Stochastic Vendor-buyer Inventory System with Defective Items

Shusheng Wu, Jinyuan Liu, Pin-Yi Lo

Abstract—With a rational criterion, we developed an analytical method that proves the uniqueness for the optimal solution for stochastic vendor-buyer inventory model with defective items. Our method approach implies a rigorous approach to locate the optimal solution. The doubtful results that derived by the iterative method are demonstrated by the numerical example of the published paper is examined to indicate the virtue of our approach. Our findings offer an analytical basis for stochastic vendor-buyer inventory models with defective items that will facilitate decision-makers to realize the solution mechanism to find the minimum average cost.

Index Terms—Integrated vendor-buyer inventory model, Minimax distribution free approach, Inventory systems, Analytical solution procedure

I. INTRODUCTION

mentioned five papers are too eager to develop new inventory models and then they did not provide a deeply examination for the solution procedure of Wu and Ouyang [27]. We will follow this research tendency to develop our analytical solution approach to find the optimal solution. Our results will help practitioners to construct new models to meet the challenge in the real world application.

II. Assumptions and Notation

To explain our method, for examples, we consider the integrated vendor-buyer inventory model of Wu and Ouyang [27], so that we use the following assumptions and notation.

Assumptions:

(1) There is a single-vendor and single-buyer for a single product in this model.
(2) The vendor’s production rate of expected non-defective items is finite and greater than the buyer’s demand, i.e., \((1 - M_\rho)p > D\).
(3) The reorder point \(r\) satisfies \(r = \mu L + k\sigma\sqrt{L}\), where \(k\) is the safety factor that is a decision variable.
(4) If we let
\[
L_0 = \sum_{j=1}^{n} b_j,
\]
and
\[
L_i = \sum_{j=1}^{n} b_j + \sum_{j=1}^{i} a_j,
\]
so the lead time crashing cost \(R(L)\) per cycle for a given \(L \in [L_0, L_1]\), is given by
\[
R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i} c_j(b_j - a_j).
\]
(5) The components of lead time are crashed one at a time starting with component 1 (because it has the minimum unit crashing cost), and then component 2, etc.
(6) The \(i\)-th component has a minimum duration \(a_i\) and normal duration \(b_i\), and a crashing cost per unit time \(c_i\).

Further, for convenience, we rearrange \(c_i\) such that
\[
c_1 \leq c_2 \leq \cdots \leq c_n.
\]
(7) The lead time \(L\) has \(n\) mutually independent components.
(8) Replenishments are made whenever the inventory level (based on the number of non-defective items) falls to the reorder point \(r\).
(9) Inventory is continuously reviewed.
(10) Upon arrival of an order, all the items are inspected and defective items in each lot will be returned to the vendor at the time of delivery of the next lot.
(11) We assume that the number of defective items in an arriving order of size \(Q\) is a binomial random variable with parameters \(Q\) and \(\rho\), where \(0 \leq \rho \leq 1\) represents the defective rate in an order lot.
(12) An arriving order may contain some defective items.

Notations:

\(\delta\) is the buyer’s proportion of quantity inspected per shipment, \(0 < \delta \leq 1\).
\(\rho\) is the defective rate in an order lot (independent of lot size) which is a random variable and has a probability density function (p.d.f.) \(g(\rho)\), \(0 < \rho < 1\), with finite mean \(M_\rho\).
\(\beta\) is the fraction of the demand during the stock-out period will be backordered, \(\beta \in [0,1]\).
\(W\) is the buyer’s unit treatment cost for uninspected defective items.
\(Y\) is the buyer’s unit inspection cost.
\(\pi_0\) is the buyer’s profit per unit.
\(\pi\) is the buyer’s shortage cost per unit short.
\(h_b\) is the buyer’s non-defective (including uninspected defective items) holding cost per item per unit time.
\(h_i\) is the vendor’s holding cost per item per unit time.
\(F\) is the transportation cost per delivery.
\(A_i\) is the vendor’s set-up cost per set-up.
\(A_o\) is the buyer’s ordering cost per order.
\(P\) is the production rate of expected non-defective items.
\(Q\) is the order quantity of the buyer (decision variable).
\(\beta\) is the fraction of the demand during the stock-out period.
\(\mu\) is the lead time demand which has a p.d.f. \(f(x)\) with finite mean \(DL\) and standard deviation \(\sigma\sqrt{L}\), where \(\sigma\) denotes the standard derivation of the demand per unit time, for lead time \(L\).
\(E(\cdot)\) is the mathematical expectation.
\(x^+\) is the maximum value of \(x\) and \(0, x^+ = \max\{x, 0\}\).

III. Review of Previous Results

For distribution-free model, we directly quote the objective function of Wu and Ouyang [27].

\[
EAC^u(Q,k,L) = \frac{AD}{Q(1-E(p))} + \frac{(Q-1)kE(p-p^2)}{1-E(p)} + \frac{h}{2}\left[Q(1-E(p))+Q\frac{E(p^2)-E^2(p)}{1-E(p)}+E(p^2)/1-E(p)\right],
\]
\[
+ h\sqrt{L}\sigma\left[k\pm\frac{1-\beta}{2}\left(\sqrt{1+k^2-k}\right)\right]+\frac{Dv}{1-E(p)}\left[\frac{D}{2Q(1-E(p))}\sum_{j=1}^{n} c_j(L_{j-1}-L)+\sum_{j=1}^{n} c_j(b_j-a_j)\right],
\]
for \(L \in [L_0, L_1]\), where \(EAC^u(Q,k,L)\) is an estimated lower bound for the exact model, \(EAC(Q,k,L)\). Wu and
Ouyang [27] claimed that \( EAC^c(Q, k, L) \) is a concave (convex down) function for the restricted sub-domain \( L \in [L_i, L_{i+1}] \) such that the minimum values will be attained on the two boundary points, the left boundary \( L_i \) or the right boundary \( L_{i+1} \). Without loss of generality, we will adopt \( L \) to represent \( L_i \) or \( L_{i+1} \) to simplify the expressions. For the approximated model of \( EAC^c(Q, k, L) \), they evaluated the first partial derivative for two variables, \( Q \) and \( k \). Based on the results of \( \frac{\partial}{\partial k}EAC^c(Q, k, L) = 0 \), and \( \frac{\partial}{\partial Q}EAC^c(Q, k, L) = 0 \), Wu and Ouyang [27] implied that

\[
Q = \sqrt{\frac{2D}{h}A + c_i(1-L_i) - \sum_{j=1}^{i-1}c_j(b_j - a_j)} + \frac{\pi + \pi_0(1-\beta)}{2}\sqrt{L} \left[ \sqrt{1+k^2} - k \right]^{1/2}, \tag{3.2}
\]

and

\[
2\sqrt{1+k^2} - k = 1 - \beta + \frac{D(\pi + \pi_0(1-\beta))}{h(1-E(p))}, \tag{3.3}
\]

where

\[
\delta = 1 - 2E(p) + E(p^2) + 2\frac{h'}{h}E(p(1-p)). \tag{3.4}
\]

is an abbreviation.

Wu and Ouyang [27] mentioned that through iterative algorithms, their optimal solution can be derived. However, we may assert that their two sequences, iterated term by term by Equations (3.1) and (3.2) repeatedly, are lack of support to verify their convergences. Moreover, even if their two sequence both are convergent, why their limits are the optimal solutions for ordering quantity and safety factor which were not examined by Wu and Ouyang [27]. In the next section, we will construct our solution procedure to illustrate that there is a pair of an optimal order quantity and a safety factor under a reasonable condition to derive our feasible domain with a lower bound and an upper bound.

IV. Our Revision

We simplify the expression in Equations (3.2) and (3.3) as follows,

\[
Q = \sqrt{\frac{2D}{h}A + c_i(1-L_i) - \sum_{j=1}^{i-1}c_j(b_j - a_j)} + \frac{\pi + \pi_0(1-\beta)}{2}\sqrt{L} \left[ \sqrt{1+k^2} - k \right]^{1/2}, \tag{4.1}
\]

and

\[
\frac{\sqrt{1+k^2} - k}{\sqrt{1+k^2}} = \frac{2\alpha_4Q}{(1-\beta)\alpha_3Q + \alpha_4}, \tag{4.2}
\]

where we assume the following four abbreviations to simplify the expression,

\[
\alpha_4 = \frac{2D}{h\delta}A + c_i(1-L_i) + \sum_{j=1}^{i-1}c_j(b_j - a_j), \tag{4.3}
\]

\[
\alpha_2 = \frac{D}{h\delta}(\pi + \pi_0(1-\beta))\sqrt{L}, \tag{4.4}
\]

\[
\alpha_3 = h(1-E(p)), \tag{4.5}
\]

and

\[
\alpha_4 = D(\pi + \pi_0(1-\beta)). \tag{4.6}
\]

Owing to Equation (4.2), if we compute

\[
1 - \frac{\sqrt{1+k^2} - k}{\sqrt{1+k^2}} = 1 - \frac{2\alpha_4Q}{(1-\beta)\alpha_3Q + \alpha_4}, \tag{4.7}
\]

to imply that

\[
\frac{k}{\sqrt{1+k^2}} = \frac{\alpha_4 - (1+\beta)\alpha_3Q}{(1-\beta)\alpha_3Q + \alpha_4}. \tag{4.8}
\]

From Equation (4.8), with \( k \geq 0 \), we recall that \( k \) is the safe factor, and then we derive an upper bound for the ordering quantity, \( Q \), as following,

\[
\alpha_4 \geq (1+\beta)\alpha_3Q. \tag{4.9}
\]

If we take square on both side of Equation (4.8), and then minus one, it yields that

\[
1 - \frac{\alpha_4 - (1+\beta)\alpha_3Q}{(1-\beta)\alpha_3Q + \alpha_4} = \frac{4\alpha_4Q(\alpha_4 - \beta\alpha_3Q)}{(\alpha_4 - (1+\beta)\alpha_3Q)^2}. \tag{4.10}
\]

According to the restriction of Equation (4.9), we know that \( \alpha_4 - \beta\alpha_3Q > 0 \) such that the numerator of Equation (4.10) is positive and then the right-hand side of Equation (4.10) is positive which is consistent with the left-hand side of Equation (4.10). Consequently, we derive a relationship to denote the safety factor \( k \) as a function in the ordering quantity \( Q \) in the following,

\[
k = \frac{\alpha_4 - (1+\beta)\alpha_3Q}{2(\alpha_4Q - \alpha_3\alpha_4Q)}. \tag{4.11}
\]

Based on Equation (4.11), we derive that

\[
\sqrt{1+k^2} = \left( 1 + \frac{(\alpha_4 - (1+\beta)\alpha_3Q)^2}{4\alpha_4Q(\alpha_4 - \beta\alpha_3Q)} \right)^{1/2}, = \frac{\alpha_4 + (1-\beta)\alpha_3Q}{2\sqrt{\alpha_3Q(\alpha_4 - \beta\alpha_3Q)}}, \tag{4.12}
\]

If we substitute Equations (4.11) and (4.12) into Equation (4.1), then it implies that

\[
Q^2 = \alpha_4 + \alpha_4\frac{\alpha_4Q}{\alpha_4 - \beta\alpha_3Q}. \tag{4.13}
\]

From Equation (4.13), we have a lower bound for \( Q \) so that

\[
Q > \sqrt{\alpha_4}. \tag{4.14}
\]

By Equation (4.9) again, it yields that

\[
\alpha_4 - \beta\alpha_3Q \geq \alpha_4Q, \tag{4.15}
\]

such that we obtain that

\[
\sqrt{\alpha_3Q(\alpha_4 - \beta\alpha_3Q)} \leq 1, \tag{4.16}
\]

and then owing to Equation (4.13), we derive the second upper bound for \( Q \),

\[
Q \leq \sqrt{\alpha_4 + \alpha_4}. \tag{4.17}
\]

In the following, we will begin to consider these two upper bounds: \( \frac{\alpha_4}{1+\beta}\alpha_3Q \) from Equation (4.9) and \( \frac{\sqrt{\alpha_4} + \alpha_4}{\alpha_4} \) from Equation (4.17). From the numerical example in Wu and Ouyang [27], the data we have collected...
include: \( v = 1.6 \), \( h' = 12 \), \( h = 20 \), \( A = 200 \) per order, \( D = 600 \) units/year, \( \pi = 50 \), \( \pi_0 = 150 \), \( \sigma = 7 \), there are three components in the lead-time, where \( L_0 = 8 \), \( R(L_0) = 0 \), \( L_1 = 6 \), \( R(L_1) = 5.6 \), \( L_2 = 4 \), \( R(L_2) = 22.4 \), \( L_3 = 3 \), and \( R(L_3) = 57.4 \). There are four values, 0, 0.5, 0.8, and 1, for the fraction of the backordered demand, \( \beta \). The defective rate, denoted by \( P \), follows a Beta distribution, and probability density function is expressed as

\[ g(p) = 4(1 - p)^3. \]  

Consequently, researchers know the expected value, \( E(p) = 1/5 \), and the expected value of \( E(p^2) = 1/15 \) to derive the variance.

In the following Table 1, we first obtain our two upper bounds, \( \frac{\alpha_4}{(1+\beta)\alpha_3} \) and \( \sqrt{\alpha_1 + \alpha_2} \), under variously proposed values of \( L_i \) and \( \beta \), and then the ratio of \( \frac{\alpha_4}{(1+\beta)\alpha_3} \) over \( \sqrt{\alpha_1 + \alpha_2} \) is expressed to point out that

\[ \frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1 + \alpha_2}. \]  

From Table 1, all results are greater than one to reveal that

\[ \frac{\alpha_4}{(1+\beta)\alpha_3} > \sqrt{\alpha_1 + \alpha_2}. \]  

Hence, by Equations (4.14) and (4.17), and our findings in Table 1, we derive a pair of lower bound and upper bound for the ordering quantity \( Q \),

\[ \alpha_1 < Q \leq \sqrt{\alpha_1 + \alpha_2}. \]  

From \( Q \leq \sqrt{\alpha_1 + \alpha_2} \) and Table 1, it derives that \( \alpha_4 > (1+\beta)\alpha_3 Q \). Owing to Equation (4.11), it yields that \( k > 0 \) to show that

\[ 1 + k^2 - k = \frac{1}{1 + k^2 + k} < 1, \]  

such that by Equation (4.1), we find that \( Q < \sqrt{\alpha_1 + \alpha_2} \).

Hence, we improve Equation (4.19) as follows,

\[ \sqrt{\alpha_1} < Q < \sqrt{\alpha_1 + \alpha_2}. \]  

We now solve Equation (4.13) under the restriction of Equation (4.21) by setting the next supplementary function,

\[ f(Q) = (Q^2 - \alpha_1)\sqrt{\alpha_4 - \beta_3 \alpha_3 Q - \alpha_2} = \alpha_3 Q. \]  

It shows that

\[ f'(Q) = 2(\alpha_4 - \beta_3 \alpha_3 Q)^{0.5}(\alpha_4 - 2 \beta_3 \alpha_3 Q) + \alpha_3 \sqrt{\alpha_4 - \beta_3 \alpha_3 Q})^{0.5} - Q^{0.5}((\beta_3 \alpha_3 Q)^2 - \alpha_1), \]

\[ \times (\alpha_4 - \beta_3 \alpha_3 Q)^{1.5} Q^{1.5}. \]  

From Equation (4.11), we havethat

\[ \alpha_4 \geq (1+\beta)\alpha_3 Q \geq 2 \beta_3 \alpha_3 Q, \]

and again by Equation (4.11),

\[ \sqrt{\alpha_4 - \alpha_3 \beta_3 \alpha_3 Q}^{0.5} \geq \alpha_3 Q^{0.5} \geq Q^{0.5}(\beta_3 \alpha_3 Q)^{0.5}, \]

together with Equation (4.21), it yields that

\[ \alpha_1 > (Q^2 - \alpha_1). \]  

By combining results of Equation (4.23) through Equation (4.26), the inequality below holds,

\[ f''(Q) > 0. \]  

Consequently, the supplementary function \( f(Q) \) is convex up. The supplementary function can be simplified as follows

\[ f(Q) = \sqrt{\alpha_4 - \beta_3 \alpha_3 Q} \times \left[ Q^2 - \alpha_1 - \alpha_2 \sqrt{\alpha_4 - \beta_3 \alpha_3 Q} \right]. \]  

Next, we will show that \( f(\sqrt{\alpha_1 + \alpha_2}) > 0 \).

If \( Q = \sqrt{\alpha_1 + \alpha_2} \), we show that

\[ Q = \sqrt{\alpha_1 + \alpha_2} < \frac{\alpha_4}{(1+\beta)\alpha_3}, \]

Hence, if \( Q = \sqrt{\alpha_1 + \alpha_2} \), then

\[ \alpha_4 Q < \alpha_4 - \beta_3 \alpha_3 Q, \]

and then we obtain that

\[ f(\sqrt{\alpha_1 + \alpha_2}) >, \]

\[ \alpha_4 - \beta_3 \alpha_3 Q \sqrt{\alpha_1 + \alpha_2} \times \left[ \left( \alpha_1 + \alpha_2 \right)^{0.5} - \alpha_1 \right] = 0. \]  

Based on the following three things: (i) From \( f(\sqrt{\alpha_1}) = -\alpha_2 \sqrt{\alpha_1 \alpha_3 \alpha_3} < 0 \), (ii) \( f(\sqrt{\alpha_1 + \alpha_2}) > 0 \), and (iii) \( f(Q) \) is a convex up function, we partition the figure of \( f(Q) \) into two sections. On the left-hand side, \( f(Q) \) decreases to its minimum, and then on the right-hand side, \( f(Q) \) increases from its minimum. Hence, \( f(Q) = 0 \) has a unique solution. for later discussion, we denote it as \( Q^* \), where satisfies \( f(Q^*) = 0 \).

Based on our above discussion, we have proved that Equation (4.13) under the restriction of Equation (4.21) have a unique solution, which is the only solution that satisfies the first partial derivative system. Hence, we show the minimum problem has a unique solution.

V. Numerical Examples

We recall those data from Wu and Ouyang [27] to assume the following numerical example:

\( v = 1.6 \), \( h' = 12 \), \( h = 20 \), \( A = 200 \) per order, \( D = 600 \) units/year, \( \pi = 50 \), \( \pi_0 = 150 \), \( \sigma = 7 \), and there are three components in the lead-time, with the following lead-time and lead-time crashable cost: \( L_0 = 8 \), \( R(L_0) = 0 \), \( L_1 = 6 \), \( R(L_1) = 5.6 \), \( L_2 = 4 \), \( R(L_2) = 22.4 \), \( L_3 = 3 \), and \( R(L_3) = 57.4 \). Four values of 0, 0.5, 0.8, and 1, are the fraction of the backordered demand, \( \beta \). The defective rate, expressed as \( P \), satisfies the Beta distribution, where the probability density function is denoted as

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\[ g(p) = 4(1 - p)^3. \]
Hence, we derive the expected value, \( E(p) = 1/5 \), and the expected value of \( E(p^2) = 1/15 \) for the variance. The numerical findings are expressed in the next table 2.

Based on the above table, we compare our findings with that of Wu and Ouyang [27] to express the results in the following table 3. Based on the first and the third column of the above table, we point out that our results are superior to that of Wu and Ouyang [27].

VI. A Detailed Examination of Their Iterative Algorithm

Next, we present a detailed examination of the solution algorithm of Wu and Ouyang [27]. Based on Equation (3.2), the ordering quantity \( Q \) is expressed as a mapping of the safety factor, \( k \). Consequently, for a given value of \( k \), then we can obtain a corresponding value of the ordering quantity, \( Q(k) \).

We check their results that was cited as Equation (3.3), then the the fraction of the backordered demand, \( \beta \). Consequently, we can assert that their iterative algorithm is too complicated such that even Wu and Ouyang [27] cannot operative their iterative algorithm.

VII. Direction for Future Research

To help researchers to locate the possible directions for future studies, we cited several recently published papers to indicate those important research trends.

In the following, we provide a brief for our cited articles. Based on block partition strategy, Wang et al. [33] studied the oblique QR decomposition with respect to block landweber scheme. Kusuma and Dirgantara [34] examined a new metaheuristic with run-catch optimizer and then applied it for addressing outsourcing optimization problems.

Referring to fractional-order backstepping strategy and input saturation, Tian et al. [35] developed a class of engineering system for finite-time control. According to conditional value at risk and copula-based value at risk, Ismail et al. [36] constructed currency exchange portfolio risk estimation. Considering an equiangular spiral bubble net predation, Liu et al. [37] derived an improved whale optimization algorithm.

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VIII. Conclusion

For the inventory model with defective items, we present a complete analysis to prove that there is a unique optimal solution for order quantity with respect to a nonnegative safety factor. Our solution method with an implicit expression formula sets up a theoretical development for the existence and uniqueness of the optimal order quantity. In this paper, we also examined the algorithm of Wu and Ouyang [27] to present revisions such our iterative algorithm also derived the same optimal solution that is the same result as that obtained by our analytic approach.
Table 1. The ratio of $\frac{\alpha_3}{(1 + \beta)\alpha_3}$ over $\sqrt{\alpha_1 + \alpha_2}$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0$</td>
<td>18.86</td>
<td>9.68</td>
<td>6.21</td>
<td>4.17</td>
</tr>
<tr>
<td>$\beta = 0.5$</td>
<td>20.09</td>
<td>10.26</td>
<td>6.55</td>
<td>4.37</td>
</tr>
<tr>
<td>$\beta = 0.8$</td>
<td>21.86</td>
<td>11.07</td>
<td>6.99</td>
<td>4.60</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>22.96</td>
<td>11.51</td>
<td>7.18</td>
<td>4.65</td>
</tr>
</tbody>
</table>

Table 2. For $i = 0, 1, 2, 3$, the local optimal solutions.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$EAC(\delta_i, k_i, L_i)$</th>
<th>$k_i$</th>
<th>$Q_i$</th>
<th>$L_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
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<td>3.0286</td>
<td>193.9755</td>
<td>8.0</td>
</tr>
<tr>
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<td>5894.2264</td>
<td>2.4307</td>
<td>179.6360</td>
<td>8.0</td>
</tr>
<tr>
<td>0.8</td>
<td>5463.9046</td>
<td>1.9391</td>
<td>168.7155</td>
<td>8.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5085.6258</td>
<td>1.4884</td>
<td>159.3316</td>
<td>3.0</td>
</tr>
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Table 3. The comparison between our findings and that of Wu and Ouyang [27].

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Our findings</th>
<th>$EAC(\delta_i, k_i, L_i)$</th>
<th>$Q_i$</th>
<th>$EAC(\delta_i, k_i, L_i)$</th>
<th>$Q_i$</th>
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<td>5697.95</td>
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<td></td>
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<tr>
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<td>171.139009</td>
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</tr>
<tr>
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<td>160.468314</td>
<td>5082.14</td>
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<td></td>
</tr>
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Table 4. An iterative procedure proposed by Wu and Ouyang [27].

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Q_n$</th>
<th>$k_n$</th>
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<tr>
<td>1</td>
<td>315.621514</td>
<td>2.291610</td>
</tr>
<tr>
<td>2</td>
<td>194.448259</td>
<td>3.024756</td>
</tr>
<tr>
<td>3</td>
<td>181.250214</td>
<td>3.138608</td>
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<td>4</td>
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<tr>
<td>5</td>
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<td>3.152298</td>
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<table>
<thead>
<tr>
<th>$n$</th>
<th>$Q_n$</th>
<th>$k_n$</th>
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