Existence of Solutions for Fractional p-Laplacian Differential Equation with Multipoint Boundary Conditions at Resonance

Xinyu Fu, Jinbo Ni, Gang Chen

Abstract—**This paper studies a class of fractional p-Laplacian differential equations, characterized by mixed fractional differential operators and multipoint boundary conditions at resonance. Utilizing the extension of Mawhin's continuation theorem, we establish the result of existence of solutions.**

Index Terms—**Fractional differential equation, p-Laplacian operator, Multipoint boundary condition, Resonance, Continuation theorem.**

I. Introduction

BOUNDARY value problems (BVPs) of ordinary differential equations (ODEs) serve as a fundamental pillar in the theoretical framework of differential equations, exhibiting a wide array of practical implementations. For example, during the early 19th century, the renowned French mathematician Fourier utilized the technique of variable separation to tackle the issue of heat conduction. This approach culminated in the formulation of a two-point BVP for second-order ODE:

$$\begin{cases} \Phi''(\varpi) + \lambda k^2 \Phi(\varpi) = 0, \\ \Phi(0) = \Phi(l) = 0, \end{cases} \tag{1}$$

where λ is a parameter [1].

The qualitative analysis concerning the existence of solutions for BVPs in ODEs has persistently attracted substantial scholarly interest [2]-[4]. For example, in [4], Lin and Zhang employed the extension of Mawhin's continuation theorem to investigate the existence results for third-order differential equation cointaining p-Laplacian operator with multipoint boundary conditions at resoance as follow:

$$\begin{cases} \left(\phi_p(\varsigma''(\varpi))\right)' = \Phi\left(\varpi, \varsigma(\varpi), \varsigma'(\varpi), \varsigma''(\varpi)\right), \ \ \varpi \in (0,1) \\ \phi_p\left(\varsigma''(0)\right) = \sum\limits_{i=1}^{r} \delta_i \phi_p\left(\varsigma''(\zeta_i)\right), \\ \varsigma'(1) = \sum\limits_{j=1}^{s} \varphi_j \varsigma'(\mu_j), \qquad \varsigma''(1) = 0, \end{cases} \tag{2}$$

where $\phi_p(\varepsilon) = |\varepsilon|^{p-2}\varepsilon$, $p > 1$, $\Phi : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ is a continuous function, $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_r < 1$, $\delta_i \in \mathbf{R}$, $i = 1, 2, \cdots r$, $r \geq 2$; $0 < \mu_1 < \mu_2 < \cdots < \mu_j < 1$, $\varphi_j \in \mathbf{R}$, $j = 1, 2, \cdots s$, $s \geq 1$.

Fractional differential equations (FDEs), constituted by fractional-order derivatives, have piqued scholarly curiosity since their origination by Liouville in 1832, drawing a global array of researchers for investigation [5]-[7]. Currently, the paradigm of FDEs pervades a multitude of research disciplines, boasting comprehensive applications in control theory, chemistry, viscoelasticity, and non-Newtonian mechanics [8]-[9]. Owing to the expansive practical utility of FDEs, the existence of solutions to fractional BVPs has surfaced as a subject of considerable interest [10]-[12]. Over the preceding 30 years, the progression of fractional calculus theory and the requisites of practical problems have instigated the proposition of numerous definitions of fractional calculus. Among these, Caputo-type and Riemann-Liouville-type fractional derivatives are predominantly utilized in the examination of BVPs of FDEs.

On the other hand, to investigate the issue of turbulence in porous media flow, Leibenson proposed a differential equation model that incorporates a p-Laplacian operator [13]. In recent years, the existence results for BVPs of FDEs which is containing p-Laplacian operator has captivated the attention of a multitude of scholars [14]-[19]. Specifically, BVPs of p-Laplacian FDEs at resonance have been a focal point of discussion among some researchers [20]-[23]. Given that the p-Laplacian operator is a quasi-linear operator, the common theoretical underpinning for such discussions is the generalized Mawhin's continuation theorem as extended by Ge and Ren (refer to preliminaries). For instance,

In 2023, Azouzi and Guedda [23] discussed the p-Laplacian equation about existence results for BVPs of FDEs at resonance as follows:

$$\begin{cases} \left(\phi_p(\mathcal{D}_{0^+}^{\nu}\varsigma(\varpi))\right)' = \Phi\left(\varpi, \varsigma(\varpi), \mathcal{D}_{0^+}^{\nu-1}\varsigma(\varpi)\right), \ \ \varpi \in [0,1] \\ \varsigma(0) = \mathcal{D}_{0^+}^{\nu-1}\varsigma(1) = 0, \\ \mathcal{D}_{0^+}^{\nu-1}\varsigma(1) = \sum\limits_{d=1}^{r-2} \delta_d \mathcal{D}_{0^+}^{\nu-1}\varsigma(\varrho_d), \end{cases} \tag{3}$$

where $\phi_p(\varepsilon) = |\varepsilon|^{p-2}s$, $p > 1$, $1 < \nu \leq 2$, $0 < \varrho_1 < \varrho_2 < \cdots < \varrho_{r-2} < 1$, $\delta_d \in \mathbf{R}_+$, $d = 1, 2, \cdots r-2 (r \geq 3)$, $\mathcal{D}_{0^+}^{\nu}$ is Riemann-Liouville fractional derivative, $\Phi : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is a continuous function.

Motivated by the above-indicted, through the use of extension of Mawhin's continuation theorem, this paper talk about the existence results for BVPs of mixed FDEs with p-Laplacian operators at resonance as follows:

Manuscript received October 24, 2023; revised February 18, 2024.

This work was supported in part by the Graduate Innovation Fund of Anhui University of Science and Technology (2023cx2146).

X. Y. Fu is a postgraduate student of the School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan, 232001, China (e-mail: fuxinyu1116@163.com).

J. B. Ni is an associate professor of the School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan, 232001, China, (Corresponding author, e-mail: nijinbo2005@126.com).

G. Chen is a postgraduate student of the School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan, 232001, China (e-mail: a15856113486@163.com).

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\beta}\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(t)\big) = f\big(t, t^{2-\alpha}x(t), \mathcal{D}_{0^+}^{\alpha-1}x(t), \\ \qquad\qquad\qquad\qquad \mathcal{D}_{0^+}^{\alpha}x(t)\big), \qquad t \in (0,1), \\ \phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(0)\big) = \sum\limits_{i=1}^{m}\gamma_i\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(\xi_i)\big), \\ \mathcal{D}_{0^+}^{\alpha-1}x(1) = \sum\limits_{j=1}^{n}\omega_j\mathcal{D}_{0^+}^{\alpha-1}x(\eta_j), \quad \mathcal{D}_{0^+}^{\alpha}x(1) = 0, \end{cases} \tag{4}$$

where $0 < \beta \le 1$, $1 < \alpha \le 2$, ${}^C\mathcal{D}_{0^+}^{\beta}$ represents Caputo fractional derivative, $\mathcal{D}_{0^+}^{\alpha}$ represents Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\gamma_i \in \mathbf{R}$, $i = 1, 2\cdots m$, $m \ge 2$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $\omega_j \in \mathbf{R}$, $j = 1, 2\cdots n, n \ge 1$, $\sum\limits_{i=1}^{m}\gamma_i = 1$, $\sum\limits_{j=1}^{n}\omega_j \ne 1$, $f : [0,1]\times\mathbf{R}^3 \to \mathbf{R}$ is a continuous function.

The main novelties in this work are as follows:

On the one hand, we generalize the work of [4] to the fractional-order case, in particular, when the α and β of the BVP (4) are integers, the problem (4) reduces to the problem (2). On the other hand, by proving Lemma 3.2 (see main results), we avoid the need to add the extra condition that the denominator is not zero when defining the projection operator $\mathcal{Q}$, which improves the results of existing literature [23] to some extent.

II. Preliminaries

In this section, we will introduce some definitions and lemmas.

Definition 2.1 ([9]) The Riemann-Liouville fractional integral of order $\kappa > 0$ of a function $\mu : (0, +\infty) \to \mathbf{R}$ is given by

$$I_{0^+}^{\kappa}\mu(\varpi) = \frac{1}{\Gamma(\kappa)}\int_0^{\varpi}(\varpi - \sigma)^{\kappa-1}\mu(\sigma)d\sigma.$$

Lemma 2.1 ([9]) Suppose that ${}^C\mathcal{D}_{0^+}^{\kappa}\mu \in C^n[0,1]$, $\kappa > 0$. Then

$$I_{0^+}^{\kappa}\,{}^C\mathcal{D}_{0^+}^{\kappa}\mu(\varpi) = \mu(\varpi) + \epsilon_0 + \epsilon_1\varpi + \epsilon_2\varpi^2 + \cdots + \epsilon_{n-1}\varpi^{n-1},$$

where $\epsilon_i = -\frac{\mu^{(i)}(0)}{i!}$, $i = 0, 1, 2, \cdots n-1$, $n = [\kappa] + 1$.

Lemma 2.2 ([9]) Suppose that ${}^C\mathcal{D}_{0^+}^{\kappa}\mu \in C^n[0,1]$, $\kappa > 0$. Then

$$I_{0^+}^{\kappa}\,{}^R\mathcal{D}_{0^+}^{\kappa}\mu(\varpi) = \mu(\varpi) + \epsilon_0\varpi^{\kappa-1} + \epsilon_1\varpi^{\kappa-2} + \epsilon_2\varpi^{\kappa-3} + \cdots + \epsilon_{n-1}\varpi^{\kappa-n+1},$$

where $\epsilon_i = -\frac{\mu^{(i)}(0)}{i!}$, $i = 0, 1, 2, \cdots n-1$, $n = [\kappa] + 1$.

Lemma 2.3 ([9]) Suppose that $\kappa > 0$, $\varrho > -1$, $\varpi > 0$. Then

$$I_{0^+}^{\kappa}\varpi^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1+\kappa)}\varpi^{\kappa+\varrho}$$

$$\mathcal{D}_{0^+}^{\kappa}\varpi^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1-\kappa)}\varpi^{\varrho-\kappa} = {}^C\mathcal{D}_{0^+}^{\kappa}\varpi^{\varrho},$$

in particular $\mathcal{D}_{0^+}^{\kappa}\varpi^{\kappa-r} = 0$, $r = 1, 2, \cdots s$; ${}^C\mathcal{D}_{0^+}^{\kappa}\varpi^{k} = 0$, $k = 0, 1, 2, \cdots s-1$, where $s = [\kappa] + 1$.

Definition 2.2 ([2]) The spaces $\mathcal{X}$ and $\mathcal{Y}$ are both Banach spaces, and the norms are $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$. Denote the operator

$$\mathcal{M} : \mathcal{X} \cap \operatorname{dom}\mathcal{M} \to \mathcal{Y}$$

called quasilinear if

(i) $\operatorname{Im}\mathcal{M} := \mathcal{M}(X \cap \operatorname{dom}\mathcal{M})$ is closed subset of $\mathcal{Y}$,

(ii) $\operatorname{Ker}\mathcal{M} := \{x \in \mathcal{X} \cap \operatorname{dom}\mathcal{M} : \mathcal{M}x = 0\}$ is linearly homeomorphic to $\mathcal{R}^n$, $n < \infty$.

Definition 2.3 ([2]) Let $\mathcal{X}_1 = \operatorname{Ker}\mathcal{M}$, and $\mathcal{X}_2$ is the complement space of $\mathcal{X}_1$ in $\mathcal{X}$, then $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$. On the other hand, presume that $\mathcal{Y}_1$ is a subspace of $\mathcal{Y}$, and $\mathcal{Y}_2$ is the complement space of $\mathcal{Y}_1$ in $\mathcal{Y}$, so we have $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$. Let $\mathcal{P} : \mathcal{X} \to \mathcal{X}_1$ and $\mathcal{Q} : \mathcal{Y} \to \mathcal{Y}_1$ are two projectors, $\Omega \subset \mathcal{X}$ be an open and bounded set with origin $\varrho \in \Omega$, where ϱ is the origin of a linear space.

Supposed that $\mathcal{N}_\lambda : \bar{\Omega} \to \mathcal{Y}$, $\lambda \in [0,1]$ is a continuous operator. Denote $\mathcal{N}_1$ by $\mathcal{N}$. Let $\Sigma_\lambda = \{u \in \bar{\Omega} : \mathcal{M}u = \mathcal{N}_\lambda u\}$. If $\mathcal{Y}_1 \subset \mathcal{Y}$ and $\dim\mathcal{Y}_1 = \dim\mathcal{X}_1$, then $\mathcal{N}_\lambda$ is $\mathcal{M}$-compact in $\bar{\Omega}$. For $\lambda \in [0,1]$, the operator $\mathcal{R} : \bar{\Omega}\times[0,1] \to \mathcal{X}$ is continuous and compact,

(i) $(\mathcal{I} - \mathcal{Q})\mathcal{N}_\lambda(\bar{\Omega}) \subset \operatorname{Im}\mathcal{M} \subset (\mathcal{I} - \mathcal{Q})\mathcal{Y}$,

(ii) $\mathcal{Q}\mathcal{N}_\lambda x = 0$, $\lambda \in (0,1) \Leftrightarrow \mathcal{Q}\mathcal{N}x = 0$,

(iii) $\mathcal{R}(\cdot, 0)$ is the zero operator and $\mathcal{R}(\cdot,\lambda)|_{\Sigma_\lambda} = (\mathcal{I} - \mathcal{P})|_{\Sigma_\lambda}$,

(iv) $\mathcal{M}[\mathcal{P} + \mathcal{R}(\cdot,\lambda)] = (\mathcal{I} - \mathcal{Q})\mathcal{N}_\lambda$.

Lemma 2.4 (Ge-Mawhin's continuation theorem) ([2]) Define two spaces $\mathcal{X}$ and $\mathcal{Y}$, both thier are belong to Banach spaces equipped with the norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$. $\Omega \subset \mathcal{X}$ is an open bounded nonempty set. Assume that a quasilinear operaror

$$\mathcal{M} : \mathcal{X} \cap \operatorname{dom}\mathcal{M} \to \mathcal{Y},$$

and define a nonlinear operator

$$\mathcal{N}_\lambda : \bar{\Omega} \to \mathcal{Y},\ \lambda \in [0,1],$$

Furthermore, the following condition are met

(C_1) $\mathcal{M}x \ne \mathcal{N}_\lambda x$, $\forall(x,\lambda) \in (\operatorname{dom}\mathcal{M} \cap \partial\Omega)\times(0,1)$,

(C_2) $\mathcal{Q}\mathcal{N}x \ne 0$, $x \in \operatorname{dom}\mathcal{M} \cap \partial\Omega$,

(C_3) $\deg(\mathcal{J}\mathcal{Q}\mathcal{N}, \operatorname{Ker}\mathcal{M} \cap \Omega, 0) \ne 0$,

where $\mathcal{N} = \mathcal{N}_1$, $\mathcal{J} : \operatorname{Im}\mathcal{Q} \to \operatorname{Ker}\mathcal{M}$ is a homeomorphism with $\mathcal{J}(\phi) = \phi$, then the equation $\mathcal{M}x = \mathcal{N}x$ has at least one solution in $\bar{\Omega}$.

III. Main results

Define two spaces $\mathcal{Y} = C[0,1]$, with the norm $\|y\|_{\mathcal{Y}} = \|y\|_{\infty}$. $\mathcal{X} = \{x : t^{2-\alpha}x(t),\ \mathcal{D}_{0^+}^{\alpha-1}x(t),\ \mathcal{D}_{0^+}^{\alpha}x(t) \in C[0,1]\}$, with the norm $||x||_{\mathcal{X}} = \max\{||t||_{\infty}, ||t^{2-\alpha}x||_{\infty}, ||\mathcal{D}_{0^+}^{\alpha-1}x||_{\infty}, ||\mathcal{D}_{0^+}^{\alpha}x||_{\infty}\}$, where $\|\cdot\|_{\infty} = \max\limits_{t\in[0,1]}|x(t)|$. It is easily to check that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are two Banach spaces.

Define the quasilinear operator $\mathcal{M} : \operatorname{dom}\mathcal{M} \subset \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{M}x = {}^C\mathcal{D}_{0^+}^{\beta}\phi_p(\mathcal{D}_{0^+}^{\alpha}x), \tag{5}$$

where

$$\operatorname{dom}\mathcal{M} = \Big\{x \in \mathcal{M} : \mathcal{M}x = {}^C\mathcal{D}_{0^+}^{\beta}\phi_p(\mathcal{D}_{0^+}^{\alpha}x(t)),$$
$$\phi_p(\mathcal{D}_{0^+}^{\alpha}x(0)) = \sum_{i=1}^{m}\gamma_i\phi_p(\mathcal{D}_{0^+}^{\alpha}x(\xi_i)),$$
$$\mathcal{D}_{0^+}^{\alpha-1}x(1) = \sum_{j=1}^{n}\eta_j\mathcal{D}_{0^+}^{\alpha-1}x(\eta_j),\ \mathcal{D}_{0^+}^{\alpha}x(1) = 0\Big\}.$$

Define the nonlinear operator $\mathcal{N}_\lambda x$ by

$$\mathcal{N}_\lambda x(t) = \lambda f\big(t,\ t^{2-\alpha}x(t),\ \mathcal{D}_{0^+}^{\alpha-1}x(t),\ \mathcal{D}_{0^+}^{\alpha}x(t)\big),\ \forall t\in(0,1).$$

Then boundary value problem (4) is equal to the operator equation

$$\mathcal{M}x = \mathcal{N}_\lambda x,\ x\in \mathrm{dom}\mathcal{M}.$$

Lemma 3.1. Let $\mathcal{M}:\mathrm{dom}\mathcal{M}\subset\mathcal{X}\to\mathcal{Y}$ given by (5), then

$$\mathrm{Ker}\mathcal{M} = \big\{x\in\mathcal{X}|x(t)=ct^{\alpha-2},\ c\in\mathbf{R},\ t\in(0,1)\big\}, \quad (6)$$

$$\mathrm{Im}\mathcal{M} = \Big\{y\in\mathcal{Y}|\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}y(s)ds=0\Big\}, \quad (7)$$

and $\mathcal{M}$ is the quasilinear operator.

Proof. By Lemma 2.1 and Lemma 2.2, we get ${}^{C}\mathcal{D}_{0^+}^{\beta}\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(t)\big)=0$ has solution:

$$x(t) = c_1t^{\alpha-1}+c_2t^{\alpha-2}+I_{0^+}^{\alpha}\phi_q(c_0),\ c_0,c_1,c_2\in\mathbf{R},$$

combining with the boundary conditions

$$\mathcal{D}_{0^+}^{\alpha}x(1)=0,\ \mathcal{D}_{0^+}^{\alpha-1}x(1)=\sum_{j=1}^{n}\eta_j\mathcal{D}_{0^+}^{\alpha-1}x(\eta_j),$$

If $y\in\mathrm{Im}\mathcal{M}$, then there exists a function $x\in\mathrm{dom}\mathcal{M}$ such that $y(t)={}^{C}\mathcal{D}_{0^+}^{\beta}\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(t)\big)$. By Lemma 2.1, we obtain

$$\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(t)\big) = I_{0^+}^{\beta}y(t)+c_3,\ c_3\in\mathbf{R},$$

according to $\sum\limits_{i=1}^{m}\gamma_i=1$ and the boundary condition

$$\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(0)\big)=\sum_{i=1}^{m}\gamma_i\phi_p\big(\mathcal{D}_{0^+}^{\alpha}x(\xi_i)\big),$$

we obtain

$$\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}y(s)ds=0.$$

On the other hand, if $y\in\mathcal{Y}$ and satisfies

$$\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}y(s)ds=0.$$

Let

$$x(t)=I_{0^+}^{\alpha}\phi_q(k(t))+\frac{t^{\alpha-1}}{(1-\sum\limits_{j=1}^{n}\beta_j)\Gamma(\alpha)} \times\Big[\sum_{j=1}^{n}\omega_j\int_0^{n_j}\phi_q(k(t))dt-\int_0^1\phi_q(k(t))dt\Big],$$

where $k(t)=I_{0^+}^{\beta}y(t)-I_{0^+}^{\beta}y(t)|_{t=1}$, then $x(t)\in\mathrm{dom}\mathcal{M}$ and $\mathcal{M}x(t)=y(t)$. Therefore, (3.2) holds. Clearly, $\dim\mathrm{Ker}\mathcal{M}=1<+\infty$, and $\mathrm{Im}\mathcal{M}:=\mathcal{M}(\mathrm{dom}\mathcal{M}\cap\mathcal{X})$ is a closed subset of $\mathcal{Y}$. So, we can obtain that $\mathcal{M}$ is the quasilinear operator.

Lemma 3.2 If $\sum\limits_{i=1}^{m}\gamma_i=1$, then there exists $l\in\{0,1,2,\cdots,m-1\}$ such that $\sum\limits_{i=1}^{m}\gamma_i\xi_i^{l+\beta}\neq 0$.

Proof. Using the method of proof to the contrary, assuming that the above conditions are not established, then for any $l\in\{0,1,2\cdots m-1\}$, we have $\sum\limits_{i=1}^{m}\gamma_i\xi_i^{l+\beta}=0$. That is

$$\begin{cases}\gamma_1\xi_1^{\beta}+\gamma_2\xi_2^{\beta}+\gamma_3\xi_3^{\beta}+\cdots+\gamma_m\xi_m^{\beta}=0,\\ \gamma_1\xi_1^{\beta+1}+\gamma_2\xi_2^{\beta+1}+\gamma_3\xi_3^{\beta+1}+\cdots+\gamma_m\xi_m^{\beta+1}=0,\\ \gamma_1\xi_1^{\beta+2}+\gamma_2\xi_2^{\beta+2}+\gamma_3\xi_3^{\beta+2}+\cdots+\gamma_m\xi_m^{\beta+2}=0,\\ \quad\vdots\\ \gamma_1\xi_1^{\beta+m-1}+\gamma_2\xi_2^{\beta+m-1}+\gamma_3\xi_3^{\beta+m-1}+\cdots+\gamma_m\xi_m^{\beta+m-1}=0,\end{cases} \quad (8)$$

the coefficient determinant of the equations (8) is

$$\begin{vmatrix}\xi_1^{\beta} & \xi_2^{\beta} & \xi_3^{\beta} & \cdots & \xi_m^{\beta}\\ \xi_1^{\beta+1} & \xi_2^{\beta+1} & \xi_3^{\beta+1} & \cdots & \xi_m^{\beta+1}\\ \xi_1^{\beta+2} & \xi_2^{\beta+2} & \xi_3^{\beta+2} & \cdots & \xi_m^{\beta+2}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \xi_1^{\beta+m-1} & \xi_2^{\beta+m-1} & \xi_3^{\beta+m-1} & \cdots & \xi_m^{\beta+m-1}\end{vmatrix} = \Pi_i^m\xi_i^{\beta}\begin{vmatrix}1 & 1 & 1 & \cdots & 1\\ \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_m\\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \cdots & \xi_m^2\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \xi_1^{m-1} & \xi_2^{m-1} & \xi_3^{m-1} & \cdots & \xi_m^{m-1}\end{vmatrix}$$

Clearly, the right end of the above equation is the Vandermonde determinant. Due to $0<\xi_1<\xi_2<\cdots<\xi_m<1$, we can get $\gamma_1=\gamma_2=\cdots=\gamma_m=0$, which is contradiction with $\sum\limits_{i=1}^{m}\gamma_i=1$. So, the conclusion is proof.

Lemma 3.3 Suppose that $\Omega\subset\mathcal{X}$ is an open bounded subset, then $\mathcal{N}_\lambda$ is $\mathcal{M}$-compact.

Proof. Define the projection operators $\mathcal{P}:\mathcal{X}\to\mathrm{Ker}\mathcal{M}$ and $\mathcal{Q}:\mathcal{Y}\to\mathrm{Im}\mathcal{Q}$ as follow

$$\mathcal{P}x(t)=\Big(\lim_{t\to0^+}t^{2-\alpha}x(t)\Big)t^{\alpha-2},\ \forall t\in(0,1),$$

$$\mathcal{Q}y(t)=\frac{1}{\sum\limits_{i=1}^{m}\gamma_i\xi_i^{\beta+l}}\frac{\Gamma(l+\beta+1)}{\Gamma(\beta)\Gamma(l+1)} \times\Big(\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}y(s)ds\Big)t^l,\ \forall t\in(0,1),$$

where l satisfies Lemma 3.2. By the definition of projection operator $\mathcal{P}$, it can see that $\mathrm{Im}\mathcal{P}=\mathrm{Ker}\mathcal{M}$ and $\mathcal{P}x(t)=\mathcal{P}^2x(t)$, for any $x\in\mathcal{X}$, we have $x=(x-\mathcal{P}x)+\mathcal{P}x$, then

$$\mathcal{X}=\mathrm{Ker}\mathcal{P}+\mathrm{Ker}\mathcal{M}.$$

Besides, we can easily proof that $\mathrm{Ker}\mathcal{P}\cap\mathrm{Ker}\mathcal{M}=\{0\}$. Then we have

$$\mathcal{X}=\mathrm{Ker}\mathcal{P}\oplus\mathrm{Ker}\mathcal{M}.$$

Based on the definition of projection operator $\mathcal{Q}$, we obtain

$$\begin{aligned}\mathcal{Q}^2y &= \mathcal{Q}(\mathcal{Q}y)\\ &= \frac{1}{\sum\limits_{i=1}^{m}\gamma_i\xi_i^{\beta+l}}\frac{\Gamma(l+\beta+1)}{\Gamma(\beta)\Gamma(l+1)}\\ &\quad\times\Big(\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}\mathcal{Q}y(s)ds\Big)t^l\\ &= \mathcal{Q}y.\end{aligned}$$

For any $y\in\mathcal{Y}$, we have $y=(y-\mathcal{Q}y)+\mathcal{Q}y$, where $y-\mathcal{Q}y\in\mathrm{Ker}\mathcal{Q}$, $\mathcal{Q}y\in\mathrm{Im}\mathcal{Q}$. Due to $\mathrm{Ker}\mathcal{Q}=\mathrm{Im}\mathcal{M}$ and $\mathcal{Q}^2y=\mathcal{Q}y$, we know that $\mathrm{Im}\mathcal{Q}\cap\mathrm{Im}\mathcal{M}=\{0\}$. Hence

$$\mathcal{Y}=\mathrm{Im}\mathcal{Q}\oplus\mathrm{Im}\mathcal{M}.$$

Defined $\mathcal{R}:\bar{\Omega}\times[0,1]\to\mathrm{Ker}\mathcal{P}$ by

$$\mathcal{R}(x,\lambda)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(h(s))ds+c_1t^{\alpha-1},$$

where

$$\begin{aligned}c_1=&\frac{1}{(1-\sum\limits_{j=1}^{n}\beta_j)\Gamma(\alpha)}\sum_{j=1}^{n}\beta_j\int_0^{n_j}\phi_q\big(h(s)\big)ds\\&-\int_0^1\phi_q\big(h(s)\big)ds,\\h(s)=&\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}(\mathcal{I}-\mathcal{Q})\mathcal{N}_\lambda x(\tau)d\tau\\&-\frac{1}{\Gamma(\beta)}\int_0^1(1-\tau)^{\beta-1}(\mathcal{I}-\mathcal{Q})\mathcal{N}_\lambda x(\tau)d\tau.\end{aligned}$$

Since $f\in C\big([0,1]\times\mathbf{R}^3,\mathbf{R}\big)$, we can easy know that $\mathcal{R}(x,\lambda)$ is continuous on $\bar{\Omega}\times[0,1]$. We proof that $\mathcal{N}_\lambda$ is $\mathcal{M}$-compact on $\bar{\Omega}$ by the following four steps.

Step 1. We prove that (i) in Definition 2.3 holds. By the definition of $\mathcal{Q}$, we obtain $\mathcal{Q}^2y=\mathcal{Q}y$, so $\mathcal{Q}(\mathcal{I}-\mathcal{Q})\mathcal{N}_\lambda(\bar{\Omega})=0$. On the other hand, for any $y\in\mathrm{Im}\mathcal{M}$, clearly $\mathcal{Q}y=0$, then $y=y-\mathcal{Q}y=(\mathcal{I}-\mathcal{Q})y$, so $y\in(\mathcal{I}-\mathcal{Q})\mathcal{Y}$. Hence

$$(\mathcal{I}-\mathcal{Q})\mathcal{N}_\lambda(\bar{\Omega})\subset\mathrm{Im}\mathcal{M}\subset(\mathcal{I}-\mathcal{Q})\mathcal{Y}.$$

Step 2. We prove that (ii) in Definition 2.3 holds. Due to $\mathcal{Q}\mathcal{N}_\lambda x=\lambda\mathcal{Q}\mathcal{N}x$, it is easily to check that $\mathcal{Q}\mathcal{N}_\lambda x=0,\ \lambda\in(0,1)\Leftrightarrow\mathcal{Q}\mathcal{N}x=0,\ \forall x\in\Omega$.

Step 3. We prove that (iii) in Definition 2.3 holds. Evidently $\mathcal{R}(\cdot,0)=0$. Besides, for any $x\in\Sigma_\lambda=\{x\in\bar{\Omega}:\mathcal{M}x=\mathcal{N}_\lambda x\}$, we get

$$\begin{aligned}&\mathcal{R}(x,\lambda)(t)\\&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\big(h(s)\big)ds+\frac{t^{\alpha-1}}{(1-\sum\limits_{j=1}^{n}\beta_j)\Gamma(\alpha)}\\&\quad\times\sum_{j=1}^{n}\omega_j\int_0^{\eta_j}\phi_q\big(h(s)\big)ds-\int_0^1\phi_q\big(h(s)\big)ds\\&=I_{0^+}^{\alpha}\mathcal{D}_{0^+}^{\alpha}x(t)+c_1t^{\alpha-1}\\&=x(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathcal{D}_{0^+}^{\alpha-1}x(0)-\big(\lim_{t\to0^+}t^{2-\alpha}x(t)\big)t^{\alpha-2}\\&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathcal{D}_{0^+}^{\alpha-1}x(0)\\&=x(t)-\big(\lim_{t\to0^+}t^{2-\alpha}x(t)\big)t^{\alpha-2}\\&=\big[(\mathcal{I}-\mathcal{P})x\big](t).\end{aligned}$$

Step 4. We prove that (iv) in Definition 2.3 holds. $\forall\ x\in\bar{\Omega}$, we get

$$\begin{aligned}&M\big[\mathcal{P}x+\mathcal{R}(x,\lambda)\big](t)\\&={}^{C}\mathcal{D}_{0^+}^{\beta}\phi_p\Big[\mathcal{D}_{0^+}^{\alpha}\Big(\big(\lim_{t\to0^+}t^{2-\alpha}x(t)\big)t^{\alpha-2}\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q\big(h(s)\big)ds+c_1t^{\alpha-1}\Big)\Big]\\&={}^{C}\mathcal{D}_{0^+}^{\beta}\phi_p\Big[\mathcal{D}_{0^+}^{\alpha}I_{0^+}^{\alpha}\phi_q\big(\mathcal{I}_{0^+}^{\beta}(\mathcal{I}-\mathcal{Q})\mathcal{N}_\lambda x(t)\big)\Big]\\&=\big[(\mathcal{I}-\mathcal{Q})\mathcal{N}_\lambda x\big](t).\end{aligned}$$

Thus, $\mathcal{N}_\lambda$ is $\mathcal{M}$-compact on $\bar{\Omega}$.

Theorem 3.1 Presume that $f:[0,1]\times\mathbf{R}^3\to\mathbf{R}$ is a continuous function and meets following conditions

$(\mathcal{H}_1)$ There exists non-negative function $\psi,\iota,\vartheta\in\mathcal{Y}$ such that for all $(\nu,\upsilon,\omega)\in\mathbf{R}^3$, $\varrho\in[0,1]$,

$$|f(\varrho,\nu,\upsilon,\omega)|\le\phi_p\big[\psi(\varrho)\,|\nu|+\iota(\varrho)|\upsilon|+\vartheta(\varrho)|\omega|+\sigma(\varrho)\big].$$

$(\mathcal{H}_2)$ There is a constant $\mathcal{A}>0$, such that $\forall x\in\mathrm{dom}\mathcal{M}$, $t\in[0,1]$, either $|t^{2-\alpha}x(t)|>\mathcal{A}$ or $|\mathcal{D}_{0^+}^{\alpha-1}x(t)|>\mathcal{A}$, we have $\mathcal{Q}\mathcal{N}x(t)\neq0$.

$(\mathcal{H}_3)$ There exists a constant $\mathcal{B}>0$, such that either

$$h\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}f(s,h,0,0)ds<0,\ \forall t\in[0,1],\ |h|>\mathcal{B},$$

or

$$h\sum_{i=1}^{m}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\beta-1}f(s,h,0,0)ds>0,\ \forall t\in[0,1],\ |h|>\mathcal{B},$$

when

$$\phi_q\Big(\frac{2}{\Gamma(\beta+1)}\Big)\Big(\frac{2||\psi||_\infty}{\Gamma(\alpha)}+||\iota||_\infty+||\vartheta||_\infty\Big)<1.\qquad(9)$$

Then the problem (4) at least one solution.

Lemma 3.4 Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ establish, the following set

$$\Omega_1=\{x\in\mathrm{dom}\mathcal{M}\backslash\mathrm{Ker}\mathcal{M}|\mathcal{M}x=\mathcal{N}_\lambda x,\ \lambda\in[0,1]\}$$

is bounded.

Proof. For $x\in\Omega_1$, we have $\mathcal{M}x=\lambda\mathcal{N}x$, then $\mathcal{N}x\in\mathrm{Im}\mathcal{M}=\mathrm{Ker}\mathcal{Q}$, so $\mathcal{Q}\mathcal{N}x=0$. Form $(\mathcal{H}_2)$, we know that there exists $t_0,\ t_1\in[0,1]$ such that $|t_0^{2-\alpha}x(t_0)|\le\mathcal{A}$, $|D_{0^+}^{\alpha-1}x(t_1)|\le\mathcal{A}$. Since

$$x(t)=I_{0^+}^{\alpha-1}\mathcal{D}_{0^+}^{\alpha-1}x(t)+ct^{\alpha-2},$$

then

$$t^{2-\alpha}x(t)=t^{2-\alpha}I_{0^+}^{\alpha-1}\mathcal{D}_{0^+}^{\alpha-1}x(t)+c.$$

Take $t=t_0$, then we have

$$c=t_0^{2-\alpha}x(t_0)-t_0^{2-\alpha}I_{0^+}^{\alpha-1}\mathcal{D}_{0^+}^{\alpha-1}x(t_0).$$

Hence,

$$\begin{aligned}|c|&\le\big|t_0^{2-\alpha}x(t_0)\big|+\big|t_0^{2-\alpha}I_{0^+}^{\alpha-1}\mathcal{D}_{0^+}^{\alpha-1}x(t_0)\big|\\&\le\mathcal{A}+\Big|\frac{t_0^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^{t_0}(t_0-s)^{\alpha-2}\mathcal{D}_{0^+}^{\alpha-1}x(s)ds\Big|\\&\le\mathcal{A}+\frac{t_0^{2-\alpha}}{\Gamma(\alpha-1)}\big\|\mathcal{D}_{0^+}^{\alpha-1}x\big\|_\infty\int_0^{t_0}(t_0-s)^{\alpha-2}ds\\&\le\mathcal{A}+\frac{\big\|\mathcal{D}_{0^+}^{\alpha-1}x\big\|_\infty}{\Gamma(\alpha)}.\end{aligned}$$

That is,

$$\left\|t^{2-\alpha}x\right\|_{\infty}\leq\mathcal{A}+\frac{2\left\|\mathcal{D}_{0^{+}}^{\alpha-1}x\right\|_{\infty}}{\Gamma(\alpha)}.\tag{10}$$

On the other hand,

$$\mathcal{D}_{0^{+}}^{\alpha-1}x(t)=\int_{t_{1}}^{t}\mathcal{D}_{0^{+}}^{\alpha}x(s)ds+\mathcal{D}_{0^{+}}^{\alpha-1}x(t_{1}),$$

therefore

$$\begin{aligned}\left|\mathcal{D}_{0^{+}}^{\alpha-1}x(t)\right|&\leq\left|\int_{t_{1}}^{t}\mathcal{D}_{0^{+}}^{\alpha}x(s)ds\right|+\left|\mathcal{D}_{0^{+}}^{\alpha-1}x(t_{1})\right|\\&\leq\mathcal{A}+\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}.\end{aligned}\tag{11}$$

Combine (10) with (11), we get

$$\left\|t^{2-\alpha}x\right\|_{\infty}\leq\Big(1+\frac{2}{\Gamma(\alpha)}\Big)\mathcal{A}+\frac{2}{\Gamma(\alpha)}\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}.\tag{12}$$

By $\mathcal{M}x=\lambda\mathcal{N}x$, we get

$$^{C}\mathcal{D}_{0^{+}}^{\beta}\phi_{p}\big(\mathcal{D}_{0^{+}}^{\alpha}x(t)\big)=\lambda f\big(t,t^{2-\alpha}x(t),\mathcal{D}_{0^{+}}^{\alpha-1}x(t),\mathcal{D}_{0^{+}}^{\alpha}x(t)\big),$$

then

$$\mathcal{D}_{0^{+}}^{\alpha}x(t)=\phi_{q}\big(\lambda I_{0^{+}}^{\beta}f-\lambda I_{0^{+}}^{\beta}f|_{t=1}\big).$$

Form $(\mathcal{H}_1)$, we have

$$\begin{aligned}&|\mathcal{D}_{0^{+}}^{\alpha}x(t)|\\&\leq\phi_{q}\big(\lambda I_{0^{+}}^{\beta}\,|f|+\lambda I_{0^{+}}^{\beta}\,|f|\,|_{t=1}\big)\\&\leq\phi_{q}\Big[\frac{1}{\Gamma(\beta)}\int_{0}^{t}(t-s)^{\beta-1}ds+\frac{1}{\Gamma(\beta)}\int_{0}^{1}(1-s)^{\beta-1}ds\Big]\\&\quad\times\Big(\|\psi\|_{\infty}\|t^{2-\alpha}x\|_{\infty}+\|\iota\|_{\infty}\|\mathcal{D}_{0^{+}}^{\alpha-1}x\|_{\infty}\\&\quad+\|\vartheta\|_{\infty}\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}+\|\sigma\|_{\infty}\Big)\\&\leq\phi_{q}\big(\frac{2}{\Gamma(\beta+1)}\big)\big(\|\psi\|_{\infty}\|t^{2-\alpha}x\|_{\infty}+\|\iota\|_{\infty}\|\mathcal{D}_{0^{+}}^{\alpha-1}x\|_{\infty}\\&\quad+\|\vartheta\|_{\infty}\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}+\|\sigma\|_{\infty}\big).\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}&\leq\phi_{q}\Big(\frac{2}{\Gamma(\beta+1)}\Big)\Big[\|\psi\|_{\infty}\Big(1+\frac{2}{\Gamma(\alpha)}\Big)\mathcal{A}\\&+\frac{2}{\Gamma(\alpha)}\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}\|\psi\|_{\infty}+\|\iota\|_{\infty}\big(\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}+\mathcal{A}\big)\\&+\|\vartheta\|_{\infty}\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}+\|\sigma\|_{\infty}\Big],\end{aligned}$$

then

$$\begin{aligned}||\mathcal{D}_{0^{+}}^{\alpha}x||_{\infty}\leq\Delta^{-1}\phi_{q}\Big(\frac{2}{\Gamma(\beta+1)}\Big)\Big[\|\psi\|_{\infty}\Big(\mathcal{A}+\frac{2\mathcal{A}}{\Gamma(\alpha)}\Big)\\+\mathcal{A}\|\iota\|_{\infty}+\|\sigma\|_{\infty}\Big]:=M_{1}.\end{aligned}\tag{13}$$

where $\Delta=1-\phi_{q}(\frac{2}{\Gamma(\beta+1)})\left(\frac{2||\psi||_{\infty}}{\Gamma(\alpha)}+\|\iota\|_{\infty}+\|\vartheta\|_{\infty}\right)$
By (11), (12) and (13), we have

$$\left\|t^{2-\alpha}x\right\|_{\infty}\leq\big(1+\frac{2}{\Gamma(\alpha)}\big)\mathcal{A}+\frac{2M_{1}}{\Gamma(\alpha)}:=M_{2}.\tag{14}$$

$$\left\|\mathcal{D}_{0^{+}}^{\alpha-1}x\right\|_{\infty}\leq\mathcal{A}+M_{1}:=M_{3}.\tag{15}$$

Combine (13), (14) with (15), we get

$$\begin{aligned}\|x\|&=\max\left\{\left\|t^{2-\alpha}x\right\|_{\infty},\left\|\mathcal{D}_{0^{+}}^{\alpha-1}x\right\|_{\infty},\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}\right\}\\&\leq\max\left\{M_{1},M_{2},M_{3}\right\}=M.\end{aligned}$$

Hence, Ω_1 is bounded.

Lemma 3.5 Assume that $(\mathcal{H}_3)$ establish, the following set

$$\Omega_{2}=\{x\in\mathrm{Ker}\mathcal{M}|\mathcal{N}x\in\mathrm{Im}\mathcal{M}\}$$

is bounded.

Proof. Let $x\in\Omega_2$, then $x(t)=ct^{\alpha-2},\ c\in\mathbf{R}$ and $\mathcal{N}x\in\mathrm{Im}\mathcal{M}$. So we obtain

$$\sum_{i=1}^{m}\gamma_{i}\int_{0}^{\xi_{i}}(\xi_{i}-s)^{\beta-1}f(s,c,0,0)ds=0.$$

By $(\mathcal{H}_3)$, we know that

$$\begin{aligned}\|x\|&=\max\left\{\left\|t^{2-\alpha}x\right\|_{\infty},\left\|\mathcal{D}_{0^{+}}^{\alpha-1}x\right\|_{\infty},\|\mathcal{D}_{0^{+}}^{\alpha}x\|_{\infty}\right\}\\&=\left\|t^{2-\alpha}x\right\|_{\infty}=|c|\leq\mathcal{B}.\end{aligned}$$

Hence, Ω_2 is bounded.

Lemma 3.6 Assume that the first part of $(\mathcal{H}_3)$ establish, then the set

$$\Omega_{3}=\{x\in\mathrm{Ker}\mathcal{M}|(1-\lambda)\mathcal{Q}\mathcal{N}x-\lambda\mathcal{J}x=0,\ \lambda\in[0,1]\}$$

is bounded, where $\mathcal{J}:\mathrm{Ker}\mathcal{M}\to\mathrm{Im}\mathcal{Q}$ is a homeomorphism defined by

$$\mathcal{J}(ct^{\alpha-2})=ct^{l},\ c\in\mathbf{R}.$$

Proof. Take $x\in\Omega_3$, then $x(t)=ct^{\alpha-2},\ c\in\mathbf{R}$ and

$$\begin{aligned}&\frac{(1-\lambda)}{\sum_{i=1}^{m}\gamma_{i}\xi_{i}^{\beta+l}}\frac{\Gamma(l+\beta+1)}{\Gamma(\beta)\Gamma(l+1)}\\&\times\Big(\sum_{i=1}^{m}\gamma_{i}\int_{0}^{\xi_{i}}(\xi_{i}-s)^{\beta-1}f(t,c,0,0)dt\Big)t^{l}-\lambda ct^{l}=0.\end{aligned}\tag{16}$$

If $\lambda=0$, then $\mathcal{Q}\mathcal{N}x=0$. By $(\mathcal{H}_3)$ of Theorem 3.1, we get $|c|\leq\mathcal{B}$. If $\lambda=1$, then $c=0$. Or else, in the event of $|c|>\mathcal{B}$, we have

$$\begin{aligned}&\frac{c(1-\lambda)}{\sum_{i=1}^{m}\gamma_{i}\xi_{i}^{\beta+l}}\frac{\Gamma(l+\beta+1)}{\Gamma(\beta)\Gamma(l+1)}\\&\times\Big(\sum_{i=1}^{m}\gamma_{i}\int_{0}^{\xi_{i}}(\xi_{i}-s)^{\beta-1}f(t,c,0,0)dt\Big)t^{l}-\lambda c^{2}t^{l}<0.\end{aligned}$$

Which is contradictory to (16). Hence, Ω_3 is bounded.

Remark 3.1 Assume that the latter part of $(\mathcal{H}_3)$ establish, the following set

$$\Omega_{3}'=\{x\in\mathrm{Ker}\mathcal{M}|(1-\lambda)\mathcal{Q}\mathcal{N}x+\lambda\mathcal{J}x=0,\ \lambda\in[0,1]\}.$$

is bounded.

Proof of Theorem 3.1 Set $\Omega=\{x\in\mathcal{X}|\|x\|_{\mathcal{X}}<\max\{M,\mathcal{B}\}+1\}$. Based on Lemma 3.1 and 3.3, we can obtain that $\mathcal{M}$ is the quasilinear operator and $\mathcal{N}_\lambda$ is $\mathcal{M}$-compact on $\bar{\Omega}$. Based on Lemma 3.4 and 3.5, the following conditions are met

(C_1) $\mathcal{M}x \neq \mathcal{N}_\lambda x,\ \forall(x,\lambda) \in (\mathrm{dom}\mathcal{M} \cap \partial\Omega) \times (0,1)$,
(C_2) $\mathcal{QN}x \neq 0,\ x \in \mathrm{dom}\mathcal{M} \cap \partial\Omega$.

Let

$$\mathcal{H}(x,\lambda) = \lambda \mathcal{J}x + (1-\lambda)\mathcal{QN}x.$$

By Lemma 3.6, we know that $\mathcal{H}(x,\lambda) \neq 0,\ x \in \mathrm{Ker}\mathcal{M} \cap \partial\Omega$. According to the homotopy property of degree, we attain

$$\begin{aligned} &\deg(\mathcal{QN}|_{\mathrm{Ker}\mathcal{M}}, \Omega \cap \mathrm{Ker}\mathcal{M}, 0) \\ &= \deg\big(\mathcal{H}(\cdot,0), \Omega \cap \mathrm{Ker}\mathcal{M}, 0\big) \\ &= \deg\big(\mathcal{H}(\cdot,1), \Omega \cap \mathrm{Ker}\mathcal{M}, 0\big) \\ &= \deg\big(\mathcal{I}, \Omega \cap \mathrm{Ker}\mathcal{M}, 0\big) \neq 0. \end{aligned}$$

Hence, the condition (C_3) of Lemma 2.1 is satisified. According to Lemma 2.1, we can obtain that the BVP (4) has at least one solution in $\mathcal{X}$. The proof is complete.

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