Inventory Systems with an Exponential Deterioration Rate

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Abstract—**The purpose of this paper is to present a complete and simplified solution procedure for the inventory model under trapezoidal type demand, Weibull-distributed deterioration, and partial backlogging. A published paper already provide a solution approach for this kind of inventory models. However, we will show that his approach is too complicated that depending on the partition the domain of a trapezoidal type demand to result in three sub-optimal problems. However, this paper will show that the already exist approach is correct but tedious. Moreover, a generalization of this kind of inventory systems from a trapezoidal type demand to any positive demand under a condition of generalized backlogged rate is also developed by this paper.**

Index Terms—**exponential deterioration, inventory systems, backlogged rate, demand is dependent on inventory level**

I. Introduction

CLASSICAL economic inventory models have assumed that the demand is almost always stable such that Harris [1] used a constant demand for his well-known EOQ model. However, this assumption does not hold true for many real world situations as it is increasing competition in modern market that results in variation and uncertainty. Hence, many modifications are constructed by researchers to improve the inventory model of Harris [1]. For example, Henery [2] developed an inventory system with increasing demand. Murdeshwar [3] extended the results of Henery [2] with shortages. Hill [4] combined an increasing demand with a stable demand to construct a ramp-type demand for his inventory model. Cheng et al. [5] further extended the ramp-type demand of Hill [4] to a trapezoidal-type demand to indicate the declining phase of a product. Recently, there is also a trend where previously published papers are improved upon. For example, Lin et al. [6] construct a new solution method to simplify the complicated solution approach proposed by Deng et al. [7]. Deng and Chao [8] showed that the new approach proposed by Julian et al. [9] contained questionable findings such that Deng and Chao [8] provided an amendment. Lin [10] extended the inventory model of Lin [11] from linear deterioration rate to a deteriorated function with a general expression. Lin et al. [12] applied a solution method from an operational management view to simplify the tedious derivation proposed by Yang et al. [13] based on differential system of calculus. Tung and Deng [14] rectified the incomplete solution procedure of Lin et al. [15] to include the optimal solution on the boundary. Chao et al. [16] improved Tung [17] to show that the optimal solution is independent of the distribution for the changing point of a ramp-type demand. Lin et al. [18] obtained a complete solution procedure to indicate that the assumption of Tung et al. [19], which was only supported by numerical examination, is unnecessary. Tung et al. [20] pointed out that the search for real solutions done in Yang et al. [21] is improper and presented an improved solution procedure. By following this trend, this paper will provide an extension for the inventory model of Zhao [22].

Manuscript received October 30, 2023; revised February 9, 2024.

This work was supported in part by the Weifang University of Science and Technology, with the unified social credit code: 52370700MJE3971020.

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II. Assumptions and Notation

To be compatible with Zhao [22], we adopt the same notation and assumptions as Zhao [22].

(a) $I(t)$ the level of inventory at time t, $0 \le t \le T$.

(b) T the fixed length of each ordering cycle.

(c) t_1 the time when the inventory level reaches zero for the inventory model.

(d) t_1^* the optimal point.

(e) S the maximum inventory level for each ordering cycle.

(f) Q^* the optimal ordering quantity.

(g) A_0 the fixed cost per order.

(h) c_1 the cost of each deteriorated item.

(i) c_2 the inventory holding cost per unit per unit of time.

(j) c_3 the shortage cost per unit per unit of time.

(k) c_4 the lost sales cost per unit.

(l) $C_i(t_1)$ $i = 1,2,3$ the average total cost per unit time under different conditions, respectively.

(m) $TC(t_1)$ the average total cost per unit time.

(n) The demand rate, $D(t)$, which is positive and consecutive, is assumed to be a trapezoidal type function of time; that is,

$$D(t) = \begin{cases} f(t), & t \le \mu_1 \\ D_0, & \mu_1 < t < \mu_2 \\ g(t), & \mu_2 \le t < T \end{cases}, \quad (2.1)$$

where μ_1 is time point changing from the increasing demand function $f(t)$ to constant demand D_0, and μ_2 is time point changing from the constant demand D_0 to the decreasing demand function $g(t)$.

(o) The replenishment rate is infinite; that is, replenishment is instantaneous.

(p) The deterioration rate of the item is defined as Weibull (α, β) that is the deterioration rate is

$$\theta(t) = \alpha\beta t^{\beta-1}, \tag{2.2}$$

for $\alpha > 0, \beta > 0, t > 0$.

(q) Shortages are allowed and they adopt the notation used in Abad [23], where the unsatisfied demand is backlogged and the fraction of shortages backordered is

$$e^{-\delta t}, \tag{2.3}$$

where t is the waiting time up to the next replenishment. We also assume that

$$te^{-\delta t}, \tag{2.4}$$

is an increasing function, which had appeared in Skouri et al. [24].

III. Review of Previous Results

For an inventory model with trapezoidal type demand, Weibull-distributed deterioration, and partial backlogging, depending on the relations among μ_1, μ_2 and t_1, Zhao [22] developed three objective functions. In the following, direct quotes of the inventory models constructed by Zhao [22] are shown. For interested readers please refer to original derivations of Zhao [22]. They constructed three objective functions, for three different domains as follows. First, for domain, $0 \le t_1 \le \mu_1$.

$$\begin{aligned} C_1(t_1) = \frac{1}{T}\Bigg\{ & A_0 + c_1\int_0^{t_1} f(x)\left(e^{\alpha x^\beta} - 1\right)dx + c_2\int_0^{t_1}\int_t^{t_1} f(x)e^{\alpha\left(x^\beta - t^\beta\right)}dxdt + c_4\int_{t_1}^{\mu_1}\left(1 - e^{\delta(t-T)}\right)f(t)dt \\ & + c_4\left[\int_{\mu_1}^{\mu_2}\left(1 - e^{\delta(t-T)}\right)D_0 dt + \int_{\mu_2}^{T}\left(1 - e^{\delta(t-T)}\right)g(t)dt\right] + c_3\left[\int_{t_1}^{\mu_1} e^{\delta(t-T)}(T-t)f(t)dt + \int_{\mu_2}^{T} e^{\delta(t-T)}(T-t)g(t)dt \right. \\ & \left. + \frac{D_0}{\delta^2}\left(e^{\delta(\mu_2 - T)} - e^{\delta(\mu_1 - T)}\right) + \frac{D_0}{\delta}\left((T-\mu_2)e^{\delta(\mu_2 - T)} - (T-\mu_1)e^{\delta(\mu_1 - T)}\right)\right]\Bigg\}, \end{aligned} \tag{3.1}$$

Second, for domain, $\mu_1 \le t_1 \le t_2$,

$$\begin{aligned} C_2(t_1) = \frac{1}{T}\Bigg\{ & A_0 + c_1\left[\int_0^{\mu_1} f(x)\left(e^{\alpha x^\beta} - 1\right)dx + D_0\int_{\mu_1}^{t_1}\left(e^{\alpha x^\beta} - 1\right)dx\right] \\ & + c_2\left[\int_0^{\mu_1}\int_t^{\mu_1} f(x)e^{\alpha\left(x^\beta - t^\beta\right)}dxdt + D_0\int_0^{t_1}\int_t^{t_1} e^{\alpha\left(x^\beta - t^\beta\right)}dxdt\right] \\ & + c_3\left[\int_{\mu_2}^{T} e^{\delta(t-T)}(T-t)g(t)dt + (D_0/\delta)\left(\left(T - t_1 + (1/\delta)\right)e^{\delta(t_1 - T)} - e^{\delta(\mu_2 - T)}\left(T - \mu_2 + (1/\delta)\right)\right)\right] \\ & + c_4\left[D_0\int_{t_1}^{\mu_2}\left(1 - e^{\delta(t-T)}\right)dt + \int_{\mu_2}^{T}\left(1 - e^{\delta(t-T)}\right)g(t)dt\right]\Bigg\}. \end{aligned} \tag{3.2}$$

Third, for domain, $\mu_2 \le t_1 \le T$,

$$\begin{aligned} C_3(t_1) = \frac{1}{T}\Bigg\{ & A_0 + c_1\left[\int_0^{\mu_1} f(x)\left(e^{\alpha x^\beta} - 1\right)dx + D_0\int_{\mu_1}^{\mu_2}\left(e^{\alpha x^\beta} - 1\right)dx + \int_{\mu_2}^{t_1}\left(e^{\alpha x^\beta} - 1\right)g(x)dx\right] \\ & + c_2\left[\int_0^{\mu_1}\int_t^{\mu_1} f(x)e^{\alpha\left(x^\beta - t^\beta\right)}dxdt + D_0\int_0^{\mu_1}\int_{\mu_1}^{\mu_2} e^{\alpha\left(x^\beta - t^\beta\right)}dxdt \right. \\ & \left. + D_0\int_{\mu_1}^{\mu_2}\int_t^{\mu_2} e^{\alpha\left(x^\beta - t^\beta\right)}dxdt + \int_0^{\mu_2}\int_{\mu_2}^{t_1} e^{\alpha\left(x^\beta - t^\beta\right)}g(x)dxdt\right] \\ & + c_3\int_{t_1}^{T} e^{\delta(t-T)}(T-t)g(t)dt + c_4\int_{t_1}^{T}\left(1 - e^{\delta(t-T)}\right)g(t)dt\Bigg\}, \end{aligned} \tag{3.3}$$

and then they assumed an auxiliary function, $h(t_1)$, to simplify the expressions,

$$h(t_1) = c_1\left(e^{\alpha t_1^\beta} - 1\right) + c_2\int_0^{t_1} e^{\alpha\left(t_1^\beta - t^\beta\right)}dt + c_3(t_1 - T)e^{\delta(t_1 - T)} + c_4\left(e^{\delta(t_1 - T)} - 1\right). \tag{3.4}$$

Based on (3.1-3.3), Zhao [22] constructed his objective function, $TC(t_1)$,

$$TC(t_1) = \begin{cases} C_1(t_1), & 0 < t_1 \le \mu_1 \\ C_2(t_1), & \mu_1 < t_1 \le \mu_2 \\ C_3(t_1), & \mu_2 < t_1 < T \end{cases}. \tag{3.5}$$

Zhao [22] derived that for $0 < t_1 < \mu_1$,

$$\frac{d}{dt_1}C_1(t_1)=\frac{f(t_1)}{T}h(t_1), \tag{3.6}$$

for $\mu_1 < t_1 < \mu_2$,

$$\frac{d}{dt_1}C_1(t_1)=\frac{D_0}{T}h(t_1), \tag{3.7}$$

and for $\mu_2 < t_1 < T$,

$$\frac{d}{dt_1}C_1(t_1)=\frac{g(t_1)}{T}h(t_1). \tag{3.8}$$

Zhao [22] obtained that

$$h(0)<0, \tag{3.9}$$

$$h(T)>0, \tag{3.10}$$

and

$$\frac{d}{dt_1}h(t_1)>0, \tag{3.11}$$

under the condition that $te^{-\delta t}$ is an increasing function. Hence, there is a point, say t_1^*, satisfying $h(t_1^*)=0$, Zhao [22] then concluded the following replenishment policy,

$$TC(t_1^*)=\begin{cases} C_1(t_1^*), & \text{if } 0<t_1^*\le\mu_1 \\ C_2(t_1^*), & \text{if } \mu_1<t_1^*\le\mu_2 \\ C_3(t_1^*), & \text{if } \mu_2<t_1^*<T \end{cases}. \tag{3.12}$$

IV. Our Generalized Solution Process

By reviewing the solution approach of Zhao [22], it is realized that the solution approach must have been divided into three different parts, because the demand is a trapezoidal type function with two changing point μ_1 and μ_2 such that the expression of demand has three different expressions $f(t)$, D_0 and $g(t)$ as denoted in (1). To obtain a simplified derivation for the model of Zhao [22], the abstract expression, $D(t)$, is used for demand. The differential system governing the inventory level is expressed as follows, for $0<t<t_1$,

$$\frac{d}{dt}I(t)=-\alpha\beta t^{\beta-1}I(t)-D(t), \tag{4.1}$$

for $t_1<t<T$,

$$\frac{d}{dt}I(t)=-e^{-\delta(T-t)}D(t), \tag{4.2}$$

with $I(t_1)=0$. We derive the inventory level, $I(t)$, as follows for $0\le t\le t_1$,

$$I(t)=e^{-\alpha t^\beta}\int_t^{t_1}D(x)e^{\alpha x^\beta}dx, \tag{4.3}$$

for $t_1\le t\le T$,

$$I(t)=\int_t^{t_1}e^{-\delta(T-x)}D(x)dx. \tag{4.4}$$

The maximum inventory level, S, is obtained as

$$S=I(0)=\int_0^{t_1}D(x)e^{\alpha x^\beta}dx. \tag{4.5}$$

The total number of deteriorated items, D_T, is derived as

$$D_T=S-\int_0^{t_1}D(x)dx=\int_0^{t_1}D(x)\left(e^{\alpha x^\beta}-1\right)dx. \tag{4.6}$$

The total number of inventory, H_T, is computed as

$$H_T=\int_0^{t_1}I(t)dt=\int_0^{t_1}e^{-\alpha t^\beta}\int_t^{t_1}D(x)e^{\alpha x^\beta}dxdt. \tag{4.7}$$

The total shortage quantity, B_T, is evaluated as

$$B_T=(-1)\int_{t_1}^T I(t)dt=\int_{t_1}^T\int_{t_1}^t e^{-\delta(T-x)}D(x)dxdt. \tag{4.8}$$

The total number of lost sales, L_T, is estimated as

$$L_T=\int_{t_1}^T\left(1-e^{-\delta(T-t)}\right)D(t)dt. \tag{4.9}$$

The average total cost per unit time is found as

$$\begin{aligned} C(t_1)&=\frac{1}{T}\left(A_0+c_1D_T+c_2H_T+c_3B_T+c_4L_T\right)\\ &=\frac{1}{T}\Big[A_0+c_1\int_0^{t_1}D(x)\left(e^{\alpha x^\beta}-1\right)dx+c_2\int_0^{t_1}e^{-\alpha t^\beta}\int_t^{t_1}D(x)e^{\alpha x^\beta}dxdt\\ &+c_3\int_{t_1}^T\int_{t_1}^t e^{-\delta(T-x)}D(x)dxdt+c_4\int_{t_1}^T\left(1-e^{-\delta(T-t)}\right)D(t)dt\Big]. \end{aligned} \tag{4.10}$$

Based on Equation (4.10), we calculate that

$$\frac{d}{dt_1}C(t_1)=\frac{D(t_1)}{T}h(t_1), \tag{4.11}$$

where $h(t_1)$ is defined as Equation (3.4), because by the Leibniz rule,

$$\begin{aligned} &\frac{d}{dt_1}\int_0^{t_1}e^{-\alpha t^\beta}\int_t^{t_1}D(x)e^{\alpha x^\beta}dxdt\\ &=e^{-\alpha t_1^\beta}\int_{t_1}^{t_1}D(x)e^{\alpha x^\beta}dx+\int_0^{t_1}\frac{\partial}{\partial t_1}\left[e^{-\alpha t^\beta}\int_t^{t_1}D(x)e^{\alpha x^\beta}dx\right]dt\\ &=\int_0^{t_1}e^{-\alpha t^\beta}D(t_1)e^{\alpha t_1^\beta}dt, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\frac{d}{dt_1}\int_{t_1}^T\int_{t_1}^t e^{-\delta(T-x)}D(x)dxdt\\ &=\int_{t_1}^{t_1}e^{-\delta(T-x)}D(x)dx+\int_{t_1}^T\frac{\partial}{\partial t_1}\left(\int_{t_1}^t e^{-\delta(T-x)}D(x)dx\right)dt\\ &=\int_{t_1}^T(-1)e^{-\delta(T-t_1)}D(t_1)dt\\ &=-(T-t_1)D(t_1)e^{-\delta(T-t_1)}. \end{aligned} \tag{4.13}$$

Now, by comparing Equation (4.11) with Equations (3.6-3.8) it can be shown that our generalization derives the same result as Zhao [22] but our approach is much more simplified than that of Zhao [22].

In Zhao [22], $h(t_1)$ is already proved to have a unique solution for $h(t_1)=0$, for $0<t_1<T$. Owing to $h(t_1)<0$, for $0<t_1<t_1^*$ and $h(t_1)>0$, for $t_1^*<t_1<T$, by (21), it yields that $\frac{d}{dt_1}C(t_1)<0$, for $0<t_1<t_1^*$ and

$\frac{d}{dt_1}C(t_1) > 0$, for $t_1^* < t_1 < T$ such that t_1^* is the minimum point and the minimum value is $C(t_1^*)$. Based on the above discussion, a generalized model of Zhao [22] is provided to extend from a trapezoidal type demand to any positive demand that will help researchers develop new inventory models.

V. A Related Problem of Centroids

In this section, we provide a short patchwork for Shieh [25]. Wang et al. [26], Abbasbandy and Asady [27], Chu and Tsao [28], Cheng [29], and Pan and Yeh [30] had applied centroid-based distance to rank fuzzy numbers. For trapezoidal fuzzy number, denoted as $A = (a,b,c,d;\omega)$ with the left-hand wing,

$$A_L(x) = \omega \frac{x-a}{b-a}, \tag{5.1}$$

for $a \le x \le b$ and the right-hand wing, for $c \le x \le d$,

$$A_R(x) = \omega \frac{d-x}{d-c}, \tag{5.2}$$

and for $b \le x \le c$,

$$A(x) = \omega . \tag{5.3}$$

Wang et al. [26] developed the centroids, $(\bar{x}_o(A), \bar{y}_o(A))$, as

$$\bar{x}_o(A) = \frac{\int_{-\infty}^{\infty} xA(x)dx}{\int_{-\infty}^{\infty} A(x)dx}, \tag{5.4}$$

and

$$\bar{y}_o(A) = \frac{\int_0^{\omega} y\left[A_R^{-1}(y) - A_L^{-1}(y)\right]dy}{\int_0^{\omega}\left[A_R^{-1}(y) - A_L^{-1}(y)\right]dy}, \tag{5.5}$$

where $A_L^{-1} : [0,\omega] \to [a,b]$ and $A_R^{-1} : [0,\omega] \to [c,d]$ are the inverse functions of A_L and A_R, defined in Equations (a.1) and (a.2), respectively.

Wang et al. [26] claimed the following two properties of correct centroids formulae:

(P1) If A and B are fuzzy numbers with their membership functions $A(x)$ and $B(x)$ have the relation of $B(x+b) = A(x)$, then

$$\bar{x}_o(B) = b + \bar{x}_0(A), \tag{5.6}$$

and

$$\bar{y}_0(B) = \bar{y}_0(A). \tag{5.7}$$

(P2) If A and B are fuzzy numbers with their membership functions $A(x)$ and $B(x)$ have the relation of $B(x) = \omega A(x)$, then

$$\bar{x}_o(B) = \bar{x}_0(A), \tag{5.8}$$

and

$$\bar{y}_0(B) = \omega\, \bar{y}_0(A). \tag{5.9}$$

Cheng [29] defined that

$$\bar{y}_o(A) = \frac{\omega \int_0^1 y\left[A_R^{-1}(y) + A_L^{-1}(y)\right]dy}{\int_0^{\omega}\left[A_R^{-1}(y) + A_L^{-1}(y)\right]dy}. \tag{5.10}$$

Chu and Tsao [28] assumed that

$$\bar{y}_o(A) = \frac{\int_0^{\omega} y\left[A_R^{-1}(y) + A_L^{-1}(y)\right]dy}{\int_0^{\omega}\left[A_R^{-1}(y) + A_L^{-1}(y)\right]dy}. \tag{5.11}$$

Shieh [25] constructed that

$$\bar{y}_o(A) = \frac{\int_0^{\omega} \alpha \left|A^{\alpha}\right| d\alpha}{\int_0^{\omega} \left|A^{\alpha}\right| d\alpha}, \tag{5.12}$$

where $\left|A^{\alpha}\right|$ is the length of the $\alpha-$cut.

If A is a crisp set with $A(x_0) = \omega$ and $A(x) = 0$, if $x \ne x_0$ then its centroids is defined by (x_0, ω) . It is well-known that (x_0, ω) satisfies properties (P1) and (P2).

Remark 1. The functions A_L and A_R for a fuzzy number need not be invertible, for instance,

$$A = 0.5/1 + 0.5/2 + 1/3 + 0.5/4, \tag{5.13}$$

where $A_L(x) = 0.5$, for $x = 1,2$ that results is the left-hand wing is not one-to-one function and then it is invertible.

We proposed the following question: For the fuzzy number in Remark 1, how to compute $\int_{-\infty}^{\infty} A(x)dx$? Our question is related to in Fuzzy set theory, how to handle discrete case?

VI. A Simplification of Previous Results

Our second question: The fuzzy number in Remark 1 is not a convex fuzzy set. We may predict that the definition of the left-hand wing should be added the following condition:

$$A_L(x) = 0.5, \tag{6.1}$$

for $1 \le x \le 2$.

Shieh [25] cited several well known results,

$$\overline{A}_R^{-1}(y) = \sup\{x | A_R(x) = y\}, \tag{6.2}$$

and

$$\underset{-L}{A}^{-1}(y) = \inf\{x | A_L(x) = y\}, \tag{6.3}$$

to imply that

$$\left|A^y\right| = \overline{A}_R^{-1}(y) - \underset{-L}{A}^{-1}(y). \tag{6.4}$$

We can simplify his approach to directly defined that $\left|A^y\right|$ is the length of $y-$cut to simplify the computations of Equations (6.2-6.4).

For two fuzzy numbers A and B, have an x-axis shift, that is $B(y) = A(x)$ for all $y = ax + b$, then we recall the following well-known result of $\alpha-$cut, Lemma 1 of Shieh et al. [25]. $B^{\alpha} = aA^{\alpha} + b$.

If A is a trapezoidal fuzzy number, $A = [a,b,c,d;\omega]$, then the centroids satisfy the next findings,

$$\bar{x}_o(A) = \frac{1}{3}\left[a+b+c+d-\frac{dc-ab}{d+c-a-b}\right], \quad (6.5)$$

and

$$\bar{y}_o(A) = \frac{\omega}{3}\left[1+\frac{c-b}{d+c-a-b}\right]. \quad (6.6)$$

VII. Discussion for Degenerated Cases

Our third question: For degenerated cases, then how to express the results of centroids. In the following, we present some findings.

If $b=c$, then for a triangular fuzzy number,

$$\bar{x}_o(A) = \frac{1}{3}[a+b+d], \quad (7.1)$$

and

$$\bar{y}_o(A) = \frac{\omega}{3}. \quad (7.2)$$

If $b=c$, and $a=b=d$, that is a crisp fuzzy number, $A = [x_0, x_0, x_0, x_0; \omega]$, then

$$\bar{x}_o(A) = x_o, \quad (7.3)$$

and

$$\bar{y}_o(A) = \frac{\omega}{3}. \quad (7.4)$$

For the crisp fuzzy number, Shieh [25] directly defined

$$\bar{y}_o(A) = \omega, \quad (7.5)$$

that is not consistent with the limiting case of his own formula for $\bar{y}_0(A)$

For trapezoidal fuzzy number, if we apply another formula of Chu and Tsao [28] as Equation (5.11), owing to

$$A^{\alpha} = \left[a+\frac{\alpha}{\omega}(b-a), d-\frac{\alpha}{\omega}(d-c)\right], \quad (7.6)$$

such that we compute that

$$A_L^{-1}(\alpha) + A_R^{-1}(\alpha) = a+d+\frac{\alpha}{\omega}(b+c-a-d), \quad (7.7)$$

and then we obtain

$$\int_0^{\omega} y\left[A_R^{-1}(y)+A_L^{-1}(y)\right]dy = \frac{\omega^2}{6}\left[a+d+2(b+c)\right], \quad (7.8)$$

and

$$\int_0^{\omega}\left[A_R^{-1}(y)+A_L^{-1}(y)\right]dy = \frac{\omega}{2}\left[a+d+b+c\right]. \quad (7.9)$$

For a trapezoidal fuzzy number, we derive that

$$\bar{y}_o(A) = \frac{\omega}{3}\left[1+\frac{b+c}{a+d+b+c}\right]. \quad (7.10)$$

For a triangular fuzzy number, $b=c$,

$$\bar{y}_o(A) = \frac{\omega}{3}\left[1+\frac{2b}{a+d+2b}\right]. \quad (7.11)$$

For a crisp fuzzy number, $b=c$ and $a=d=b$,

$$\bar{y}_o(A) = \frac{\omega}{2}. \quad (7.12)$$

We may claim that the adjust factor for $\bar{y}_o(A)$ of Wang et al. [26] and Shieh [25] is 3 and the adjust factor for $\bar{y}_o(A)$ of Chu and Tsao [28] is 2. Based on our above discussion, we compare different definitions of Shieh [25] and Chu and Tsao [28] to help researchers understand the concept of centroids.

VIII. A related problem with Inventory Model

In this section, we study an EOQ model with ramp-type demand, shortage, and Weibull deterioration distribution that was studied by Giri et al. [31]. We will point out some questionable results in their paper in this section, and then we will present our improvements in the next section.

Giri et al. [31] assumed that $\mu < t_1$, and constructed two models depending on the relationship between μ and δ: (a) $\mu \le \delta$, and (b) $\delta \le \mu$.

For the first model with $\mu \le \delta$, Giri et al. [31] implicitly assumed that $\delta \le t_1$ such that Giri et al. [31] overlooked the case for $\mu \le t_1 < \delta$.

We know that

$$f(0) = A > 0, \quad (8.1)$$

such that the demand rate in Figure 1 of Giri et al. [31] is questionable. To save the precious space of this journal, we will not duplicate Figure 1 of Giri et al. [31] in this paper. For those interested readers, please directly refer to Giri et al. [31].

For the inventory level, we know that there are two different results in Giri et al. [31],

$$q(t) = \frac{AQ_0}{b}\left[e^{bt}-1\right], \quad (8.2)$$

and

$$q(t) = Q_0 - \frac{A}{b}\left[e^{bt}-1\right]. \quad (8.3)$$

The cost of shortage occurred in the interval $[t_1, T]$ is denoted as follows,

$$AC_2\int_{t_1}^{T} e^{b\mu}(T-t)dt. \quad (8.4)$$

In the Proposition 1 of Giri et al. [31], they added an extra condition:

$$\frac{(C_1+C_2)\delta}{C_2} + \frac{C_1}{bC_2}\left(1-\frac{b\mu}{2}-e^{-b\mu}\right) < T. \quad (8.5)$$

In their proof of Proposition 1, Giri et al. [31] mentioned that

$$G(0) < 0. \quad (8.6)$$

IX. Our Improvement

We claim that for any given K $(0 < K < 1)$, under the assumption $\mu < \delta < t_1 = KT \le T$, hence the value of T cannot be too small so the minimum value of T equals $\frac{\delta}{K}$. Therefore, to evaluate at $G(0)$ in their proof of Proposition 1 is meaningless.

We compute

$$G\left(T=\frac{\delta}{K}\right)=\frac{A}{2}e^{b\mu}\delta^2\left(C_1+C_2\left(\frac{1}{K}-1\right)^2\right)-C_3. \quad (9.1)$$

Based on the numerical example in Giri et al. [31], we know that $A = 200$, $b = 0.08$, $\mu = 0.2$, $C_1 = 3$, $C_2 = 5$, $\delta = 0.3$, $C_3 = 90$.

Consequently, we derive that

$$G\left(T=\frac{\delta}{K}\right)<0, \quad (9.2)$$

if and only if

$$0.46 < K. \quad (9.3)$$

From the optimal solution of Giri et al. [31], they obtained that

$$t_1^* = 0.43, \quad (9.4)$$

and

$$T^* = 0.69. \quad (9.5)$$

Hence, we derive that

$$K=\frac{0.43}{0.69}=0.62>0.46, \quad (9.6)$$

and then the inequality of Equation (9.2) is verified through a numerical method.

We may predict that if we change the value of the parameter, from $C_4 = 10$ to a small value, then we expect that the percentage of items in the stock will reduce. Consequently, we may construct an example that $0.46 < K$ is invalid.

Based on our above discussion, we can list the following directions for future researchers:

(i) Change to the exact holding cost, $C_1 H$ that is not to use the approximation $\left(\frac{Q_0 t_1}{2}-D\right)C_1$.

(ii) Using the approximation $\left(\frac{Q_0 t_1}{2}-D\right)C_1$, to prove $\Psi_1(t_1,T)$ convex with the following approach: (a) Positive definite at critical point, (b) Existence and unique of critical point, or (c) Discussion of the case of $\mu < t_1 \le \delta$.

X. Our Proposed Exact Model

We derive the exact total cost per unit time, denoted as $\Psi(t_1,T)$ in the following,

$$\Psi(t_1,T)=\frac{1}{T}\left\{C_3+C_1H+C_4D+\frac{A}{2}C_2e^{b\mu}(T-t_1)^2\right\}, \quad (10.1)$$

where we assume three abbreviations: Q_0, H, and D to simplify our expressions:

$$Q_0 = Ae^{b\mu}\left[\delta-\mu+\int_\delta^{t_1} e^{\alpha(t-\delta)^\beta}dt\right]+\frac{A}{b}\left(e^{b\mu}-1\right), \quad (10.2)$$

$$H = Q_0\delta-\frac{A}{b^2}\left(e^{b\mu}-1-b\mu\right)-(\delta-\mu)\frac{A}{b}\left(e^{b\mu}-1\right)^2$$
$$-\frac{A}{2}e^{b\mu}(\delta-\mu)^2$$
$$+Ae^{b\mu}\int_\delta^{t_1} e^{-\alpha(t-\delta)^\beta}\int_t^{t_1} e^{\alpha(x-\delta)^\beta}dxdt, \quad (10.3)$$

and

$$D = Q_0-\frac{A}{b}\left(e^{b\mu}-1\right)-Ae^{b\mu}(t_1-\mu). \quad (10.4)$$

We obtain the partial derivatives with respect to t_1 and T,

$$\frac{\partial\Psi}{\partial t_1}=\frac{A}{T}e^{b\mu}\left\{C_1e^{\alpha(t_1-\delta)^\beta}\left(\int_\delta^{t_1}e^{-\alpha(t-\delta)^\beta}dt+\delta\right),\right.$$
$$\left.+C_4\left(e^{\alpha(t_1-\delta)^\beta}-1\right)-C_2(T-t_1)\right\}, \quad (10.5)$$

and

$$\frac{\partial\Psi}{\partial T}=\frac{1}{T^2}\left\{C_2Ae^{b\mu}T(T-t_1)-C_3,\right.$$
$$\left.-C_1H-C_4D-\frac{A}{2}C_2e^{b\mu}(T-t_1)^2\right\}. \quad (10.6)$$

We further derive the second derivatives with respect to t_1 and T,

$$\frac{\partial^2\Psi}{\partial t_1{}^2}=\frac{A}{T}e^{b\mu}\left[\Delta(t_1-\delta)^{\beta-1}\left[\Lambda+C_4\right]+C_1+C_2\right], \quad (10.7)$$

with two abbreviations: Δ and H, where

$$\Delta=\alpha\beta e^{\alpha(t_1-\delta)^\beta}, \quad (10.8)$$

and

$$\Lambda=C_1\left(\int_\delta^{t_1}e^{-\alpha(t-\delta)^\beta}dt+\delta\right). \quad (10.9)$$

$$\frac{\partial^2\Psi}{\partial T\partial t_1}=\frac{-A}{T^2}e^{b\mu}\left[\frac{\Delta}{\alpha\beta}(\Lambda+C_4)-C_4+C_2t_1\right], \quad (10.10)$$

and

$$\frac{\partial^2\Psi}{\partial T^2}=\frac{2}{T^3}\left[C_3+C_1H+C_4D+C_2\frac{A}{2}e^{b\mu}t_1^2\right]. \quad (10.11)$$

Based on $\frac{\partial\Psi}{\partial T}=0$ of Equation (10.6), we derive that

$$\frac{A}{2}C_2e^{b\mu}\left(T^2-t_1^2\right)=C_3+C_1H+C_4D. \quad (10.12)$$

Based on Equation (10.5), from $\frac{\partial\Psi}{\partial t_1}=0$, then we show that

$$\frac{C_1}{C_2} e^{\alpha (t_1-\delta)^\beta} \left(\int_\delta^{t_1} e^{-\alpha (t-\delta)^\beta} dt + \delta \right) + ,$$

$$\frac{C_4}{C_2} \left(e^{\alpha (t_1-\delta)^\beta} - 1 \right) + t_1 = T \,. \tag{10.13}$$

How to solve the system consisting Equations (10.12) and (10.13) will be an interesting research topic in the future.

XI. Direction for Future Research

In this section, a possible direction for the future research is provided. Zhao [22] used $xe^{-\delta x}$ as an increasing function, that is $xB(x)$ increases where $B(x)$ is the backlogged rate, which was mentioned in Hung [32][32]. Lin et al. [33] relaxed the condition of Lin [10] to remove the condition proposed by Hung [32]. Consequently, Lin et al. [33] derived that the critical points for the first derivative of the objective function is not unique. Future researchers can remove the condition of $xe^{-\delta x}$ as an increasing function thus allowing an interesting direction for the future research.

There are several important papers that were recently published. We list them in the following to point our possible directions for future research: Li et al. [34] developed a service supply chain to reduce carbon emissions. Wan et al. [35] examined a location strategy for electric commerce warehouse. Chen et al. [36] studied a revised optimization procedure for renewable energy. Unyapoti and Pochai [37] combined a wave crest model with a shoreline evolution. Purwani et al. [38] showed the secant method superior to the Newton method. Alomari and Massoun [39]obtained numerical solutions for some specific equations. Yang et al. [40] constructed a new model with vector continued fractions. Tang et al. [41] executed a numerical examination to analyze customer behavior during the Chinese new year. Petcharat [42] developed a control chart to study seasonal moving average procedure.

XII. Conclusion

By extending the inventory model of Zhao [22] from a trapezoidal type demand to a positive demand, the three complicated objective functions proposed by Zhao [22] is simplified to one objective function. Our derivation will help researchers realize inventory models with Weibull distributed deterioration and negative exponential partial backlogging.

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