Sinc Wavelet Collocation Method for Solving BVPs and the Fractional Diffusion Equation

Nabendra Parumasur and Pravin Singh

Abstract—**In this paper we employ Sinc functions as the basis functions and call the resulting method Sinc wavelet collocation method. The main distinguishing feature of the present method, in contrast to the standard Sinc collocation method, is that we do not use a mapping to transform the Sinc functions to a finite domain on which the BVPs are defined to an infinite domain in order to apply them. This results in a much simplified algorithm which is easy to implement while maintaining the high accuracy of the traditional Sinc collocation method. We apply the Sinc collocation method to BVPs and the fractional diffusion equation.**

Index Terms—**wavelets, Sinc functions, collocation method, BVP, fractional diffusion equation.**

I. Introduction

Two point boundary value problems (BVPs) are ubiquitous in science and engineering. One of the simplest and most popular methods for solving these problems numerically are collocation methods [3], [9] . The classical collocation methods involved orthogonal collocation which use orthogonal functions as basis functions, such as Legendre and Chebyshev basis functions (see [12] and references therein). They are specifically designed to solve problems involving steep gradients. The main feature which sets these methods apart from other collocation methods are their high order of accuracy. In this paper we employ Sinc functions as the basis functions and call the resulting method Sinc wavelet collocation method. The main distinguishing feature of the present method in contrast to the standard Sinc collocation method [1] [2], [10] and [11] is that the method mimics a wavelet method and we do not use a mapping to transform the Sinc functions to a finite domain on which the BVPs are defined to an infinite domain in order to apply them. This results in a much simplified algorithm which is easy to implement while maintaining the high accuracy of the traditional Sinc collocation method. We apply the Sinc collocation method to BVPs and the fractional diffusion equation.

II. Sinc Scaling Function and Wavelets

We consider the Sinc function which is defined by [10]:

$$\phi(x) = \frac{\sin \pi x}{\pi x}. \quad (1)$$

The following formulae follow directly from (1):

$$\begin{aligned} \phi'(x) &= \frac{\cos \pi x}{x} - \frac{\sin \pi x}{\pi x^2}, \quad (2) \\ \phi''(x) &= \frac{2\sin \pi x}{\pi x^3} - \frac{\pi \sin \pi x}{x} - \frac{2\cos \pi x}{x^2}, \\ \phi'(0) &= 0, \quad \phi''(0) = -\frac{\pi^2}{3}. \end{aligned}$$

In the context of wavelets (1) is known as the scaling function [4]. The functions

$$\phi_{j,k}(x) = \phi(2^j x - k), \quad (3)$$

have infinite support. It is well known that these functions form a basis and $V_j = span\{\phi_{j,k}(x) : k \in \mathbb{Z}\} \subset L^2[a,b]$. In particular, V_3 is spanned by the functions $\phi(8x-k)$ and support of $\phi(8x) \sim [-5,5]$ (see Figure 1).

The Sinc generating wavelet function can be obtained from the Sinc scaling function as follows [4]:

$$\psi(x) = 2\phi(2x) - \phi(x), \quad (4)$$

and the associated wavelets are given by the family of functions:

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k). \quad (5)$$

In Figure 2 we illustrate some of the wavelet functions.

We observe that the functions $\phi_{j,k}(x)$ and the wavelet functions $\psi_{j,k}(x)$ are qualitatively very similar. Since, the functions $\phi(x)$ are naturally defined in $(-\infty,\infty)$ they can be used directly to solve problems on unbounded domains. There are essentially two ways of using the functions $\phi_{j,k}(x)$ in collocation for problems on bounded domains: i) Domain extension can be applied when the exact solution is assumed to be zero outside the domain$[a,b]$. ii) Use of composition mappings to transform the Sinc functions to a new basis defined in $[a,b]$. These mappings are commonly referred to as Single-Exponential transformation and Double-Exponential transformations due to their exponential rates of convergence [10]. In the next section we describe an alternative strategy for solving problems posed on a bounded domain by using the functions $\phi_{j,k}(x)$ instead of the scaling function (1) which is normally utilized.

III. SINC WAVELET COLLOCATION METHOD

We consider the following class of two-point boundary value problems (BVPs):

$$\begin{aligned} a(x)u''(x) + b(x)u'(x) + c(x)u(x) &= f(x), \quad (6) \\ u(0) = 0, \quad u(1) &= 0, \quad (7) \end{aligned}$$

where $a(x)$, $b(x)$, $c(x)$, $f(x))$ are continuous functions.

We present a collocation method for solving BVPs (6)-(7) on the interval $[0,1]$. It is straightforward to extend the

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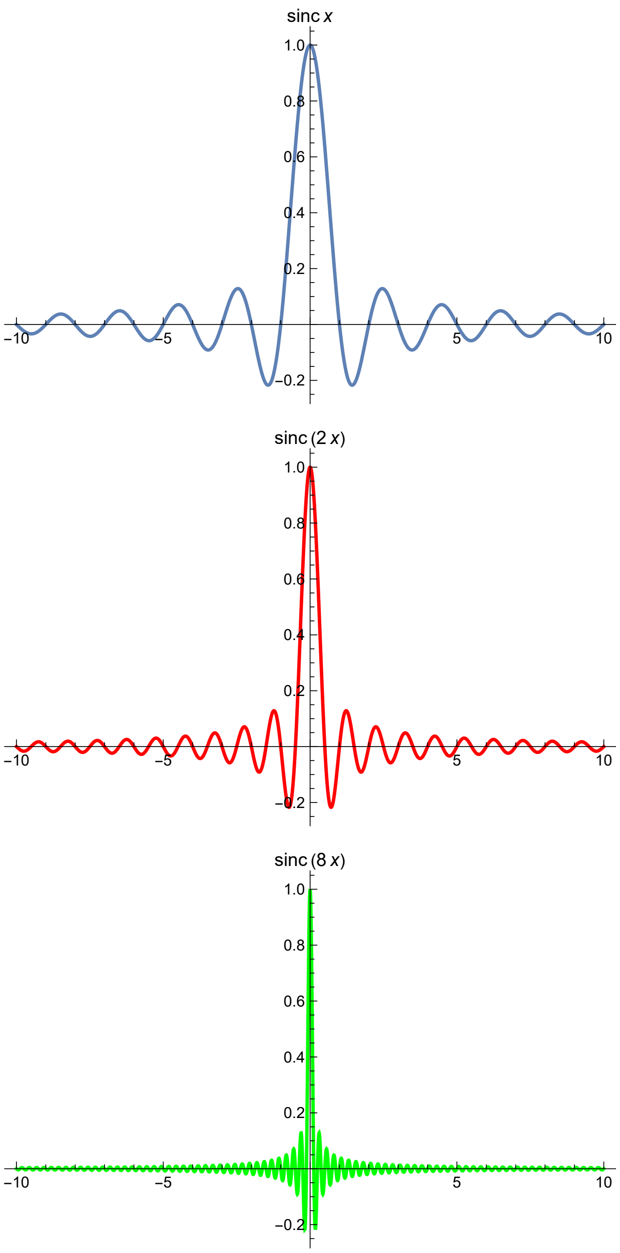

Fig. 1: Sinc scaling function and the functions $\phi_{j,0}(x)$

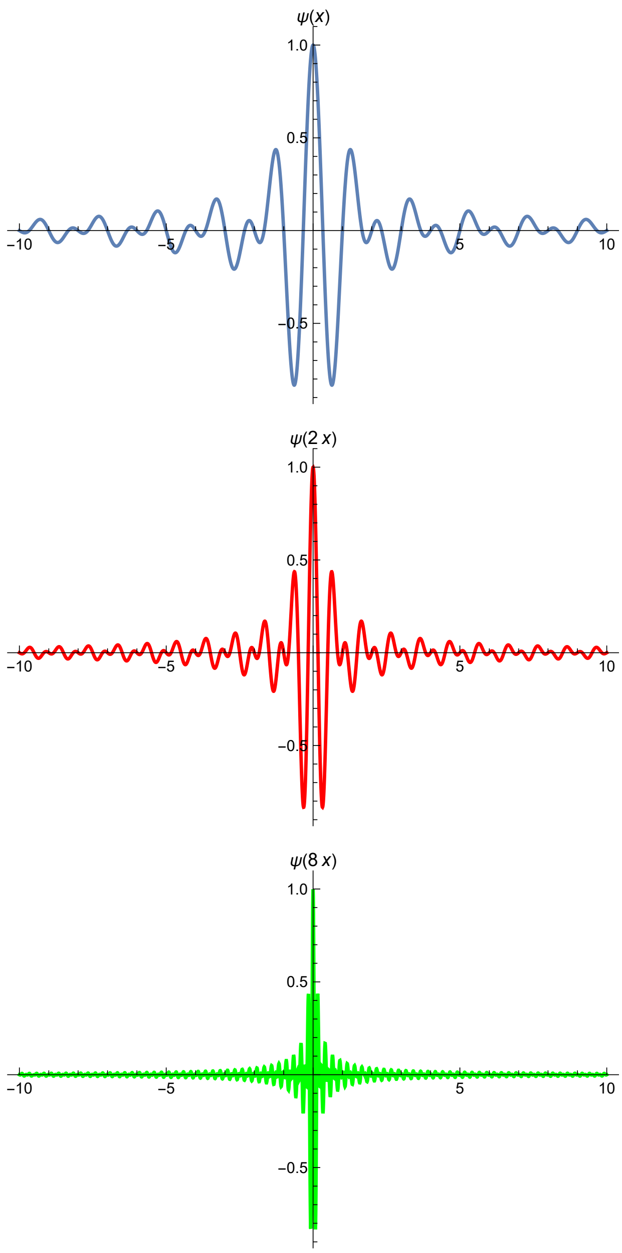

Fig. 2: Sinc generating wavelet function and the wavelet functions $\psi_{j,0}(x)$

method to solve BVPs posed in $[a,b]$ by using an appropriate linear transformation.

In general, we truncate the support of $\phi(px)$ to $[-L,L]$ where $L > 1$ is an integer, p - integer. Hence $[0,1] \subset [-L,L]$.

Hence, $\phi(px-k)$ has support $[k/p-L, k/p+L]$. We note, $\phi(px-k) = \phi[p(x-k/p)]$ is centered at k/p.

Now

$$k/p + L = 1 \;\Rightarrow\; k = (-L+1)p,$$

and

$$k/p - L = 0 \;\Rightarrow\; k = Lp.$$

Hence, $\phi(px-k)$, $\quad k = (-L+1)p, ..., Lp$, all have support that intersects $[0,1]$ (see Figure 3). In particular, we note that the values on the $y-$ scale of the functions $\phi_{3,k}(x)$ becomes small as k increases.

Hence, we assume a solution of the form:

$$y(x) = \sum_{k=(-L+1)p}^{Lp} c_k \phi(px-k) \tag{8}$$

Clearly, using (8) in (6) - (7) yields $c_0 = c_p = 0$ and

$$\begin{aligned} & a(x)p^2 \sum_{k=(-L+1)p}^{Lp} c_k \phi''(px-k) \\ + \; & b(x)p \sum_{k=(-L+1)p}^{Lp} c_k \phi'(px-k) \\ + \; & c(x) \sum_{k=(-L+1)p}^{Lp} c_k \phi(px-k) = f(x). \end{aligned}$$

There are $2pL - p + 1$ unknowns minus 2 boundary conditions. Hence, we require $2pL - p - 1$ collocations points. We

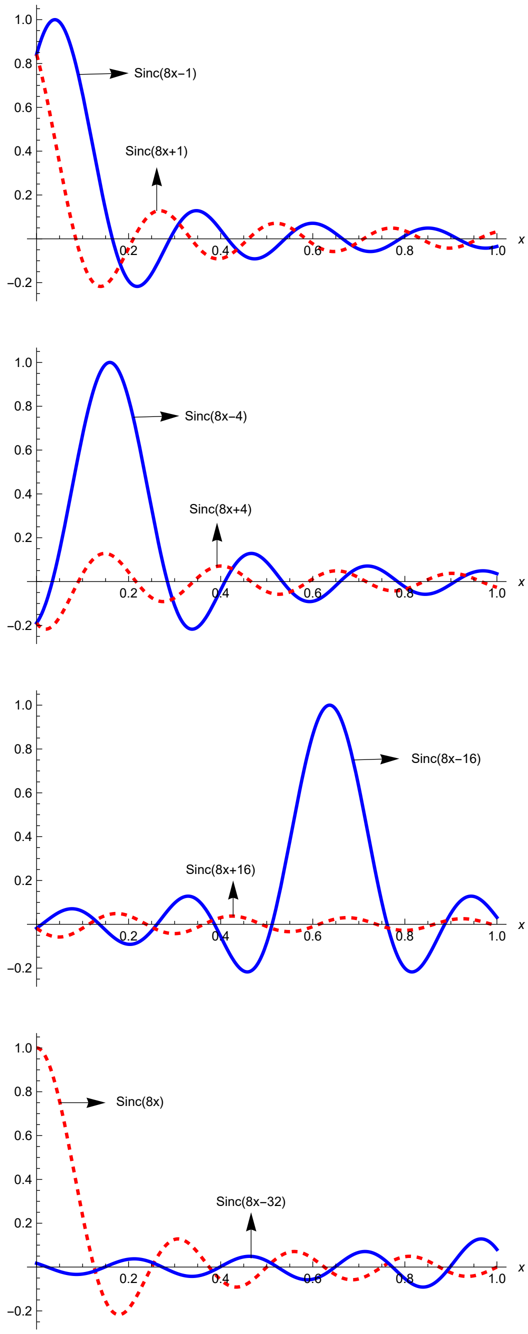

Fig. 3: Shifted Sinc functions $\phi_{3,k}(x)$.

choose $x_j \in (0,1)$ equally spaced points:

$$x_j = \frac{j}{2pL-p}, \quad j = 1, 2, ..., 2pL-p-1. \tag{9}$$

Hence, we solve the linear system:

$$\sum_{\substack{k=p(1-L)\\ k\neq 0,p}}^{Lp} c_k[p^2 a(x_j)\phi''(px_j-k) + pb(x_j)\phi'(px_j-k) + c(x_j)\phi(px-k)] = f(x_j),$$

$j = 1, 2, ..., 2pL-p-1$. The derivative functions are easily evaluated using the equations defined in (3).

In the next section, we consider a series of examples which are standard in the literature e (see e.g. [3], [9]).

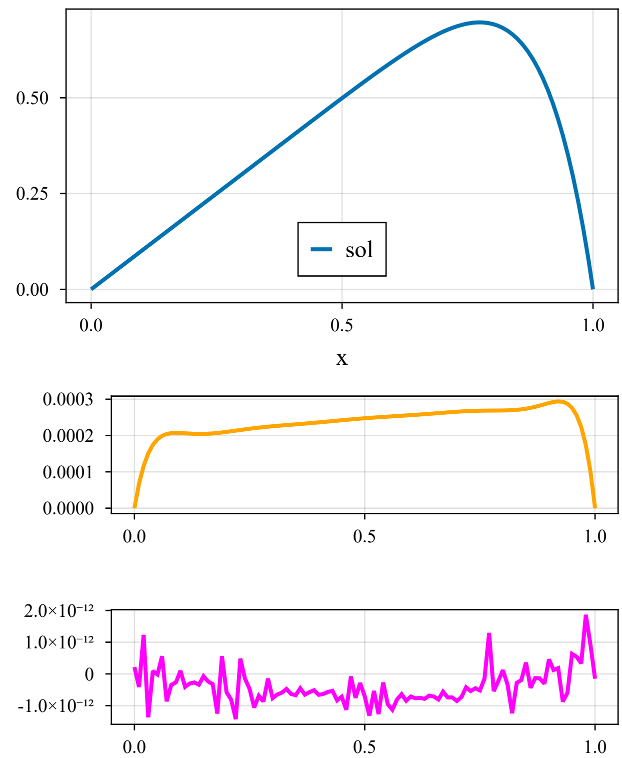

Fig. 4: Example 1: Solution (top). Errors for $p = 1$ (middle) and $p = 4$ (bottom).

IV. Examples for BVPs

Example 1: Consider a polynomial exact solution:

$$a(x) = 2,\ b(x) = 0,\ c(x) = 1,$$

with right hand side:

$$f(x) = x - x^{10} - 180x^8,$$

and exact solution:

$$u(x) = x - x^{10}.$$

The solution and errors for $p = 1$ and $p = 4$ are shown in figure 4.

Example 2: Consider an exact solution containing trigonometric functions:

$$a(x) = -1,\ b(x) = 0,\ c(x) = 1,$$

with right hand side:

$$f(x) = (1+\pi^2)\sin(\pi x)$$

and exact solution:

$$u(x) = \sin(\pi x).$$

The solution and errors for $p = 1$ and $p = 4$ are shown in figure 5.

Example 3: Consider an exact solution with hyperbolic trigonometric functions:

$$a(x) = 1,\ b(x) = 0,\ c(x) = 4,$$

with right hand side:

$$f(x)) = 4\cosh(1),$$

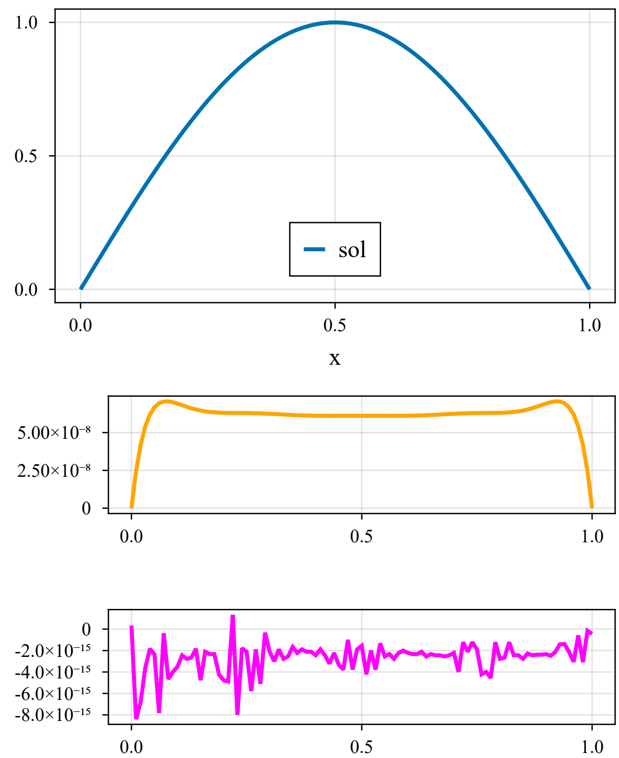

Fig. 5: Example 2: Solution (top). Errors for $p = 1$ (middle) and $p = 4$ (bottom).

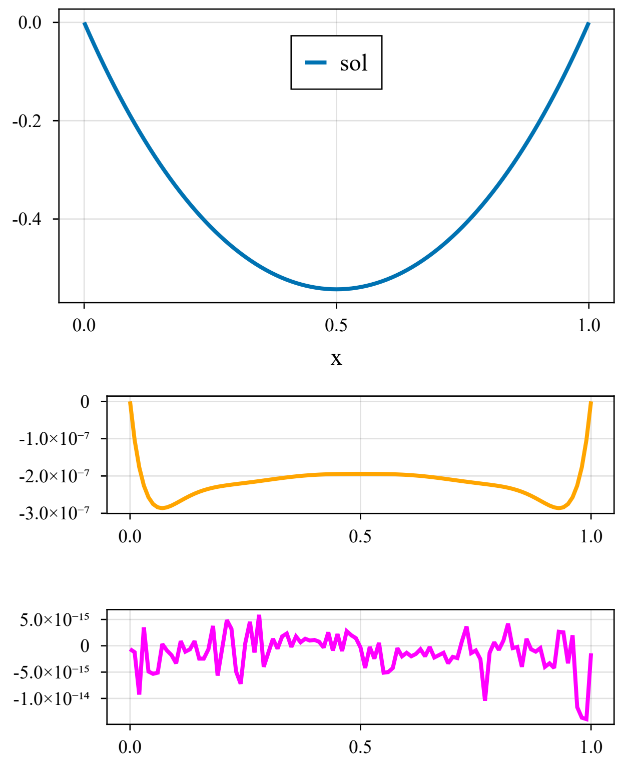

Fig. 6: Example 3: Solution (top). Errors for $p = 1$ (middle) and $p = 4$ (bottom).

and exact solution:

$$u(x) = \cosh(2x - 1) - \cosh(1).$$

The solution and errors for $p = 1$ and $p = 4$ are shown in figure 6.

Example 4: Consider an exact solution having a combination of a polynomial and trigonometric functions. Here we take:

$$a(x) = 1,\ b(x) = 0,\ c(x) = 0,$$

with right hand side:

$$f(x) = 2\sin^2(6x)+12(2x-1)\sin(12x)+72x\cos(12x)(x-1),$$

and exact solution:

$$u(x) = x(x - 1)\sin^2(6x).$$

The solution and the errors for $p = 1$ and $p = 4$ are shown in figure 7.

Example 5: Consider an exact solution with fractional power. Here we take:

$$a(x) = 1,\ b(x) = \frac{1}{6x},\ c(x) = \frac{-1}{x^2},$$

with right hand side:

$$f(x)) = -19\frac{\sqrt{x}}{6},$$

and exact solution:

$$u(x) = x^{3/2}(1 - x).$$

The solution and errors for $p = 1$ and $p = 4$ are shown in 8.

Example 6: Consider an exact solution with exponentials:

$$a(x) = -1,\ b(x) = 20,\ c(x) = 10,$$

with right hand side:

$$f(x) = 1$$

and exact solution:

$$u(x) = c_1\exp(\lambda_1 x) + c_2\exp(\lambda_2 x) + 0.1,$$

where

$$\lambda_1 = 10 + \sqrt{110},\quad \lambda_2 = 10 - \sqrt{110},$$

and

$$c_1 = 0.1\frac{\exp(\lambda_2) - 1}{\exp(\lambda_1) - \exp(\lambda_2)},\quad c_2 = 0.1\frac{1 - \exp(\lambda_1)}{\exp(\lambda_1) - \exp(\lambda_2)}.$$

The solution and errors for $p = 1$ and $p = 4$ are shown in figure 9.

In Examples 1-6 above we see that the Sinc wavelet collocation method using $p = 4$ produces a highly accurate solution (in some cases with machine accuracy) as compared to the standard Sinc collocation method $p = 1$.

V. APPLICATION TO THE FRACTIONAL DIFFUSION EQUATION

We consider the time-fractional Diffusion equation which maybe reduced to a system of BVPs of the form (6) after discretization in time. For this purpose we solve the following equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + f(x,t) \quad x \in (0,1), \quad 0 < t < 1, \tag{10}$$

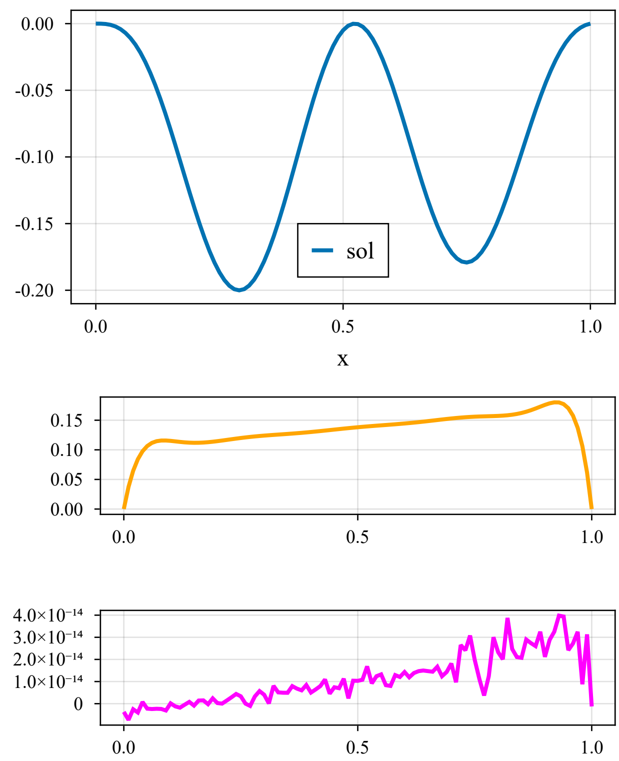

Fig. 7: Example 4: Solution (top). Errors for $p = 1$ (middle) and $p = 4$ (bottom).

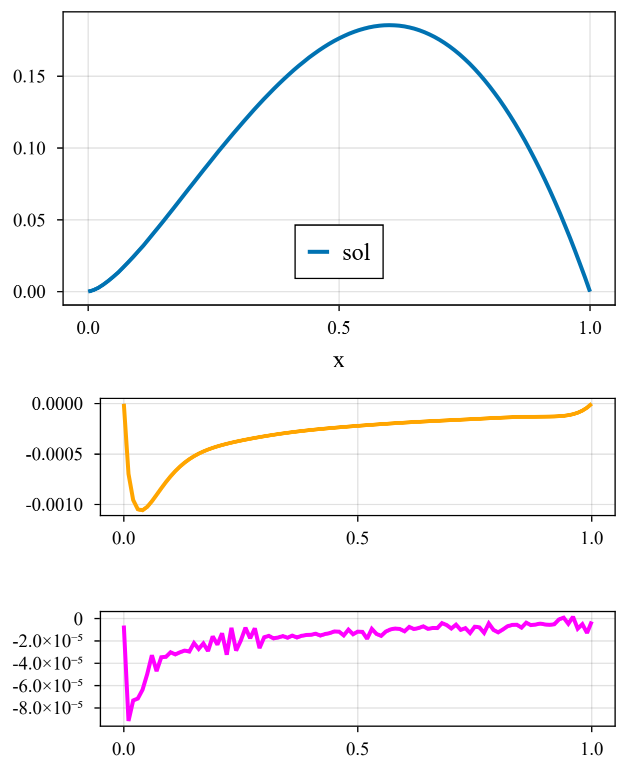

Fig. 8: Example 5: Solution (top). Errors for $p = 1$ (middle) and $p = 4$ (bottom).

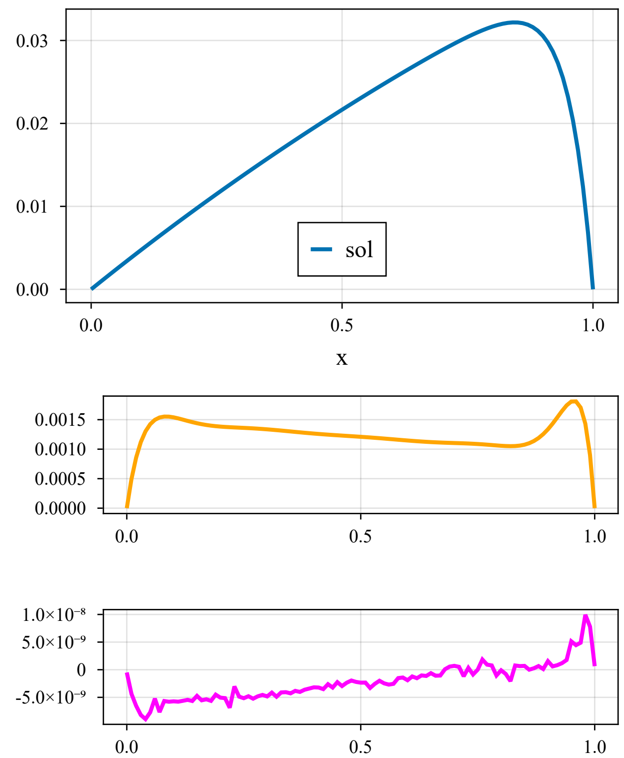

Fig. 9: Example 6: solution (top). Errors for $p = 1$ (middle) and $p = 4$ (bottom).

subject to the following initial and boundary conditions:

$$\begin{aligned} u(x,0) &= g(x), \quad x \in (0,1), \\ u(0,t) &= u(1,t) = 0, \quad 0 \leq t \leq 1, \end{aligned}$$

where α is the order of the time-fractional derivative. The Caputo fractional derivative of order α (see [5] [7], [8], and references therin) is used in (10) and is defined by:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,t)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1. \tag{11}$$

For the temporal discretization of (10) we use a standard finite difference approach [6]. We intoduce the grid points $t_k = k\Delta t, k = 0, 1, ..., K$, where $\Delta t = 1/K$ is the time step and replace the fractional derivative (11) by the discrete approximation:

$$\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{K} b_j \frac{u(x,t_{k+1-j} - u(x,t_{k-j})}{\Delta t^\alpha}, \tag{12}$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, \ j = 0, 1, ..., k$. The approximate solution is given by:

$$u(x,t_k) = \sum_{q=(-L+1)p}^{Lp} c_q^k \phi(px - q), \quad k = 0, 1, 2, ..., K \tag{13}$$

Clearly upon substitution of (13) into (10) results in a system of BVPs of the form (6) with an appropriately defined right hand side and boundary conditions. By using

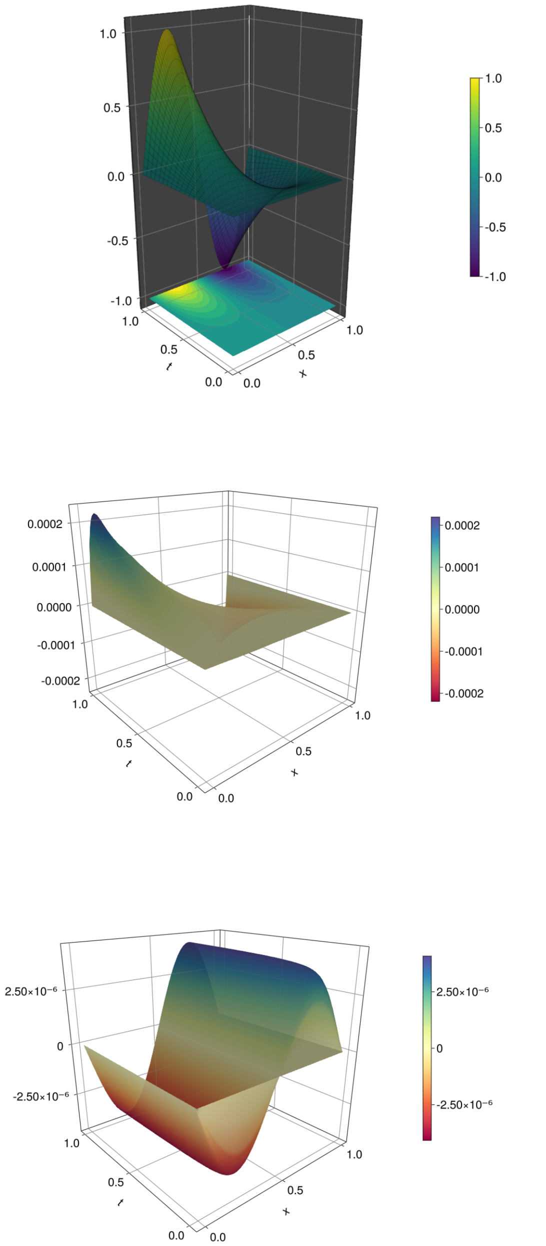

Fig. 10: Example 7: $3D$ plots of solution and errors

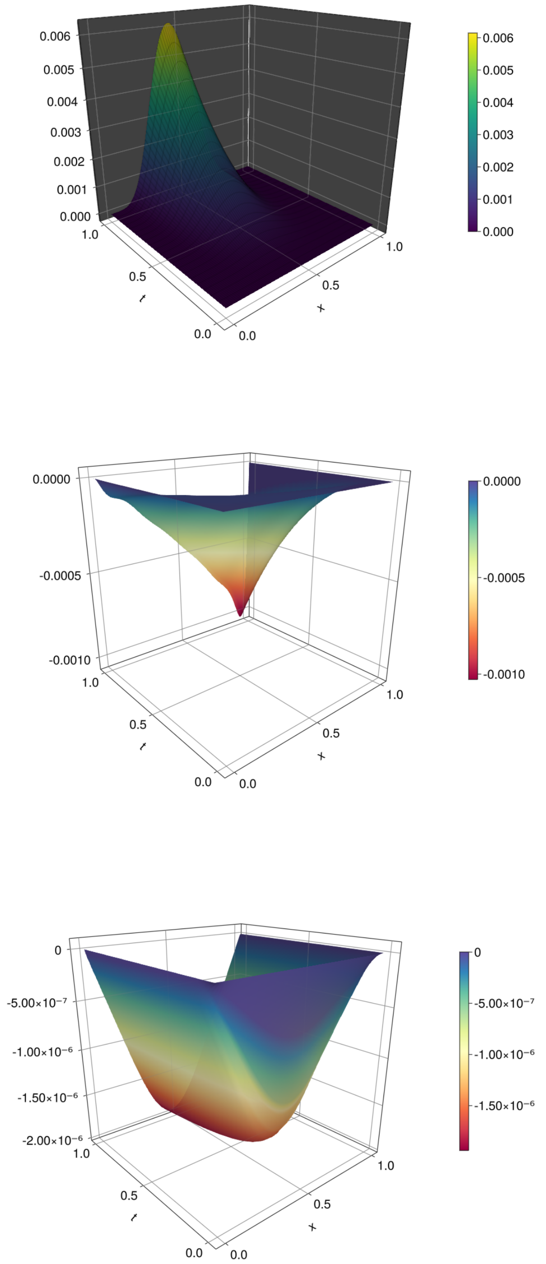

Fig. 11: Example 8: $3D$ plots of solution and errors

the collocation points at $t = 0$ we obtain

$$u(x, t_0) = \sum_{q=(-L+1)p}^{Lp} c_q^0 \phi(px_j - q), \; j = 1, 2, ..., p(2L-1)-1, \tag{14}$$

and together with the two boundary conditions a linear system which is solved for c_q^0.

In the next section we provide numerical examples to demonstrate the efficacy of the Sinc wavelet collocation method for solving the fractional diffusion equation.

Example 7: As a first example for the time fractional Diffusion equation we consider the following test case [6]:

$$u(x, t) = t^2 \sin(2\pi x),$$

$$f(x, t) = \frac{1}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x).$$

We use $\alpha = 0.5$ and $K = 200$.

The top plot in figure 10 depicts a $3D$ mesh plot of the exact solution superimposed on the surface plot of the approximate solution. There is a perfect matching between the exact and approximate solutions. A $3D$ plot for the errors for $p = 1$ (middle plot) and $p = 2$ (bottom plot) are given in Figure 10, respectively. Clearly, the error is much better for the case $p = 2$ as compared to the conventional Sinc method using $p = 1$.

Example 8: As a second example, we consider the following test case [6]:

$$u(x, t) = t^2 (x(1-x)(2-x))^{\frac{16}{3}},$$

$$f(x,t) = \frac{2}{\gamma}t^{2-\alpha}(3-\alpha)(x^3-3x^2+2x)^{\frac{16}{3}} - \frac{16}{3}t^2(x^3-3x^2+2x)^{\frac{10}{3}}(\frac{13}{3}(3x^2-6x+2)^2 + (x^3-3x^2+2x)(6x-6))$$

We use $\alpha = 0.5$ and $K = 200$.

The top plot in Figure 11 depicts a $3D$ mesh plot of the exact solution superimposed on the surface plot of the approximate solution. There is a perfect matching between the exact and approximate solutions. A $3D$ plot for the errors for $p = 1$ (middle) and $p = 4$ (bottom) are given in Figure 11, respectively. Clearly, the error is much better for the case $p = 4$ as compared to the conventional Sinc method using $p = 1$.

VI. CONCLUSION

In the past, Sinc collocation methods have been applied successfully to solve problems on unbounded domains using domain extension [1], [2]. In this case the solution has to decay to zero outside [a,b]. In the case of bounded domains several composition mappings, such as the single-exponential and double exponential mappings, have been employed to transform the Sinc basis to a bounded domain [10]. In this paper, we presented a powerful numerical method for problems on bounded domains using Sinc scaling functions which do not require the solution to decay to zero and the use of composition mappings. The Sinc wavelet collocation method performed better than the standard Sinc collocation method for solving BVPs and the time-fractional diffusion equation. The authors plan to extend the method to other fractional PDEs.

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