On Generalized Matrix Mittag-Leffler Function

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***Abstract*—This paper aims first to recall the generalized Mittag-Leffler function and propose several properties of the generalized matrix Mittag-Leffler function. Afterward, we set a definition for a further extension of the generalized matrix Mittag-Leffler function and then show that this function is absolutely convergent under a certain condition.**

***Index Terms*—Mittag-Leffler function, Matrix Mittag-Leffler function.**

I. Introduction

THE generalized Mittag-Leffler function is an extension of the classical Mittag-Leffler function, which is a special function widely used in the field of fractional calculus. The Mittag-Leffler function, denoted as $E_a(z)$, is defined for complex number z and positive real number α as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1}$$

where $\Gamma(\cdot)$ denotes the gamma function. The Mittag-Leffler function arises in various areas of mathematics and physics, including fractional calculus, probability theory, and viscoelasticity [1]. It can be explored more in terms of different theoretical aspects like the results found in [2], [3].

The generalized version allows for matrices as arguments and provides a powerful tool for solving fractional differential and integral equations in matrix form [4]. The generalized matrix Mittag-Leffler function extends the concept to matrices. Given a matrix $A \in \mathbb{C}^{n\times n}$, the function $E_\alpha(A)$ is defined as

$$E_\alpha(A) = \sum_{n=0}^{\infty} \frac{A^n}{\Gamma(\alpha n + 1)}, \tag{2}$$

where $\alpha > 0$ and A^n denotes the matrix power in which the sum is taken over the powers of A. The generalized matrix Mittag-Leffler function has similar properties as its scalar counterpart such as holomorphicity, asymptotic behavior, and integral representations. This function has found applications in various fields, including fractional calculus, fractional differential equations, and mathematical physics. It provides a valuable tool for solving and analyzing systems involving fractional operators or fractional derivatives. In summary, the generalized matrix Mittag-Leffler function is an extension of the classical Mittag-Leffler function to matrices. It has numerous applications in fractional calculus and related areas, offering a powerful tool for solving fractional differential and integral equations in matrix form.

Additionally, its application extends to fractional integral equations, aiding in the examination of integral operators with memory effects and contributing to a deeper understanding of solution properties. In the field of viscoelasticity, the function is employed to model materials with time-dependent behavior and memory effects, assisting in predicting mechanical responses. Furthermore, it finds utility in signal processing for analyzing signals with fractional dynamics, leading to the development of advanced techniques considering long-term memory and non-local dependencies. In control theory, the generalized matrix Mittag-Leffler function is applied to design and analyze control systems involving fractional operators, enabling the development of control strategies that account for memory effects and enhance system performance. Its use also extends to fractional integral equations, where it facilitates the analysis of integral operators with memory effects and adds to a better knowledge of solution properties. The function is used to represent materials having time-dependent behavior and memory effects in the field of viscoelasticity, which helps predict mechanical responses [5], [6]. Moreover, it is useful in signal processing to analyze signals with fractional dynamics, which stimulates the creation of sophisticated methods taking non-local dependencies and long-term memory into account. Control systems containing fractional operators can be designed and analyzed using the generalized matrix Mittag-Leffler function in control theory [7], [8]. This allows control techniques to be developed that take memory effects into account and improve system performance. These are just a few examples of the wide range of applications for the generalized matrix Mittag-Leffler function. Its versatility and usefulness make it a valuable tool in various fields of mathematics, physics, and engineering [9], [10].

II. On generalized Mittag-Leffler function

The generalized Mittag-Leffler function for two parameters, denoted as $E_{\alpha,\beta}(z)$, is a special function that arises in the theory of fractional calculus and has various applications in mathematical physics, engineering, and other scientific fields. It is an extension of the classical Mittag-Leffler function, which corresponds to the case when $\alpha = \beta = 1$. The generalized Mittag-Leffler function is defined by the following power series representation:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0). \tag{3}$$

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Note that whenever $\beta = 1$, the one parameter of the Mittag-Leffler function will be yielded. In fact, $E_\alpha(z)$ was introduced by Mittag-Leffler [11] and $E_{\alpha,\beta}(z)$ was introduced by Wiman [12]. The main results in the classical theory of these functions can be found in the handbook by Erdelyi [13], and more results are given in the books by Dzherbashyan [14]. In the following content, we will recall an extension of the Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$, which has the form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!}, \tag{4}$$

where $z, \alpha, \beta, \gamma \in \mathbb{C}$ such that $Re(\alpha) > 0, Re(\beta) > 0$, and where $(\gamma)_k$ is the Pchhammer symbol, which is defined as

$$\begin{aligned}(\gamma)_r &= \frac{\Gamma(\gamma + r)}{\Gamma(\gamma)} \\ &= \begin{cases} 1, & r = 0, \gamma \in \mathbb{C}\backslash\{0\} \\ \gamma(\gamma+1)\cdots(\gamma+k-1), & r = k \in \mathbb{N}, \gamma \in \mathbb{C} \end{cases},\end{aligned} \tag{5}$$

where $\mathbb{N}$ is the set of natural numbers. Herein, by taking $\gamma = \beta = 1$, the one parameter of the Mittag-Leffler function will be also yielded. A further extension of the matrix Mittag-Leffler function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ can be given in the following form:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)(\delta)_k}, \tag{6}$$

where $z, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $Re(\alpha) > 0, Re(\beta) > 0$, and where $(\gamma)_k$ is as defined above in (5), and $(\delta)_k$ is defined as

$$\begin{aligned}(\delta)_r &= \frac{\Gamma(\delta + r)}{\Gamma(\delta)} \\ &= \begin{cases} 1, & r = 0, \delta \in \mathbb{C}\backslash\{0\} \\ \delta(\delta+1)\cdots(\delta+k-1), & r = k \in \mathbb{N}, \delta \in \mathbb{C} \end{cases}.\end{aligned} \tag{7}$$

Remark 1: Based on the previous discussion, the following special cases can be immediately yielded:

- If $\delta = 1$, we have
$$E^{\gamma,1}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)(1)_k} = E^{\gamma}_{\alpha,\beta}(z). \tag{8}$$
- If $\gamma = \delta = 1$, we have
$$E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z). \tag{9}$$
- If $\gamma = \delta = \beta = 1$, we have
$$E^{1,1}_{\alpha,1}(z) = E_{\alpha}(z). \tag{10}$$

III. On generalized matrix Mittag-Leffler function

In this section, we first recall the definition of the generalized matrix Mittag-Leffler function, and then we establish some further properties of its construction.

Definition 1: Let $A \in \mathbb{C}^{n\times n}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $Re(\alpha) > 0$ and $Re(\beta) > 0$. Then the generalized matrix Mittag-Leffler function is defined by

$$E^{\gamma,\delta}_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{(\gamma)_k A^k}{\Gamma(\alpha k + \beta)(\delta)_k}, \tag{11}$$

where $(\gamma)_k$ and $(\delta)_k$ are previously defined in (5) and (7) respectively.

Corollary 1: In light of Definition 1, the following properties of the generalized matrix Mittag-Leffler function are held:

1) If $\alpha = \beta = \gamma = \delta = 1$, we have
$$E^{1,1}_{1,1}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = e^A. \tag{12}$$
2) If $\alpha = 0$ and $\beta = \gamma = \delta = 1$, we have
$$E^{1,1}_{0,1}(A) = (I - A)^{-1}. \tag{13}$$
3) If $\alpha = \gamma = \delta = 1$ and $\beta = 2$, we have
$$E^{1,1}_{1,2}(A) = A^{-1}(e^A - I). \tag{14}$$
4) If $\alpha = 2$ and $\beta = \gamma = \delta = 1$, then we have
$$E^{1,1}_{2,1}(A^2) = cosh(A).$$
5) If $\alpha = 2$and $\beta = \gamma = \delta = 1$, then we have
$$E^{1,1}_{2,1}((-A)^2) = cos(A).$$
6) If $\alpha = \beta = 2$ and $\gamma = \delta = 1$, then we have
$$E^{1,1}_{2,2}(A^2) = A^{-1}(e^A - e^{-A}).$$
7) If $\alpha = \beta = 2$ and $\gamma = \delta = 1$, then we have
$$E^{1,1}_{2,2}(-A^2) = A^{-1} sin(A).$$
8) If $\beta = \gamma = \delta = 1$, then we have
$$E^{1,1}_{\alpha,1}(At) = \frac{A}{\alpha} E^{1,1}_{\alpha,\alpha}(At), \; t \in \mathbb{R}.$$

Proof: We prove some parts of Corollary 1 for completeness, as the proofs of the remaining parts are trivial.
2) It should be noted that if one takes the given assumption into account, we get

$$E^{1,1}_{0,1}(A) = \sum_{k=0}^{\infty} A^k = I + A + A^2 + A^3 + \cdots . \tag{15}$$

Now, let us assume

$$B = I + A + A^2 + A^3 + \cdots . \tag{16}$$

Then by multiplying, from the left, equation (16) by A, we obtain

$$AB = A + A^2 + A^3 + A^4 + \cdots . \tag{17}$$

Subtracting (17) from (16) yields

$$B - AB = I,$$

or

$$B = (I - A)^{-1}, \tag{18}$$

which finishes the proof.
3) Suppose $\alpha = \gamma = \delta = 1$ and $\beta = 2$. Then we get

$$E^{1,1}_{1,2}(A) = I + \frac{A}{2!} + \frac{A^2}{3!} + \frac{A^3}{4!} + \cdots = e^A. \tag{19}$$

Now, multiplying, from the left, equation (19) by A^{-1} yields

$$\begin{aligned} A^{-1}e^A &= A^{-1} + I + \frac{A}{2!} + \frac{A^2}{3!} + \frac{A^3}{4!} + \cdots \\ &= A^{-1} + E^{1,1}_{1,2}(A), \end{aligned} \tag{20}$$

which is equivalent to the desired result.
6) To prove this result, one might notice

$$E_{2,2}^{1,1}(A^2) = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k+1)!} = I + \frac{A^2}{3!} + \frac{A^4}{5!} + \frac{A^6}{7!} + \frac{A^8}{9!} + \cdots . \tag{21}$$

But, we have

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots , \tag{22}$$

and

$$e^{-A} = I - \frac{A}{1!} + \frac{A^2}{2!} - \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots . \tag{23}$$

Subtracting (23) from (22) yields

$$e^A - e^{-A} = 2A + 2\frac{A^3}{3!} + 2\frac{A^5}{5!} + \cdots . \tag{24}$$

Thus, multiplying (24), from the left, by A^{-1} gives consequently the desired result.
8) Herein, we have

$$E_{\alpha,1}^{1,1}(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k+1)}. \tag{25}$$

By taking the derivative to the both sides of the above equality, we get

$$\frac{d}{dt}E_{\alpha,1}^{1,1}(At) = \sum_{k=0}^{\infty} \frac{(k+1)A^{k+1}t^k}{\Gamma(\alpha(k+1)+1)}, \tag{26}$$

or

$$\frac{d}{dt}E_{\alpha,1}^{1,1}(At) = \frac{A}{\alpha}\sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k+\alpha)}, \tag{27}$$

which implies the result. ■

Corollary 2: In light of Definition 1, the following properties of the generalized matrix Mittag-Leffler function are held:

1) If $AB = BA$, then we have
$$E_{\alpha,\beta}^{\gamma,\delta}(A)B = BE_{\alpha,\beta}^{\gamma,\delta}(A).$$
2) For $A \in \mathbb{C}^{n\times n}$, we have
$$AE_{\alpha,\beta}^{\gamma,\delta}(A) = E_{\alpha,\beta}^{\gamma,\delta}(A)A. \tag{28}$$
3) For $A \in \mathbb{C}^{n\times n}$, we have
$$E_{\alpha,\beta}^{\gamma,\delta}(A)^T = \left(E_{\alpha,\beta}^{\gamma,\delta}(A)\right)^T. \tag{29}$$
4) For $A \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{Z}$, we have
$$A^m E_{\alpha,\beta}^{\gamma,\delta}(A) = E_{\alpha,\beta}^{\gamma,\delta}(A)A^m. \tag{30}$$
5) If A and B are two nilpotent matrices with index 2 such that $AB = BA = 0$. Then we have
$$E_{\alpha,1}^{\gamma,\delta}(A+B) = E_{\alpha,1}^{\gamma,\delta}(A) + E_{\alpha,1}^{\gamma,\delta}(B). \tag{31}$$

Proof: In what follows, we also prove some parts of Corollary 2 for completeness.

3) We have

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,\delta}(A)^T &= \sum_{k=0}^{\infty} \frac{(\gamma)_k (A^T)^k}{\Gamma(\alpha k+\beta)(\delta)_k} \\ &= \sum_{k=0}^{\infty} \frac{(\gamma)_k (A^k)^T}{\Gamma(\alpha k+\beta)(\delta)_k} \\ &= \left(\sum_{k=0}^{\infty} \frac{(\gamma)_k A^k}{\Gamma(\alpha k+\beta)(\delta)_k}\right)^T = \left(E_{\alpha,\beta}^{\gamma,\delta}(A)\right)^T. \end{aligned} \tag{32}$$

5) To prove this result, we should note that

$$\begin{aligned} E_{\alpha,1}^{\gamma,\delta}(A+B) &= \sum_{k=0}^{\infty} \frac{(\gamma)_k (A+B)^k}{\Gamma(\alpha k+1)(\delta)_A} \\ &= I + \frac{(\gamma)_1(A+B)}{\Gamma(\alpha+1)(\delta)_1} + \frac{(\gamma)_2(A+B)^2}{\Gamma(2\alpha+1)(\delta)_2} \\ &+ \frac{(\gamma)_3(A+B)^3}{\Gamma(3\alpha+1)(\delta)_3} + \cdots . \end{aligned} \tag{33}$$

Due to $AB = BA = 0$, we get

$$\begin{aligned} E_{\alpha,1}^{\gamma,\delta}(A+B) &= I + \frac{(\gamma)_1(A+B)}{\Gamma(\alpha+1)(\delta)_1} + \frac{(\gamma)_2(A^2+B^2)}{\Gamma(2\alpha+1)(\delta)_2} \\ &+ \frac{(\gamma)_3(A^3+B^3)}{\Gamma(3\alpha+1)(\delta)_3} + \cdots . \end{aligned} \tag{34}$$

Also, since $A^2 = B^2 = 0$, we obtain

$$E_{\alpha,1}^{\gamma,\delta}(A+B) = I + \frac{(\gamma)_1(A+B)}{\Gamma(\alpha+1)(\delta)_1}. \tag{35}$$

On the other hand, if one takes the right-hand side of (31), we obtain

$$\begin{aligned} E_{\alpha,1}^{\gamma,\delta}(A) + E_{\alpha,1}^{\gamma,\delta}(B) &= \left(\sum_{k=0}^{\infty} \frac{(\gamma)_k A^k}{\Gamma(\alpha k+1)(\delta)_k}\right) \\ &+ \left(\sum_{k=0}^{\infty} \frac{(\gamma)_k B^k}{\Gamma(\alpha k+1)(\delta)_k}\right), \end{aligned} \tag{36}$$

or

$$\begin{aligned} E_{\alpha,1}^{\gamma,\delta}(A) + E_{\alpha,1}^{\gamma,\delta}(B) &= \sum_{k=0}^{\infty} \frac{(\gamma)_k (A^k+B^k)}{\Gamma(\alpha k+1)(\delta)_k} \\ &= I + \frac{(\gamma)_1(A+B)}{\Gamma(\alpha+1)(\delta)_1} \\ &+ \frac{(\gamma)_2(A^2+B^2)}{\Gamma(2\alpha+1)(\delta)_2} \\ &+ \frac{(\gamma)_3(A^3+B^3)}{\Gamma(3\alpha+1)(\delta)_3} + \cdots . \end{aligned} \tag{37}$$

This consequently means

$$E_{\alpha,1}^{\gamma,\delta}(A) + E_{\alpha,1}^{\gamma,\delta}(B) = I + \frac{(\gamma)_1(A+B)}{\Gamma(\alpha+1)(\delta)_1}. \tag{38}$$

Thus, comparing (35) with (38) gives immediately the desired result. ■

IV. A FURTHER GENERALIZATION OF A MATRIX MITTAG-LEFFLER FUNCTION

In this part, we aim to set a new definition for a further extension of the generalized matrix Mittag-Leffler function and then show that this function is absolutely convergent under a certain condition.

Definition 2: Let $A, B \in \mathbb{C}^{n\times n}$ and z, α, β, γ be complex values with $min\{Re(\alpha), Re(\beta), Re(B)\} > 0$. Then a further extension of the generalized matrix Mittag-Leffler function can be defined as

$$E_{\alpha,\beta}^{A,B}(z) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k^{-1} z^k}{\Gamma(\alpha k+\beta)}, \tag{39}$$

where

$$\Gamma(A) = \int_0^\infty e^{-t} t^{A-1} dt,$$

$$\Gamma^{-1}(A) = A(A+I)\cdots(A+(k-1)I)\Gamma^{-1}(A+kI),$$

for $k \geq 1$, and

$$(A)_k = \begin{cases} I, & k=0. \\ A(A+I)\cdots(A+(k-1)I), & k\geq 1, \end{cases} \tag{40}$$

or

$$(A)_k = \Gamma^{-1}(A)\Gamma(A+kI), \ k \geq 1. \tag{41}$$

Theorem 1: The extension version of the generalized matrix Mittag-Leffler function $E_{\alpha,\beta}^{A,B}(z)$ is absolutely convergent for any $z \in \mathbb{C}$.

Proof: In order to prove this result, we rewrite $E_{\alpha,\beta}^{A,B}(z)$ again as

$$E_{\alpha,\beta}^{A,B}(z) = \sum_{k=0}^{\infty} M_k z^k, \tag{42}$$

where

$$M_k = \frac{(A)_k (B)_k^{-1}}{\Gamma(\alpha k+\beta)}. \tag{43}$$

Now, by applying the convergence ratio test, we obtain

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{M_{k+1}}{M_k}\right| &\leq \frac{||\Gamma^{-1}(A)||\,||\Gamma(A+kI+I)||}{||\Gamma^{-1}(A)||\,||\Gamma(A+kI)||} \\ &\frac{||\Gamma^{-1}(B+kI+I)||\,||\Gamma(B)||}{||\Gamma^{-1}(A)||\,||\Gamma(A+kI)||} \\ &\frac{||\Gamma(\alpha k+\beta)||}{||\Gamma^{-1}(B+kI)||\,||\Gamma(B)||\,\Gamma(\alpha k+\alpha+\beta)} \\ &= \frac{||\Gamma(A+kI+I)||\,||\Gamma^{-1}(B+kI+I)||}{||\Gamma(A+kI)||\,||\Gamma^{-1}(B+kI)||} \\ &\frac{\Gamma(\alpha k+\beta)}{||\Gamma(\alpha k+\alpha+\beta)||}\end{aligned}$$

This implies

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{M_{k+1}}{M_k}\right| &\leq \frac{||(A+kI)||\,||\Gamma(A+kI)||}{||\Gamma(A+kI)||\,||\Gamma^{-1}(B+kI)||} \\ &\frac{||((B+kI)\Gamma(B+kI))^{-1}||\,\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\alpha+\beta)} \\ &= \frac{||(A+kI)||\,||\Gamma^{-1}(B+kI)||}{||\Gamma^{-1}(B+kI)||} \\ &\frac{||(B+kI)^{-1}||\,\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\alpha+\beta)} \\ &= \frac{||(A+kI)||\,||(B+kI)^{-1}||\,\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\alpha+\beta)}.\end{aligned}$$

Therefore, $\left|\frac{M_{k+1}}{M_k}\right| \to 0$ as $k \to \infty$, which yields the convergence of $E_{\alpha,\beta}^{A,B}(z)$ for any $z \in \mathbb{C}$. ■

In the following content, we list some straightforward special cases of the extended version of the generalized matrix Mittag-Leffler function $E_{\alpha,\beta}^{A,B}(z)$ defined previously in (2).

Remark 2: For $A \in \mathbb{C}^{n\times n}$, the following properties are hold for $E_{\alpha,\beta}^{A,B}(z)$:

1) If $\alpha = \beta = 0$, then we have
$$E_{0,0}^{A,A}(z) = \left(\frac{1}{(1-z)} I\right); \ |z| < 1. \tag{44}$$
2) If $\alpha = \beta = 1$, then we have
$$E_{1,1}^{A,A}(z) = \left(\frac{z^k}{k!}\right) = e^A. \tag{45}$$
3) If $\alpha = 1$ and $\beta = 2$, then we have
$$E_{1,2}^{A,A}(z) = \left(\frac{e^z-1}{z}\right). \tag{46}$$
4) If $\alpha = 2$ and $\beta = 1$, then we have
$$E_{2,1}^{A,A}(z^2) = cosh(z). \tag{47}$$
5) If $\alpha = \beta = 2$, then we have
$$E_{2,2}^{A,A}(-z^2) = cos(z). \tag{48}$$
6) If $\alpha = 2$ and $\beta = 1$, then we have
$$E_{2,2}^{A,A}(-z^2) = cos(A). \tag{49}$$
7) If $\alpha = \beta = 2$, then we have
$$E_{2,2}^{A,A}(z^2) = \frac{sinh(z)}{z}. \tag{50}$$
8) If $\alpha = \beta = 2$, then we have
$$E_{2,2}^{A,A}(-z^2) = \frac{sin(z)}{z}. \tag{51}$$

V. Conclusion

In this article, we have reviewed the generalized matrix Mittag-Leffler function and proposed a further number of its properties. Consequently, we have also set a definition for an extended version of the generalized matrix Mittag-Leffler function and hence demonstrated that this function is absolutely convergent under a specific circumstance.

References

[1] A. Sadeghi and J.R. Cardoso, "Some notes on properties of the matrix Mittag-Leffler function," Applied Mathematics and Computation, vol. 338, no. 1, pp733-738, 2018

[2] I.M. Batiha, L. Ben Aoua, T.E. Oussaeif, A. Ouannas, S. Alshorman, I.H. Jebril and S. Momani, "Common fixed point theorem in non-Archimedean menger PM-spaces using CLR property with application to functional equations," IAENG International Journal of Applied Mathematics, vol. 53, no. 1, pp360-368, 2023

[3] I.M. Batiha, Z. Chebana, T.E. Oussaeif, A. Ouannas, I.H. Jebril and M. Shatnawi, "Solvability of nonlinear wave equation with nonlinear integral Neumann conditions," International Journal of Analysis and Applications, vol. 21, Article ID 34, 2023

[4] R. Garrappa and M. Popolizio, "Computing the matrix Mittag-Leffler function with applications to fractional calculus," Journal of Scientific Computing, vol. 77, no. 1, pp129-153, 2018

[5] I.M. Batiha, A. Bataihah, A.A. Al-Nana, S. Alshorm, I.H. Jebril and A. Zraiqat, "A numerical scheme for dealing with fractional initial value problem," International Journal of Innovative Computing, Information and Control, vol. 19, no. 3, pp763-774, 2023

[6] I.M. Batiha, S. Alshorm, I.H. Jebril and M. Abu Hammad, "A brief review about fractional calculus," International Journal of Open Problems in Computer Science and Mathematics, vol. 15, no. 4, pp39-56, 2022

[7] T. Hamadneh, A. Abbes, I. Abu Falahah, Y.A. AL-Khassawneh, A.S. Heilat, A. Al-Husban and A. Ouannas, "Complexity and chaos analysis for two-dimensional discrete-time predator–prey Leslie–Gower model with fractional orders," Axioms, vol. 12, no. 6, Article ID 561, 2023

[8] S. Doley, A.V. Kumar, K.R. Singh and L. Jino, "Study of time fractional Burgers' equation using Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives," Engineering Letters, vol. 30, no. 3, pp1017-1024, 2022

[9] M. Popolizio, "On the matrix Mittag–Leffler function: theoretical properties and numerical computation," Mathematics, vol. 7, no. 12, Article ID 1140, 2019

[10] I. Matychyn and V. Onyshchenko, "Matrix Mittag-Leffler function in fractional systems and its computation," Bulletin of the Polish Academy of Sciences Technical Sciences, vol. 66, no. 4, pp495-500, 2018

[11] G. Mittag-Leffler, "Sur la nouvelle fonction $E_\alpha(x)$," Comptes Rendus de l'Academie des Sciences Paris, vol. 137, no. 1, pp554-558, 1903

[12] A. Wiman, "Über den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$," Acta Mathematica, vol. 29, no. 1, pp191-201, 1905

[13] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, "Higher Transcendental Functions, Vol. III," New York, USA: McGraw-Hill, 1955

[14] H.M. Srivastava, K.C. Gupta and S.P. Goyal, "The H-Functions of One and Two Variables, with Applications," Andhra Pradesh, India: South Asian Publishers, 1982