisfying the given constraints is known as the C^*-algebra valued fuzzy soft metric utilising fuzzy soft points.

(i) $\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_2})$ for all $\psi_{\alpha_1}, \psi_{\alpha_2} \in \tilde{\Theta}$,

(ii) $\tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_2}) = \tilde{0}_{\tilde{C}} \Leftrightarrow \psi_{\alpha_1} = \psi_{\alpha_2}$,

(iii) $\tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_2}) = \tilde{d_{c^*}}(\psi_{\alpha_2}, \psi_{\alpha_1})$,

(iv) $\tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_3}) \preceq \tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_2}) + \tilde{d_{c^*}}(\psi_{\alpha_2}, \psi_{\alpha_3})$ $\forall\ \psi_{\alpha_1}, \psi_{\alpha_2}, \psi_{\alpha_3} \in \tilde{\Theta}$.

The C^*-algebra valued fuzzy soft metric space is made up of the fuzzy soft set $\tilde{\Theta}$ and the fuzzy soft metric $\tilde{d_{c^*}}$. It is represented by the symbol $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$.

Remark II.1: ([15]) It is clear that fuzzy soft metric spaces with C^*-algebra valued fuzzy soft metrics generalise the idea of fuzzy soft metric spaces by substituting the set of fuzzy soft real numbers with $\tilde{C}_+$. The idea of a fuzzy soft metric space with C^*-algebra values is similar to the definition of real metric spaces if we assume that $\tilde{C}_+ = \mathcal{R}$.

Example II.1:([15]) If C and Θ are subsets of $\mathcal{R}$, then $\tilde{\Theta}$ is an absolute fuzzy soft set, where $\tilde{\Theta}(\alpha) = \tilde{1}$ for every α in Θ, and $\tilde{C}$ is defined as $M_2(\mathcal{R}(C)^*)$.

Define $\tilde{d_{c^*}}: \tilde{\Theta} \times \tilde{\Theta} \to \tilde{C}$ by $\tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_2}) = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}$,

where $\kappa = \inf\{|\mu^a_{\psi_{\alpha_1}}(t) - \mu^a_{\psi_{\alpha_2}}(t)|/t \in C\}$ and $\psi_{\alpha_1}, \psi_{\alpha_2} \in \tilde{\Theta}$. Then, by the completeness of $\mathcal{R}(C)^*$, $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ is a complete C^* algebra valued fuzzy soft metric space and $\tilde{d_{c^*}}$ is a C^* - algebra valued fuzzy soft metric.

Definition II.2:([15]) Assume that $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ is a C^*-algebra valued fuzzy soft metric space. According to $\tilde{C}$ a sequence $\{\psi_{\alpha_k}\}$ in $\tilde{\Theta}$ is defined as:

(1) C^*-algebra valued fuzzy soft Cauchy sequence if, for each $\tilde{0}_{\tilde{C}} \prec \tilde{\epsilon}$, there exist $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\epsilon})$ such that $||\tilde{d_{c^*}}(\psi_{\alpha_k}, \psi_{\alpha_l})|| < \tilde{\delta}$ implies that $||\mu^a_{\psi_{\alpha_k}}(t) - \mu^a_{\psi_{\alpha_l}}(s)|| < \tilde{\epsilon}$ whenever $k, l \geq N$. That is $||\tilde{d_{c^*}}(\psi_{\alpha_k}, \psi_{\alpha_l})||_{\tilde{C}} \to \tilde{0}_{\tilde{C}}$ as $k, l \to \infty$.

(2) C^*-algebra valued fuzzy soft convergent to a point $\psi_{\alpha'} \in \tilde{\Theta}$ if, for each $\tilde{0}_{\tilde{C}} \prec \tilde{\epsilon}$, there exist $\tilde{0}_{\tilde{C}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\epsilon})$ such that $||\tilde{d_{c^*}}(\psi_{\alpha_k}, \psi_{\alpha'})|| < \tilde{\delta} \Rightarrow ||\mu^a_{\psi_{\alpha_k}}(t) - \mu^a_{\psi_{\alpha'}}(t)|| < \tilde{\epsilon}$ whenever $k \geq N$. It is usually denoted as $\lim_{k\to\infty} \psi_{\alpha_k} = \psi_{\alpha'}$.

(3) It is referred to as being complete when a C^*-algebra valued fuzzy soft metric space $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ is present. If each Cauchy sequence in $\tilde{\Theta}$ converges to a fuzzy soft point in $\tilde{\Theta}$.

Lemma II.1:([15]) Let $\tilde{C}$ be a C^*-algebra with the identity element $\tilde{I}_{\tilde{C}}$ and $\tilde{\theta}$ be a positive element of $\tilde{C}$. If $\tilde{\lambda} \in \tilde{C}$ is such that $||\tilde{\lambda}|| < 1$ then for $p < q$, we have

(a) $\lim_{q\to\infty} \sum_{k=p}^{q} (\tilde{\lambda}^*)^k \tilde{\theta}(\tilde{\lambda})^k = \tilde{I}_{\tilde{C}} ||(\tilde{\theta})^{\frac{1}{2}}||^2 \left(\frac{||\tilde{\lambda}||^p}{1-||\tilde{\lambda}||}\right)$.

(b) $\sum_{k=p}^{q} (\tilde{\lambda}^*)^k \tilde{\theta}(\tilde{\lambda})^k \to \tilde{0}_{\tilde{C}}$ as $q \to \infty$.

Definition II.2: ([15]) Suppose that $\tilde{C}$ is a unital C^*-algebra with unit $\tilde{1}$.

(i) If $\tilde{\kappa} \in \tilde{C_+}$ with $||\tilde{\kappa}|| < \frac{1}{2}$ then $\tilde{I} - \tilde{\kappa}$ is invertible and $||\tilde{\kappa}(\tilde{I} - \tilde{\kappa})^{-1}|| < 1$,

(ii) Suppose that $\tilde{\kappa}, \tilde{\lambda} \in \tilde{C}$ with $\tilde{\kappa}, \tilde{\lambda} \succeq \tilde{0}_{\tilde{C}}$ and $\tilde{\kappa}\tilde{\lambda} = \tilde{\lambda}\tilde{\kappa}$ then $\tilde{\kappa}\tilde{\lambda} \succeq \tilde{0}_{\tilde{C}}$,

(iii) Let $\tilde{C}' = \{\tilde{\kappa} \in \tilde{C} / \tilde{\kappa}\tilde{\lambda} = \tilde{\lambda}\tilde{\kappa} \ \forall \ \tilde{\lambda} \in \tilde{C}\}$. Let $\tilde{\kappa} \in \tilde{C}'$, if $\tilde{\lambda}, \tilde{\theta} \in \tilde{C}$ with $\tilde{\lambda} \succeq \tilde{\theta} \succeq \tilde{0}$ and $\tilde{I} - \tilde{\kappa} \in \tilde{C}'_+$ is an invertible operator, then $(\tilde{I} - \tilde{\kappa})^{-1}\tilde{\lambda} \succeq (\tilde{I} - \tilde{\kappa})^{-1}\tilde{\theta}$, where $\tilde{C_+}' = \tilde{C_+} \cap \tilde{C}'$

Notice that in c^*-algebra , if $\tilde{0} \preceq \tilde{\kappa}, \tilde{\lambda}$, one can't conclude that $\tilde{0} \preceq \tilde{\kappa}\tilde{\lambda}$. Indeed, consider the c^*-algebra $M_2(\mathcal{R}(C)^*)$ and set

$$\tilde{\kappa} = \begin{bmatrix} \psi_{\alpha_1}(a) & \psi_{\alpha_2}(a) \\ \psi_{\alpha_2}(a) & \psi_{\alpha_1}(b) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$$

and $$\tilde{\lambda} = \begin{bmatrix} \psi_{\alpha_1}(c) & \psi_{\alpha_2}(c) \\ \psi_{\alpha_2}(c) & \psi_{\alpha_1}(d) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.6 \end{bmatrix}$$

then clearly $\tilde{\kappa} \succeq \tilde{0}$ and $\tilde{\lambda} \succeq \tilde{0}$ but $\tilde{\kappa}, \tilde{\lambda} \in M_2(\mathcal{R}(C)^*)_+$ while $\tilde{\kappa}\tilde{\lambda}$ is not.

For more properties of a C^*-algebra valued fuzzy soft metric and C^* -algebra we refer the reader to ([15], [20]).

III. Main Results

For C_*-class functions in C^*-algebra valued fuzzy soft metric spaces, we will demonstrate various coupled fixed point theorems in this section.

Definition III.1: Let $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ be a C^*-algebra valued fuzzy soft metric space. Let $S : \tilde{\Theta} \times \tilde{\Theta} \to \tilde{\Theta}$ be a mapping. Then an element $(\psi_{\alpha_1}, \phi_{\alpha_1}) \in \tilde{\Theta} \times \tilde{\Theta}$ is called coupled fixed point of S if $S(\psi_{\alpha_1}, \phi_{\alpha_1}) = \psi_{\alpha_1}$ and $S(\phi_{\alpha_1}, \psi_{\alpha_1}) = \phi_{\alpha_1}$

Definition III.2: Let $\tilde{\Theta}$ be absolute fuzzy soft set and $S : \tilde{\Theta} \times \tilde{\Theta} \to \tilde{\Theta}$ and $f : \tilde{\Theta} \to \tilde{\Theta}$ be two mappings. An element $(\psi_{\alpha_1}, \phi_{\alpha_1}) \in \tilde{\Theta} \times \tilde{\Theta}$ is called

(i) a coupled coincidence point of S and f if $f\psi_{\alpha_1} = S(\psi_{\alpha_1}, \phi_{\alpha_1})$ and $f\phi_{\alpha_1} = S(\phi_{\alpha_1}, \psi_{\alpha_1})$

(ii) a common coupled fixed point of S and f if $\psi_{\alpha_1} = f\psi_{\alpha_1} = S(\psi_{\alpha_1}, \phi_{\alpha_1})$ and $\phi_{\alpha_1} = f\phi_{\alpha_1} = S(\phi_{\alpha_1}, \psi_{\alpha_1})$.

Definition III.3: Let $\tilde{\Theta}$ be absolute fuzzy soft set and $S : \tilde{\Theta} \times \tilde{\Theta} \to \tilde{\Theta}$ and $f : \tilde{\Theta} \to \tilde{\Theta}$. Then $\{S, f\}$ is said to be ω-compatible pairs if $f(S(\psi_{\alpha_1}, \phi_{\alpha_1})) = S(f\psi_{\alpha_1}, f\phi_{\alpha_1})$ and $f(S(\phi_{\alpha_1}, \psi_{\alpha_1})) = S(f\phi_{\alpha_1}, f\psi_{\alpha_1})$

Definition III.4: Let $\tilde{C}$ is a unital C^*-algebra. Then a continuous function $\Gamma : \tilde{C_+} \times \tilde{C_+} \to \tilde{C_+}$ is called a C_*-class function if for all $A, B \in \tilde{C_+}$,

(a) $\Gamma(\tilde{A}, \tilde{B}) \preceq \tilde{A}$;

(b) $\Gamma(\tilde{A}, \tilde{B}) = \tilde{A} \Rightarrow \tilde{A} = \tilde{0}_{\tilde{C}}$ or $\tilde{B} = \tilde{0}_{\tilde{C}}$.

We denote C_* as the family of all C_*-class functions.

Definition III.5: A function $\eta : \tilde{C_+} \to \tilde{C_+}$ is called an altering distance function if the following properties are satisfied:

(a) η is nondecreasing and continuous,

(b) $\eta(\tilde{A}) = \tilde{0}_{\tilde{C}}$ if and only if $\tilde{A} = \tilde{0}_{\tilde{C}}$.

The family of all altering distance functions is denoted by Ω.

Theorem III.1: Assume that C^*-algebra valued fuzzy soft metric space $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ and suppose two mappings $S: \tilde{\Theta} \times \tilde{\Theta} \to \tilde{\Theta}$ and $f: \tilde{\Theta} \to \tilde{\Theta}$ be satisfying

$$\eta\left(\tilde{d_{c^*}}\left(S(\psi_{\alpha_1}, \phi_{\alpha_1}), S(\psi_{\alpha_2}, \phi_{\alpha_2})\right)\right) \preceq \Gamma\left(\eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\psi_{\alpha_1}, f\psi_{\alpha_2})\tilde{\kappa}\right), \theta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\phi_{\alpha_1}, f\phi_{\alpha_2})\tilde{\kappa}\right)\right) \quad (1)$$

for all $\psi_{\alpha_1}, \psi_{\alpha_2}, \phi_{\alpha_1}, \phi_{\alpha_2} \in \tilde{\Theta}$, where $\tilde{\kappa} \in \tilde{C}$ with $||\tilde{\kappa}|| < 1$ and $\eta, \theta \in \Omega$ and $\Gamma \in C_*$.

(i) $S(\tilde{\Theta} \times \tilde{\Theta}) \subseteq f(\tilde{\Theta})$,

(ii) $\{S, f\}$ is ω-compatible pairs,

(iii) $f(\tilde{\Theta})$ is complete C^*-algebra valued fuzzy soft metrics of $\tilde{\Theta}$.

Then, in $\tilde{\Theta}$, S and f have a unique common coupled fixed point.

Proof: Let $\psi_{\alpha_0}, \phi_{\alpha_0} \in \tilde{\Theta}$. From (i) we can construct the sequences $\{\psi_{\alpha_n}\}_{n=1}^{\infty}, \{\varphi_{\alpha_n}\}_{n=1}^{\infty}, \{\xi_{\alpha_n}\}_{n=1}^{\infty}, \{\zeta_{\alpha_n}\}_{n=1}^{\infty}$ such that

$$S(\psi_{\alpha_n}, \phi_{\alpha_n}) = f\psi_{\alpha_{n+1}} = \xi_{\alpha_n}, S(\phi_{\alpha_n}, \psi_{\alpha_n}) = f\phi_{\alpha_{n+1}} = \zeta_{\alpha_n} \quad \text{for } n = 0, 1, 2, \ldots$$

Observes that in C^*-algebra, if $\tilde{\kappa}, \tilde{b} \in \tilde{C_+}$ and $\tilde{\kappa} \preceq \tilde{b}$, then for any $\tilde{x} \in \tilde{C_+}$ both $\tilde{x}^*\tilde{\kappa}\tilde{x}$ and $\tilde{x}^*\tilde{b}\tilde{x}$ are positive. We conveniently refer to the element $\tilde{d_{c^*}}(\xi_{\alpha_0}, \xi_{\alpha_1})$ in $\tilde{C}$ as Q. From (1), we get

$$\begin{aligned} & \eta\left(\tilde{d_{c^*}}\left(\xi_{\alpha_n}, \xi_{\alpha_{n+1}}\right)\right) \\ = \ & \eta\left(\tilde{d_{c^*}}\left(S(\psi_{\alpha_n}, \phi_{\alpha_n}), S(\psi_{\alpha_{n+1}}, \phi_{\alpha_{n+1}})\right)\right) \\ \preceq \ & \Gamma\left(\begin{array}{c} \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\psi_{\alpha_n}, f\psi_{\alpha_{n+1}})\tilde{\kappa}\right), \\ \theta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\phi_{\alpha_n}, f\phi_{\alpha_{n+1}})\tilde{\kappa}\right) \end{array}\right) \\ \preceq \ & \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\psi_{\alpha_n}, f\psi_{\alpha_{n+1}})\tilde{\kappa}\right) \\ \preceq \ & \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(\xi_{\alpha_{n-1}}, \xi_{\alpha_n})\tilde{\kappa}\right). \end{aligned}$$

By the definition of η, we have

$$\begin{aligned} \tilde{d_{c^*}}\left(\xi_{\alpha_n}, \xi_{\alpha_{n+1}}\right) & \preceq \tilde{\kappa}^* \tilde{d_{c^*}}(\xi_{\alpha_{n-1}}, \xi_{\alpha_n})\tilde{\kappa} \\ & \preceq (\tilde{\kappa}^*)^2 \tilde{d_{c^*}}(\xi_{\alpha_{n-2}}, \xi_{\alpha_{n-1}})\tilde{\kappa}^2 \\ & \preceq \cdots \\ & \preceq (\tilde{\kappa}^*)^n \tilde{d_{c^*}}(\xi_{\alpha_0}, \xi_{\alpha_1})\tilde{\kappa}^n \preceq (\tilde{\kappa}^*)^n Q \tilde{\kappa}^n. \end{aligned}$$

So for $n + 1 > m$

$$\begin{aligned} & \tilde{d_{c^*}}\left(\xi_{\alpha_{n+1}}, \xi_{\alpha_m}\right) \\ \preceq \ & \tilde{d_{c^*}}\left(\xi_{\alpha_{n+1}}, \xi_{\alpha_n}\right) + \tilde{d_{c^*}}\left(\xi_{\alpha_n}, \xi_{\alpha_{n-1}}\right) + \cdots \\ & + \tilde{d_{c^*}}\left(\xi_{\alpha_{m+1}}, \xi_{\alpha_m}\right) \\ \preceq \ & (\tilde{\kappa}^*)^n Q \tilde{\kappa}^n + (\tilde{\kappa}^*)^{n-1} Q \tilde{\kappa}^{n-1} + \cdots + (\tilde{\kappa}^*)^m Q \tilde{\kappa}^m \\ \preceq \ & \sum_{k=m}^{n} (\tilde{\kappa}^*)^k Q \tilde{\kappa}^k = \sum_{k=m}^{n} (\tilde{\kappa}^*)^k Q^{\frac{1}{2}} Q^{\frac{1}{2}} \tilde{\kappa}^k \\ \preceq \ & \sum_{k=m}^{n} (\tilde{\kappa}^k Q^{\frac{1}{2}})^* (Q^{\frac{1}{2}} \tilde{\kappa}^k) = \sum_{k=m}^{n} |Q^{\frac{1}{2}} \tilde{\kappa}^k|^2 \\ \preceq \ & \| \sum_{k=m}^{n} |Q^{\frac{1}{2}} \tilde{\kappa}^k|^2 \| \tilde{I}_{\tilde{C}} \preceq \sum_{k=m}^{n} \|Q^{\frac{1}{2}}\|^2 \|\tilde{\kappa}\|^{2k} \tilde{I}_{\tilde{C}} \\ \preceq \ & \|Q^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|\tilde{\kappa}\|^{2k} \tilde{I}_{\tilde{C}} \preceq \|Q\| \frac{||\tilde{\kappa}||^{2m}}{1 - ||\tilde{\kappa}||} \tilde{I}_{\tilde{C}} \end{aligned}$$

$\to \tilde{0}_{\tilde{C}}$ as $m \to \infty$.

As a result, $\{\xi_{\alpha_n}\}$ is a Cauchy sequence in $\tilde{\Theta}$ with regard to $\tilde{C}$. We can also demonstrate that $\{\zeta_{\alpha_n}\}$ is a Cauchy sequence with regard to $\tilde{C}$. Let's say $f(\tilde{\Theta})$ the complete subspace of $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$. Then the sequences $\{\xi_{\alpha_n}\}$ and $\{\zeta_{\alpha_n}\}$ are converge to $\xi_{\alpha'}, \zeta_{\alpha'}$ respectively in $f(\tilde{\Theta})$. Thus there exist $\psi_{\alpha'}, \phi_{\alpha'}$ in $f(\tilde{\Theta})$ Such that

$$\lim_{n\to\infty} \xi_{\alpha_n} = \xi_{\alpha'} = f\psi_{\alpha'} \text{ and } \lim_{n\to\infty} \zeta_{\alpha_n} = \zeta_{\alpha'} = f\phi_{\alpha'}. \quad (2)$$

Now we claim that $S(\psi_{\alpha'}, \phi_{\alpha'}) = \xi_{\alpha'}$ and $S(\phi_{\alpha'}, \xi_{\alpha'}) = \zeta_{\alpha'}$. From (1) and using the triangular inequality

$$\begin{aligned}&\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), \xi_{\alpha'})\\ &\preceq \tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), \xi_{\alpha_{n+1}}) + \tilde{d_{c^*}}(\xi_{\alpha_{n+1}}, \xi_{\alpha'})\\ &\preceq \tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), S(\psi_{\alpha_{n+1}}, \phi_{\alpha_{n+1}})) + \tilde{d_{c^*}}(\xi_{\alpha_{n+1}}, \xi_{\alpha'}).\end{aligned}$$

If we assume that the relation's limit is $n \to \infty$, we get

$$\begin{aligned}\tilde{0}_{\tilde{C}} &\preceq \tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), \xi_{\alpha'})\\ &\preceq \lim_{n\to\infty} \tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), S(\psi_{\alpha_{n+1}}, \phi_{\alpha_{n+1}})).\end{aligned}$$

By the definition of η, we have

$$\begin{aligned}&\eta\left(\tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), \xi_{\alpha'})\right)\\ \preceq\ &\lim_{n\to\infty} \eta\left(\tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), S(\psi_{\alpha_{n+1}}, \phi_{\alpha_{n+1}}))\right)\\ \preceq\ &\lim_{n\to\infty} \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\psi_{\alpha'}, f\psi_{\alpha_{n+1}})\tilde{\kappa}\right)\\ \preceq\ &\lim_{n\to\infty} \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\psi_{\alpha'}, \xi_{\alpha_n})\tilde{\kappa}\right) = \tilde{0}_{\tilde{C}}.\end{aligned}$$

Therefore, we have $\tilde{d_{c^*}}(S(\psi_{\alpha'}, \phi_{\alpha'}), \xi_{\alpha'}) = \tilde{0}_{\tilde{C}}$ implies that $S(\psi_{\alpha'}, \phi_{\alpha'}) = \xi_{\alpha'}$.
Similarly, we prove $S(\phi_{\alpha'}, \xi_{\alpha'}) = \zeta_{\alpha'}$.Therefore, it follows $S(\psi_{\alpha'}, \phi_{\alpha'}) = \xi_{\alpha'} = f\psi_{\alpha'}$ and $S(\phi_{\alpha'}, \psi_{\alpha'}) = \zeta_{\alpha'} = f\phi_{\alpha'}$.
Since $\{S, f\}$ is ω-compatible pair, we have
$S(\xi_{\alpha'}, \zeta_{\alpha'}) = f\xi_{\alpha'}$ and $S(\zeta_{\alpha'}, \xi_{\alpha'}) = f\zeta_{\alpha'}$.
Now to prove that $f\xi_{\alpha'} = \xi_{\alpha'}$ and $f\zeta_{\alpha'} = \zeta_{\alpha'}$. We have

$$\begin{aligned}&\tilde{0}_{\tilde{C}} \preceq \eta\left(\tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha_{n+1}})\right)\\ &\preceq \eta\left(\tilde{d_{c^*}}(S(\xi_{\alpha'}, \zeta_{\alpha'}), S(\psi_{\alpha_{n+1}}, \phi_{\alpha_{n+1}}))\right)\\ &\preceq \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, f\psi_{\alpha_{n+1}})\tilde{\kappa}\right)\\ &\preceq \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha_n})\tilde{\kappa}\right).\end{aligned}$$

By the definition of η and taking the limit as $n \to \infty$ in the above relation, we obtain

$$\tilde{0}_{\tilde{C}} \preceq \tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'}) \preceq \tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'})\tilde{\kappa}$$

we have

$$\begin{aligned}&0 \leq ||\tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'})|| \leq ||\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'})\tilde{\kappa}||\\ &\leq ||\tilde{\kappa}^*|| ||\tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'})|| ||\tilde{\kappa}||\\ &\leq ||\tilde{\kappa}||^2 ||\tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'})|| < ||\tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'})||.\end{aligned}$$

It is impossible. So $\tilde{d_{c^*}}(f\xi_{\alpha'}, \xi_{\alpha'}) = 0$ implies that
$f\xi_{\alpha'} = \xi_{\alpha'}$. Similarly, we show that $f\zeta_{\alpha'} = \zeta_{\alpha'}$. Therefore, $S(\xi_{\alpha'}, \zeta_{\alpha'}) = f\xi_{\alpha'} = \xi_{\alpha'}$ and $S(\zeta_{\alpha'}, \xi_{\alpha'}) = f\zeta_{\alpha'} = \zeta_{\alpha'}$.
Thus $(\xi_{\alpha'}, \zeta_{\alpha'})$ is common coupled fixed point of S and f. The following will demonstrate the distinctness of the common coupled fixed point in $\tilde{\Theta}$. Take into account that there is a second coupled fixed point $(\xi_{\alpha''}, \zeta_{\alpha''})$ for S and f. Then

$$\begin{aligned}&\eta\left(\tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''})\right)\\ &= \eta\left(\tilde{d_{c^*}}(S(\xi_{\alpha'}, \zeta_{\alpha'}), S(\xi_{\alpha''}, \zeta_{\alpha''}))\right)\\ &\preceq \Gamma\left(\eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, f\xi_{\alpha''})\tilde{\kappa}\right), \theta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\zeta_{\alpha'}, f\zeta_{\alpha''})\tilde{\kappa}\right)\right)\\ &\preceq \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, f\xi_{\alpha''})\tilde{\kappa}\right) \preceq \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''})\tilde{\kappa}\right).\end{aligned}$$

By the definition of η, which further induces that

$$\begin{aligned}||\tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''})|| &\leq ||\tilde{\kappa}^* \tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''})\tilde{\kappa}||\\ &\leq ||\tilde{\kappa}||^2 ||\tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''})||\\ &< ||\tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''})||.\end{aligned}$$

It is impossible. So $\tilde{d_{c^*}}(\xi_{\alpha'}, \xi_{\alpha''}) = 0$ implies $\xi_{\alpha'} = \xi_{\alpha''}$. Similarly, we show that
$\zeta_{\alpha'} = \zeta_{\alpha''}$ and hence $(\xi_{\alpha'}, \zeta_{\alpha'}) = (\xi_{\alpha''}, \zeta_{\alpha''})$ which means the coupled fixed point is unique. In order to prove that S and f have a unique fixed point, we only have to prove $\xi_{\alpha'} = \zeta_{\alpha'}$. We have

$$\begin{aligned}&\eta\left(\tilde{d_{c^*}}(\xi_{\alpha'}, \zeta_{\alpha'})\right)\\ &= \eta\left(\tilde{d_{c^*}}(S(\xi_{\alpha'}, \zeta_{\alpha'}), S(\zeta_{\alpha'}, \xi_{\alpha'}))\right)\\ &\preceq \Gamma\left(\eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, f\zeta_{\alpha'})\tilde{\kappa}\right), \theta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\zeta_{\alpha'}, f\xi_{\alpha'})\tilde{\kappa}\right)\right)\\ &\preceq \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(f\xi_{\alpha'}, f\zeta_{\alpha'})\tilde{\kappa}\right) \preceq \eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(\xi_{\alpha'}, \zeta_{\alpha'})\tilde{\kappa}\right).\end{aligned}$$

By the definition of η, which further induces that

$$||\tilde{d_{c^*}}(\xi_{\alpha'}, \zeta_{\alpha'})|| \leq ||\tilde{\kappa}^* \tilde{d_{c^*}}(\xi_{\alpha'}, \zeta_{\alpha'})\tilde{\kappa}|| \leq ||\tilde{\kappa}||^2 ||\tilde{d_{c^*}}(\xi_{\alpha'}, \zeta_{\alpha'})||.$$

It follows from the fact $||\tilde{\kappa}|| < 1$ that $||\tilde{d_{c^*}}(\xi_{\alpha'}, \zeta_{\alpha'})|| = 0$, thus $\xi_{\alpha'} = \zeta_{\alpha'}$. Which means that S and f have a unique fixed point in $\tilde{\Theta}$.

Corollary III.1: Let $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ be a complete C^*-algebra valued fuzzy soft metric space.Suppose $S\colon \tilde{\Theta} \times \tilde{\Theta} \to \tilde{\Theta}$ satisfies

$$\begin{aligned}&\eta\left(\tilde{d_{c^*}}\left(S(\psi_{\alpha_1}, \phi_{\alpha_1}), S(\psi_{\alpha_2}, \phi_{\alpha_2})\right)\right)\\ \preceq\ &\Gamma\left(\eta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(\psi_{\alpha_1}, \psi_{\alpha_2})\tilde{\kappa}\right), \theta\left(\tilde{\kappa}^* \tilde{d_{c^*}}(\phi_{\alpha_1}, \phi_{\alpha_2})\tilde{\kappa}\right)\right)\end{aligned} \quad (3)$$

for all $\psi_{\alpha_1}, \psi_{\alpha_2}, \phi_{\alpha_1}, \phi_{\alpha_2} \in \tilde{\Theta}$, where $\tilde{\kappa} \in \tilde{C}$ with $||\tilde{\kappa}|| < 1$ and $\eta, \theta \in \Omega$ and $\Gamma \in C_*$. Then S has a unique fixed point in $\tilde{\Theta}$.

Example III.1: Let $\Theta = \{\alpha_1, \alpha_2, \alpha_3\}, U = \{x, y, z, w\}$ and C and D are two subset of Θ where $C = \{\alpha_1, \alpha_2, \alpha_3\}$, $D = \{\alpha_1, \alpha_2, \}$. Define fuzzy soft set as,

$$(\psi_\Theta, C) = \left\{\begin{array}{c}\alpha_1 = \{x_{0.7}, y_{0.6}, z_{0.6}, w_{0.5}\},\\ \alpha_2 = \{x_{0.8}, y_{0.7}, z_{0.8}, w_{0.6}\},\\ \alpha_3 = \{x_{0.9}, y_{0.7}, z_{0.9}, w_{0.8}\}\end{array}\right\}$$

$$(\phi_\Theta, D) = \left\{\begin{array}{c}\alpha_1 = \{x_{0.5}, y_{0.6}, z_{0.5}, w_{0.3}\},\\ \alpha_2 = \{x_{0.7}, y_{0.7}, z_{0.8}, w_{0.5}\}\end{array}\right\}$$

$$\psi_{\alpha_1} = \mu_{\psi_{\alpha_1}} = \{x_{0.7}, y_{0.6}, z_{0.6}, w_{0.5}\}$$
$$\psi_{\alpha_2} = \mu_{\psi_{\alpha_2}} = \{x_{0.8}, y_{0.7}, z_{0.8}, w_{0.6}\}$$

$$\psi_{\alpha_3} = \mu_{\psi_{\alpha_3}} = \{x_{0.9}, y_{0.7}, z_{0.9}, w_{0.8}\}$$

$$\phi_{\alpha_1} = \mu_{\phi_{\alpha_1}} = \{x_{0.5}, y_{0.6}, z_{0.5}, w_{0.3}\}$$
$$\phi_{\alpha_2} = \mu_{\phi_{\alpha_2}} = \{x_{0.7}, y_{0.7}, z_{0.8}, w_{0.5}\}$$

and $FSC(F_\Theta)=\{\psi_{\alpha_1},\psi_{\alpha_2},\psi_{\alpha_3},\phi_{\alpha_1},\phi_{\alpha_2}\}$, let $\tilde{\Theta}$ be a absolute fuzzy soft set, that is $\tilde{\Theta}(\alpha)=\tilde{1}$ for all $\alpha\in\Theta$, and $\tilde{C}=M_2(\mathcal{R}(C)^*)$, be a C^*-algebra.
Define $\tilde{d}_{c^*}\colon\tilde{\Theta}\times\tilde{\Theta}\to\tilde{C}$ by
$\tilde{d}_{c^*}(\psi_{\alpha_1},\psi_{\alpha_2}) = \begin{pmatrix} \inf\{|\psi_{\alpha_1}(x)-\psi_{\alpha_2}(x)|/x\in U\} & 0 \end{pmatrix}$, then obviously $(\tilde{\Theta},\tilde{C},\tilde{d}_{c^*})$ is a complete C^*-algebra valued fuzzy soft metric space. We define $S\colon\tilde{\Theta}\times\tilde{\Theta}\to\tilde{\Theta}$ by $S(\psi_{\alpha_1},\phi_{\alpha_1})(x)=\frac{\psi_{\alpha_1}+2\phi_{\alpha_1}+3}{12}$, $f\colon\tilde{\Theta}\to\tilde{\Theta}$ by $f\psi_{\alpha_1}=\frac{\psi_{2\alpha_1}+1}{5}$ for all $x\in U$ and $\psi_{\alpha_1},\phi_{\alpha_1}\in\tilde{\Theta}$.
Let two continuous functions $\eta,\theta:\tilde{C_+}\to\tilde{C_+}$ as $\eta(\tilde{\kappa})=\tilde{\kappa}$ and $\theta(\tilde{\kappa})=\frac{\tilde{\kappa}}{5}$ for all $\tilde{\kappa}\in\tilde{C_+}$ and $\Gamma:\tilde{C_+}\times\tilde{C_+}\to\tilde{C_+}$ by $\Gamma(\tilde{\kappa},\tilde{b})=\tilde{\kappa}-\theta(\tilde{\kappa})$ for all $\tilde{\kappa},\tilde{b}\in\tilde{C_+}$. Then obviously, $S(\tilde{\Theta}\times\tilde{\Theta})\subseteq f(\tilde{\Theta})$ and $\{S,f\}$ is ω-compatible pair.
Observe that $f\psi_{\alpha_1}=\frac{2\psi_{\alpha_1}+1}{5}=\{0.48,0.44,0.44,0.4\}$ and $f\psi_{\alpha_2}=\frac{2\psi_{\alpha_2}+1}{5}=\{0.52,0.48,0.52,0.44\}$. Thus,

$$\begin{aligned}&\inf\{|\mu^x_{f\psi_{\alpha_1}}(s)-\mu^x_{f\psi_{\alpha_2}}(s)|/s\in C\}\\ =\ &\inf\{0.04,0.04,0.08,0.04\}=0.04.\end{aligned}$$

Therefore, $\tilde{d}_{c^*}(f\psi_{\alpha_1},f\psi_{\alpha_2})=\begin{bmatrix}0.04&0\\0&0.04\end{bmatrix}$
also, $f\phi_{\alpha_1}=\frac{2\phi_{\alpha_1}+1}{5}=\{0.4,0.44,0.4,0.32\}$ and $f\phi_{\alpha_2}=\frac{2\phi_{\alpha_2}+1}{5}=\{0.48,0.48,0.52,0.4\}$.
Thus, $\inf\{|\mu^x_{f\phi_{\alpha_1}}(s)-\mu^x_{f\phi_{\alpha_2}}(s)|/s\in C\}$
$=\inf\{0.08,0.04,0.12,0.08\}=0.04$ and
$\tilde{d}_{c^*}(f\phi_{\alpha_1},f\phi_{\alpha_2})=\begin{bmatrix}0.04&0\\0&0.04\end{bmatrix}$.
Moreover,
$S(\psi_{\alpha_1},\phi_{\alpha_1})(x)=\frac{\psi_{\alpha_1}+2\phi_{\alpha_1}+3}{12}=\{0.391,0.4,0.383,0.341\}$ and $S(\psi_{\alpha_2},\phi_{\alpha_2})(x)=\frac{\psi_{\alpha_2}+2\phi_{\alpha_2}+3}{12}$
$=\{0.433,0.425,0.45,0.383\}$.
Then

$$\begin{aligned}&\eta\left(\tilde{d}_{c^*}(S(\psi_{\alpha_1},\phi_{\alpha_1}),S(\psi_{\alpha_2},\phi_{\alpha_2}))\right)\\
&=\begin{bmatrix}0.025&0\\0&0.025\end{bmatrix}\\
&\preceq\tfrac{4}{5}\left(\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}\begin{bmatrix}0.04&0\\0&0.04\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}\right)\\
&\preceq\left(\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}\begin{bmatrix}0.04&0\\0&0.04\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}\right)\\
&\quad-\tfrac{1}{5}\left(\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}\begin{bmatrix}0.04&0\\0&0.04\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}\right)\\
&\preceq\Gamma\left(\eta\left(\tilde{\kappa}^*\tilde{d_{c^*}}(\psi_{\alpha_1},\psi_{\alpha_2})\tilde{\kappa}\right),\theta\left(\tilde{\kappa}^*\tilde{d_{c^*}}(\phi_{\alpha_1},\phi_{\alpha_2})\tilde{\kappa}\right)\right).\end{aligned}$$

Here $\tilde{\kappa}=\begin{bmatrix}\frac{2}{\sqrt5}&0\\0&\frac{2}{\sqrt5}\end{bmatrix}$ with $||\tilde{\kappa}||=\frac{2}{\sqrt5}<1$. Therefore, all the conditions of Theorem III.1 satisfied and $(\frac13,\frac13)$ is coupled fixed point of S and f.

IV. APPLICATION TO HOMOTOPY

In this part, we examine the possibility that homotopy theory has a unique solution.

Theorem IV.1: Let $(\tilde{\Theta},\tilde{C},\tilde{d_{c^*}})$ be complete C^*-algebra valued fuzzy soft metric space, Δ and $\overline{\Delta}$ be an open and closed subset of $\tilde{\Theta}$ such that $\Delta\subseteq\overline{\Delta}$. Suppose $\mathcal{H}:\overline{\Delta}^2\times[0,1]\to\tilde{\Theta}$ be an operator with following conditions are satisfying,
τ_0) $\wp_\alpha\neq\mathcal{H}(\wp_\alpha,\varpi_\alpha,s)$, $\varpi_\alpha\neq\mathcal{H}(\varpi_\alpha,\wp_\alpha,s)$, for each $\wp_\alpha,\varpi_\alpha\in\partial\Delta$ and $s\in[0,1]$;
τ_1) for all $\wp_\alpha,\varpi_\alpha,\imath_\alpha,\jmath_\alpha\in\overline{\Delta}$, $s\in[0,1]$ and $\eta,\theta\in\Omega$, $\Gamma\in C_*$ and $\tilde{\kappa}\in\tilde{C}$ with $||\tilde{\kappa}||<1$such that

$$\begin{aligned}&\eta\left(\tilde{d_{c^*}}\left(\mathcal{H}(\wp_\alpha,\varpi_\alpha,s),\mathcal{H}(\imath_\alpha,\jmath_\alpha,s)\right)\right)\\ \preceq\ &\Gamma\left(\eta\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_\alpha,\imath_\alpha)\tilde{\kappa}^*\right),\theta\left(\tilde{\kappa}\tilde{d_{c^*}}(\varpi_\alpha,\jmath_\alpha)\tilde{\kappa}^*\right)\right).\end{aligned}$$

τ_2) $\exists\ \tilde{M}\in\tilde{C_+}\ni$
$\tilde{d_{c^*}}(\mathcal{H}(\wp_\alpha,\varpi_\alpha,s),\mathcal{H}(\wp_\alpha,\varpi_\alpha,t))\preceq||\tilde{M}|||s-t|$
for every $\wp_\alpha,\varpi_\alpha\in\overline{\Delta}$and $s,t\in[0,1]$.
Then $\mathcal{H}(.,0)$ has a coupled fixed point $\iff$ $\mathcal{H}(.,1)$ has a coupled fixed point.

Proof: Let the set

$$\Theta=\left\{\begin{array}{c}s\in[0,1]:\mathcal{H}(\wp_\alpha,\varpi_\alpha,s)=\wp_\alpha,\\ \mathcal{H}(\varpi_\alpha,\wp_\alpha,s)=\varpi_\alpha,\\ \text{for some }\wp_\alpha,\varpi_\alpha\in\Delta\end{array}\right\}$$

Suppose that $\mathcal{H}(.,0)$ has a coupled fixed point in Δ^2, we have that $(0,0)\in\Theta^2$. So that Θ is non-empty set. Now we show that Θ is both closed and open in $[0,1]$ and hence by the connectedness $\Theta=[0,1]$. As a result, $\mathcal{H}(.,1)$ has a coupled fixed point in Δ^2. First we show that Θ closed in $[0,1]$. To see this, Let $\{s_{\alpha_p}\}_{p=1}^\infty\subseteq\Theta$ with $s_{\alpha_p}\to s_{\alpha'}\in[0,1]$ as $p\to\infty$. We must show that $s_{\alpha'}\in\Theta$. Since $s_{\alpha_p}\in\Theta$ for $p=0,1,2,3,\cdots$, there exists sequences $\{\wp_{\alpha_p}\},\{\varpi_{\alpha_p}\}$ with $\wp_{\alpha_p}=\mathcal{H}(\wp_{\alpha_p},\varpi_{\alpha_p},s_{\alpha_p})$, $\varpi_{\alpha_p}=\mathcal{H}(\varpi_{\alpha_p},\wp_{\alpha_p},s_{\alpha_p})$.
Consider

$$\begin{aligned}&\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})\\
=\ &\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_p},\varpi_{\alpha_p},s_{\alpha_p}),\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_{p+1}})\right)\\
\preceq\ &\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_p},\varpi_{\alpha_p},s_{\alpha_p}),\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_p})\right)\\
&+\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_p}),\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_{p+1}})\right)\\
\preceq\ &\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_p},\varpi_{\alpha_p},s_{\alpha_p}),\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_p})\right)\\
&+||\tilde{M}|||s_{\alpha_p}-s_{\alpha_{p+1}}|.\end{aligned}$$

Letting $p\to\infty$, we get

$$\begin{aligned}&\lim_{p\to\infty}\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})\\
\preceq\ &\lim_{p\to\infty}\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_p},\varpi_{\alpha_p},s_{\alpha_p}),\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_p})\right).\end{aligned}$$

Since η,θ are continuous and non-decreasing, we obtain

$$\begin{aligned}&\lim_{p\to\infty}\eta\left(\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})\right)\\
\preceq\ &\lim_{p\to\infty}\eta\left(\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_p},\varpi_{\alpha_p},s_{\alpha_p}),\mathcal{H}(\wp_{\alpha_{p+1}},\varpi_{\alpha_{p+1}},s_{\alpha_p})\right)\right)\\
\preceq\ &\lim_{p\to\infty}\Gamma\left(\begin{array}{c}\eta\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})\tilde{\kappa}^*\right),\\ \theta\left(\tilde{\kappa}\tilde{d_{c^*}}(\varpi_{\alpha_p},\varpi_{\alpha_{p+1}})\tilde{\kappa}^*\right)\end{array}\right)\\
\preceq\ &\lim_{p\to\infty}\eta\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})\tilde{\kappa}^*\right).\end{aligned}$$

By the definition of η, and $||\tilde{\kappa}||<1$ it follows that

$$\begin{aligned}\lim_{p\to\infty}||\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})||&\leq\lim_{p\to\infty}||\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})\tilde{\kappa}^*||\\
&\leq||\tilde{\kappa}||^2\lim_{p\to\infty}||\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})||.\end{aligned}$$

So that

$$\lim_{p\to\infty}\tilde{d_{c^*}}(\wp_{\alpha_p},\wp_{\alpha_{p+1}})=\tilde{0}_{\tilde{C}}.$$

Now for $q > p$, by use of triangular inequalilty , we have

$$\begin{aligned} & \tilde{d_{c^*}}\left(\wp_{\alpha_p}, \wp_{\alpha_q}\right) \\ \preceq\ & \tilde{d_{c^*}}\left(\wp_{\alpha_p}, \wp_{\alpha_{p+1}}\right) + \tilde{d_{c^*}}\left(\wp_{\alpha_{p+1}}, \wp_{\alpha_{p+2}}\right) \\ & + \tilde{d_{c^*}}\left(\wp_{\alpha_{p+2}}, \wp_{\alpha_{p+3}}\right) + \cdots + \tilde{d_{c^*}}\left(\wp_{\alpha_{q-2}}, \wp_{\alpha_{q-1}}\right) \\ & + \tilde{d_{c^*}}\left(\wp_{\alpha_{q-1}}, \wp_{\alpha_q}\right) \to 0 \text{ as } p, q \to \infty. \end{aligned}$$

Hence $\left\{\wp_{\alpha_p}\right\}$ is a Cauchy sequence in C^*-algebra valued fuzzy soft metric spaces $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$. Similarly we can show that $\left\{\varpi_{\alpha_p}\right\}$, is Cauchy sequence in $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$ and by the completeness of $(\tilde{\Theta}, \tilde{C}, \tilde{d_{c^*}})$, there exist $u_{\alpha'}, v_{\alpha'} \in \Theta$ with

$$\lim_{p\to\infty} \wp_{\alpha_{p+1}} = u_{\alpha'} \lim_{p\to\infty} \wp_{\alpha_p} \quad \lim_{p\to\infty} \varpi_{\alpha_{p+1}} = v_{\alpha'} = \lim_{p\to\infty} \varpi_{\alpha_p}$$

we have

$$\begin{aligned} & \eta\left(\tilde{d_{c^*}}\left(u_{\alpha'}, \mathcal{H}(u_{\alpha'}, v_{\alpha'}, s_{\alpha'})\right)\right) \\ =\ & \lim_{p\to\infty} \eta\left(\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha_p}, \varpi_{\alpha_p}, s_{\alpha'}), \mathcal{H}(u_{\alpha'}, v_{\alpha'}, s_{\alpha'})\right)\right) \\ \preceq\ & \lim_{n\to\infty} \eta\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha_p}, u_{\alpha'})\tilde{\kappa}^*\right) = 0. \end{aligned}$$

It follows that $\mathcal{H}(u_{\alpha'}, v_{\alpha'}, s_{\alpha'}) = u_{\alpha'}$. Similarly, we can prove $\mathcal{H}(v_{\alpha'}, u_{\alpha'}, s_{\alpha'}) = v_{\alpha'}$. Thus $s_{\alpha'} \in \Theta$. Hence Θ is closed in $[0, 1]$. Let $s_{\alpha_0} \in \Theta$, then there exist $\wp_{\alpha_0}, \varpi_{\alpha_0} \in \Delta$ with $\wp_{\alpha_0} = \mathcal{H}(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})$, $\varpi_{\alpha_0} = \mathcal{H}(\varpi_{\alpha_0}, \wp_{\alpha_0}, s_{\alpha_0})$. Since Δ is open, then there exist $\tilde{r} > 0$ such that $B_{d_{c^*}}(\wp_{\alpha_0}, \tilde{r}) \subseteq \Delta$. Choose $s_{\alpha'} \in (s_{\alpha_0} - \epsilon, s_{\alpha_0} + \epsilon)$ such that $|s_{\alpha'} - s_{\alpha_0}| \leq \frac{1}{||\tilde{M}^p||} < \frac{\epsilon}{2}$, then for

$$\begin{aligned} & \wp_{\alpha'} \in \overline{B_{\tilde{d_{c^*}}}(\wp_{\alpha_0}, \tilde{r})} \\ =\ & \left\{\wp_{\alpha'} \in \Theta / \tilde{d_{c^*}}(\wp_{\alpha'}, \wp_{\alpha_0}) \leq \tilde{r} + \tilde{d_{c^*}}(\wp_{\alpha_0}, \wp_{\alpha_0})\right\}. \end{aligned}$$

Now we have

$$\begin{aligned} & \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha'}), \wp_{\alpha_0}\right) \\ =\ & \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha'}), \mathcal{H}_b(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})\right) \\ \preceq\ & \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha'}), \mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha_0})\right) \\ & + \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha_0}), \mathcal{H}(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})\right) \\ \preceq\ & ||\tilde{M}|| |s_{\alpha'} - s_{\alpha_0}| \\ & + \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha_0}), \mathcal{H}(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})\right) \\ \preceq\ & \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha_0}), \mathcal{H}(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})\right) \\ & + \frac{1}{||\tilde{M}^{p-1}||}. \end{aligned}$$

Letting $p \to \infty$, we obtain

$$\begin{aligned} & \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha'}), \wp_{\alpha_0}\right) \\ \preceq\ & \tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha_0}), \mathcal{H}(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})\right). \end{aligned}$$

Since η, θ are continuous and non-decreasing, we obtain

$$\begin{aligned} & \eta\left(\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha'}), \wp_{\alpha_0}\right)\right) \\ \preceq\ & \eta\left(\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha_0}), \mathcal{H}(\wp_{\alpha_0}, \varpi_{\alpha_0}, s_{\alpha_0})\right)\right) \\ \preceq\ & \Gamma\left(\eta(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha'}, \wp_{\alpha_0})\tilde{\kappa}^*), \theta(\tilde{\kappa}\tilde{d_{c^*}}(\varpi_{\alpha'}, \varpi_{\alpha_0})\tilde{\kappa}^*)\right) \\ \preceq\ & \eta\left(\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha'}, \wp_{\alpha_0}))\tilde{\kappa}^*\right). \end{aligned}$$

Since η is non-decreasing, we have

$$\begin{aligned} ||\tilde{d_{c^*}}\left(\mathcal{H}(\wp_{\alpha'}, \varpi_{\alpha'}, s_{\alpha'}), \wp_{\alpha_0}\right)|| & \leq ||\tilde{\kappa}\tilde{d_{c^*}}(\wp_{\alpha'}, \wp_{\alpha_0}))\tilde{\kappa}^*|| \\ & \leq ||\tilde{\kappa}||^2 ||\tilde{d_{c^*}}(\wp_{\alpha'}, \wp_{\alpha_0}))|| \\ & \leq r + ||\tilde{d_{c^*}}(\wp_{\alpha_0}, \wp_{\alpha_0})||. \end{aligned}$$

Similarly, we can prove,

$$||\tilde{d_{c^*}}\left(\mathcal{H}(\varpi_{\alpha'}, \wp_{\alpha'}, s_{\alpha'}), \varpi_{\alpha_0}\right)|| \leq r + ||\tilde{d_{c^*}}(\varpi_{\alpha_0}, \varpi_{\alpha_0})||.$$

Thus for each fixed $s_{\alpha'} \in (s_{\alpha_0} - \epsilon, s_{\alpha_0} + \epsilon)$, $\mathcal{H}(., s_{\alpha'}) : \overline{B_{\tilde{d_{c^*}}}(\wp_{\alpha_0}, \tilde{r})} \to \overline{B_{\tilde{d_{c^*}}}(\wp_{\alpha_0}, \tilde{r})}$, $\mathcal{H}(., s_{\alpha'}) : \overline{B_{\tilde{d_{c^*}}}(\varpi_{\alpha_0}, \tilde{r})} \to \overline{B_{\tilde{d_{c^*}}}(\varpi_{\alpha_0}, \tilde{r})}$. Then all conditions of Theorem IV are satisfied. Thus we conclude that $\mathcal{H}(., s_{\alpha'})$ has a coupled fixed point in $\overline{\Delta}^2$. But this must be in Δ^2 since (τ_0) holds. Thus, $s_{\alpha'} \in \Theta$ for any $s_{\alpha'} \in (s_{\alpha_0} - \epsilon, s_{\alpha_0} + \epsilon)$. Hence $(s_{\alpha_0} - \epsilon, s_{\alpha_0} + \epsilon) \subseteq \Theta$. Clearly Θ is open in [0, 1]. For the reverse implication, we use the same strategy.

V. Conclusion

This paper finishes various applications to homotopy theory via coupled fixed point theorems for C_*-class functions in the setting up of C^*-algebra valued fuzzy soft metric spaces.

Significance Statement

This study proposed a framework for establishing fixed point results in C^*-algebra valued fuzzy soft metric spaces using generalised contractions of C^*-class functions. The findings of this study will help to broaden the generalisation of various contractions in C^*-algebra valued fuzzy soft metric spaces and other metric spaces, facilitating their use in Homotopy theory. As a result, a novel framework for fuzzy soft metric spaces with C^*-algebra values can be established.

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