Analysis of an Efficient Second-order Strang Splitting Scheme for Solving the Perturbed FitzHugh-Nagumo Neuron Model

Ying Ye, Lingzhi Qian*

Abstract—**In this study, we examine the perturbed FitzHugh-Nagumo (FHN) neuron model, which incorporates a minor parameter. Owing to its pronounced nonlinearity, devising a numerical scheme for the perturbed FHN model that is both highly accurate and stable, as well as efficient, presents a significant challenge. To overcome this challenge, we have developed an effective numerical scheme by employing the Strang splitting technique, which leverages the Fourier spectral method combined with the second-order Strong Stability Preserving Runge-Kutta (SSP-RK2) method. Additionally, we will perform a thorough and rigorous convergence analysis of our proposed scheme. To conclude, we will conduct extensive numerical experiments to demonstrate the scheme's accuracy and efficiency.**

Index Terms—**FitzHugh-Nagumo neuron model, Nonlinearity, Strang splitting, Fourier spectral method, SSP-RK2 method.**

I. Introduction

In this paper, our focus is directed towards the perturbed FHN neuron model for the nerve impulse propagation, as referenced in [10]. This model is characterized by a partial differential equation with pronounced nonlinearity,

$$\begin{cases} \frac{\partial u}{\partial t} = \varepsilon \Delta u + f(u), \quad (x,t) \in \Omega \times (0,T], \\ u|_{t=0} = u_0, \quad x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^d$ denotes a bounded polygonal domain, Δ represents the Laplacian operator, ε is a non-negative perturbation constant, $u_0(x)$ specifies the initial condition, the nonlinear term $f(u)$ is the cubic polynomial given by $f(u) = u(u-1)(\alpha - u)$ with $\alpha \in (-1, 0)$ being a real constant. In domain Ω, the electric potential analog across the cell membrane is depicted by a unknown function $u(x,t)$, which is dependent on both the spatial variable x and the temporal variable $t \geq 0$. We assume that u and Δu are periodic on $\partial\Omega$. Notably, when reduced, Eq.(1) corresponds to the local Allen-Cahn (LAC) equation as discussed in [16]. Furthermore, with $\alpha = -1$, Eq.(1) aligns with the real Newell-Whitehead equation as delineated in [24].

In recent years, the perturbed FHN neuron model has garnered significant attention due to its applications in various phenomena such as nerve impulse propagation, logistic population dynamics and flame propagation, among others ([3], [10], [11], [14], [15]). The complexity of the FHN model makes it exceedingly difficult to derive exact solutions, prompting many researchers to explore a multitude of numerical methods to approximate solutions to this model. Inan [3] introduced a Crank-Nicolson exponential finite difference scheme for the FHN equation and its generalized counterpart, tackling the nonlinear system with a time-stepping Newton's method. Chapwanya [1] crafted three explicit nonstandard finite difference schemes by systematically integrating sub-schemes and conducting a meticulous error analysis. The study in [15] presented a suite of finite difference approaches, including explicit, implicit (Crank-Nicolson), fully implicit, and exponential methods, to solve the FHN equation and conducted a comparative analysis of these techniques. Beyond finite difference methods, the spectral Galerkin method has also been employed to address the FHN equation [5], [6], [8], [9]. Li [10] devised first- and second-order schemes that combine semi-implicit Euler and Crank-Nicolson methods for temporal discretization with the spectral Galerkin approach for spatial discretization. Furthermore, Li provided a rigorous proof of error bounds and optimal convergence rates for the numerical solutions. Other methodologies applied to the FHN model include the variational iteration method and the Adomian decomposition method, as utilized in [17].

The principal objective of this study is to devise and scrutinize a second-order Strang splitting algorithm aimed at approximating the solution to the perturbed FHN neuron model (1). This is achieved by employing the Fourier spectral method for spatial discretization coupled with the Strong Stability Preserving Runge-Kutta (SSP-RK2) method for temporal discretization. Strang splitting, a variant of operator splitting techniques, effectively reduces complexity by decomposing the original problem into a set of more manageable subproblems. Consequently, Strang splitting approaches [12], [20], [21], [23], [27] have been successfully applied to a variety of intricate issues [4], [7], [18], [22], [25]. Nonetheless, adapting Strang splitting to nonlinear partial differential equations presents a significant challenge due to its inherently multi-stage process. To our knowledge, there has been no previous investigation into the Strang splitting analysis specifically for the nonlinear perturbed FHN model. Our contributions are twofold. Firstly, we introduce an efficacious second-order Strang splitting scheme, which integrates the Fourier spectral method with the SSP-RK2 method. Secondly, we offer a comprehensive theoretical analysis of the scheme's convergence with respect to the L^2-norm.

The remainder of this paper is structured as follows: Section 2 outlines a practical and straightforward second-

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order numerical scheme for the perturbed FHN model, accommodating minor parameter perturbations. In Section 3, we delve into the convergence properties of our proposed method. Section 4 presents a suite of numerical experiments to corroborate the scheme's precision and efficiency. We conclude with Section 5, summarizing our findings and contributions.

II. An effective and easy-to-implement second-order scheme

In this section, we introduce an easy-to-implement second-order Strang splitting method for the perturbed FHN equation (1). The second-order scheme is achieved by adopting the Fourier spectral method [28] for the spatial discretization and second-order Runge-Kutta method [2] for the temporal discretization.

A. Strang splitting method

The Strang splitting method is predominantly applied to decompose complex problems into simpler subproblems. Our method aims to reformulate the original problem into a linear subproblem and a nonlinear subproblem, specifically

$$\mathcal{S}_{\mathcal{L}}(u_1): \ u_t = \varepsilon \Delta u, \qquad u|_{t=0} = u_1, \tag{2}$$

$$\mathcal{S}_{\mathcal{N}}(u_2): \ u_t = f(u), \qquad u|_{t=0} = u_2. \tag{3}$$

The canonical form of the symmetric second-order Strang splitting method [23], [28] is expressed by

$$u^{n+1} = \mathcal{S}_{\mathcal{L}}(\frac{\tau}{2})\mathcal{S}_{\mathcal{N}}(\tau)\mathcal{S}_{\mathcal{L}}(\frac{\tau}{2})u^n, \tag{4}$$

where $\tau > 0$ is the time step.

We will next apply the standard Strang splitting (4) to approximate the perturbed FHN equation (1).

B. Numerical approximation of $u_t = \varepsilon \Delta u$

Initially, to numerically approximate the perturbed FHN equation (1), we discretize the geometric domain $\Omega^{per}(d = 2)$. The set of the grid points Ω_h^{per}, is defined as:

$$\Omega_h^{per} = \left\{(x_i, y_j) = (a + ih, b + jh), 0 \le i, j \le N-1\right\},$$

where $h = \frac{b-a}{N}$ is the space step, N is the number of grid nodes. Let u_{mn}^k denote the numerical solution $u(x_m, y_n, t_k)$, where $t_k = k\tau$, $\tau = T/M$ is the time step and M is time iteration number. The discrete Fourier transform and its inverse transform are defined by [13], [26]

$$\mathcal{F}_N: \quad \tilde{u}_{pq}^k = \frac{h^2}{c_p c_q (b-a)^2} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} u_{mn}^k e^{-i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-b)}{b-a}\right)}, \tag{5}$$

$$\mathcal{F}_N^{-1}: \quad u_{mn}^k = \sum_{p=-N/2}^{N/2}\sum_{q=-N/2}^{N/2} \tilde{u}_{pq}^k e^{i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-b)}{b-a}\right)}, \tag{6}$$

where $p, q = 0, \pm 1, \pm 2, ...$ and c_p and c_q are respectively defined as

$$c_r = \begin{cases} 2, & |r| = \frac{N}{2}, \\ 1, & |r| < \frac{N}{2}, \end{cases} \qquad r = p \text{ or } q.$$

In line with spectral method theory [13], [26], the discrete Laplacian term is given by

$$\Delta u_{mn}^k = -\sum_{p=-N/2}^{N/2}\sum_{q=-N/2}^{N/2} (\xi_p^2 + \eta_q^2)\tilde{u}_{pq}^k e^{i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-b)}{b-a}\right)},$$

where $\xi_p = \frac{2\pi p}{b-a}$ and $\eta_q = \frac{2\pi q}{b-a}$. Substituting (5) into (2), we can deduce that

$$\frac{d\tilde{u}_{pq}(t)}{dt} = K\tilde{u}_{pq}(t), \tag{7}$$

where

$$K = \varepsilon \lambda_{pq}$$

and

$$\lambda_{pq} = -\left[\left(\frac{p\pi}{b-a}\right)^2 + \left(\frac{q\pi}{b-a}\right)^2\right]. \tag{8}$$

Employing the method of separation of variables, we can resolve the problem (7), with the solution given by

$$\tilde{u}_{pq}^{k+1} = exp\,(\tau K)\,\tilde{u}_{pq}^k.$$

Consequently, we obtain

$$u^{k+1} = \mathcal{F}_N^{-1}\left\{\exp(\tau K)\mathcal{F}_N[u^k](p,q)\right\}. \tag{9}$$

C. Numerical approximation of $u_t = f(u)$

Now, we concentrate on the nonlinear subproblem (3). By utilizing the second-order SSP-RK method [2], we obtain that

$$\begin{cases} u_{mn}^{(1)} = u_{mn}^k + \tau f\left(u_{mn}^k\right), \\ u_{mn}^{k+1} = \frac{1}{2}u_{mn}^k + \frac{1}{2}u_{mn}^{(1)} + \frac{1}{2}\tau f\left(u_{mn}^{(1)}\right). \end{cases} \tag{10}$$

Based on the above methods, an effective and easy-to-implement second-order scheme for problem (1) is given by

$$\begin{cases} u_{mn}^{(1)} = \mathcal{F}_N^{-1}\left\{exp(\frac{\tau}{2}K)\mathcal{F}_N[u_{mn}^k](p,q)\right\}, \\ u_{mn}^{(2)} = u_{mn}^{(1)} + \tau f\left(u_{mn}^{(1)}\right), \\ u_{mn}^{(3)} = \frac{1}{2}u_{mn}^{(1)} + \frac{1}{2}u_{mn}^{(2)} + \frac{1}{2}\tau f\left(u_{mn}^{(2)}\right), \\ u_{mn}^{k+1} = \mathcal{F}_N^{-1}\left\{exp(\frac{\tau}{2}K)\mathcal{F}_N[u_{mn}^{(3)}](p,q)\right\}. \end{cases} \tag{11}$$

In the following, $\mathcal{S}_{\mathcal{L}}^h$ and $\mathcal{S}_{\mathcal{N}}^h$ represent the numerical approximations of $\mathcal{S}_{\mathcal{L}}$ and $\mathcal{S}_{\mathcal{N}}$, respectively.

III. Theoretical analysis of the convergence

In this section, we analyse the convergence of our proposed scheme (11). Before we proceed, we need some helpful definitions and notations. We denote by U^m the solution of (11) at t_m and define a grid function space on Ω_h^{per}

$$\mathcal{W}^h = \left\{U | U = \{u_{ij} | 0 \le i, j \le N-1\}\right\}$$

and a mapping $I^h : H_{per}^s(\Omega) \to \mathcal{W}^h$ by $I^h(u) = U$, where

$$H_{per}^s(\Omega) = \{u|_{\bar{\Omega}} : u \in H^s(\Omega) \ and\ u\ is\ \Omega - periodic\}.$$

Lemma 3.1: [28] For any grid function $U \in \mathcal{W}^h$, it holds that

$$\left\|\mathcal{S}_{\mathcal{L}}^h U\right\| \le \|U\|, \left\|\mathcal{S}_{\mathcal{N}}^h U\right\| \le \|U\|.$$

To prove the convergence, we first establish some important results.

Theorem 3.1: For any function $u \in H^s_{per}(\Omega)$ with $s > 1$. Suppose the semi-norm $|f(u)|_s$ of $H^s(\Omega)$ and functions u and $f(u)$ are bounded, it holds that

$$\left\| I^h \mathcal{S}_{\mathcal{N}} u - \mathcal{S}^h_{\mathcal{N}} I^h u \right\| \le C\tau^3,$$

where $C > 0$ is a constant independent of τ and h.

Proof: The theorem will be proved in the similar way that used in [12]. Denote $w(x, y, t)$ as the exact solution of subproblem (3) and $W(x, y, t)$ as the numerical solution of the (10). It's observed that $w(x, y, t)$ and $W(x, y, t)$ respectively satisfy

$$\begin{cases} w^* = w^m + \tau f(w^m), \\ w^{m+1} = \frac{1}{2} w^m + \frac{1}{2} w^* + \frac{\tau}{2} f(w^*) + R^{m+1}_\tau, \end{cases}$$

and

$$\begin{cases} W^* = W^m + \tau f(W^m), \\ W^{m+1} = \frac{1}{2} W^m + \frac{1}{2} W^* + \frac{\tau}{2} f(W^*), \end{cases}$$

where $R^{m+1}_\tau = O(\tau^3)$ due to the use of SSP-RK2 method [2]. Thus, we get

$$\begin{aligned} \left\| w^{m+1} - W^{m+1} \right\| &\le \frac{1}{2} \|w^* - W^*\| + \frac{\tau}{2} \|f(w^*) - f(W^*)\| \\ &\le \frac{\tau}{2} C \|w^* - W^*\| + C\tau^3 \\ &\le C\tau^3, \end{aligned}$$

in which $\|w^* - W^*\| = 0$ was applied in the final step.

Lemma 3.2: [28] For any function $u \in H^s_{per}(\Omega)$ with $s > 1$, it holds that

$$\left\| I^h \mathcal{S}_{\mathcal{L}} u - \mathcal{S}^h_{\mathcal{L}} I^h u \right\| \le Ch^s |u|_s,$$

where $C > 0$ is a constant independent of τ and h.

In the following theorem, we are interested in the order of estimate for the solution u.

Theorem 3.2: Assume that $u_0 \in H^s_{per}(\Omega)$ with $s > 1$ and the solution to the problem (1) belongs to $H^1(0, T; H^s_{per}(\Omega))$. Under the conditions of Lemma 3.1 and Theorem 3.1, there exists a positive constant C such that

$$\left\| U^{m+1} - I^h u^{m+1} \right\| \le C\left(\tau^2 + \frac{h^s}{\tau}\right).$$

Furthermore, for the case $\tau \sim h$, we have

$$\left\| U^{m+1} - I^h u^{m+1} \right\| \le C\left(\tau^2 + h^{s-1}\right).$$

Proof: Using the triangle inequality, we get

$$\left\| U^{m+1} - I^h u^{m+1} \right\| \le \left\| U^{m+1} - I^h \bar{u}^{m+1} \right\| + \left\| I^h \bar{u}^{m+1} - I^h u^{m+1} \right\|. \tag{12}$$

According to [19], we have

$$\left\| I^h \bar{u}^{m+1} - I^h u^{m+1} \right\| \le C\tau^2. \tag{13}$$

By Lemma 3.1 and Lemma 3.2, the first term on the right of (12) is bounded by

$$\begin{aligned} \left\| U^{m+1} - I^h \bar{u}^{m+1} \right\| &= \left\| \mathcal{S}^h_{\mathcal{L}} \mathcal{S}^h_{\mathcal{N}} \mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{L}} \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &\le \left\| \mathcal{S}^h_{\mathcal{L}} \mathcal{S}^h_{\mathcal{N}} \mathcal{S}^h_{\mathcal{L}} U^m - \mathcal{S}^h_{\mathcal{L}} I^h \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &+ \left\| \mathcal{S}^h_{\mathcal{L}} I^h \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m - I^h \mathcal{S}_{\mathcal{L}} \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &= \left\| \mathcal{S}^h_{\mathcal{L}} \left(\mathcal{S}^h_{\mathcal{N}} \mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right) \right\| \\ &+ \left\| \left(\mathcal{S}^h_{\mathcal{L}} I^h - I^h \mathcal{S}_{\mathcal{L}} \right) \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &\le \left\| \mathcal{S}^h_{\mathcal{N}} \mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &+ Ch^s \left| \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right|_s . \end{aligned} \tag{14}$$

We deduce from Lemma 3.1 and Theorem 3.1 that

$$\begin{aligned} \left\| \mathcal{S}^h_{\mathcal{N}} \mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| &\le \left\| \mathcal{S}^h_{\mathcal{N}} \mathcal{S}^h_{\mathcal{L}} U^m - \mathcal{S}^h_{\mathcal{N}} I^h \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &+ \left\| \mathcal{S}^h_{\mathcal{N}} I^h \mathcal{S}_{\mathcal{L}} \bar{u}^m - I^h \mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &\le \left\| \mathcal{S}^h_{\mathcal{N}} \left(\mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{L}} \bar{u}^m \right) \right\| \\ &+ \left\| \left(\mathcal{S}^h_{\mathcal{N}} I^h - I^h \mathcal{S}_{\mathcal{N}} \right) \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &\le \left\| \mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| + C\tau^3. \end{aligned} \tag{15}$$

Then, Lemma 3.1 and Lemma 3.2 imply

$$\begin{aligned} \left\| \mathcal{S}^h_{\mathcal{L}} U^m - I^h \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| &\le \left\| \mathcal{S}^h_{\mathcal{L}} U^m - \mathcal{S}^h_{\mathcal{L}} I^h \bar{u}^m \right\| \\ &+ \left\| \mathcal{S}^h_{\mathcal{L}} I^h \bar{u}^m - I^h \mathcal{S}_{\mathcal{L}} \bar{u}^m \right\| \\ &\le \left\| U^m - I^h \bar{u}^m \right\| + Ch^s |\bar{u}^m|_s . \end{aligned} \tag{16}$$

Using the above results, we have

$$\left\| U^{m+1} - I^h \bar{u}^{m+1} \right\| \le \left\| U^m - I^h \bar{u}^m \right\| + Ch^s \left(|\mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m|_s + |\bar{u}^m|_s \right) + C\tau^3. \tag{17}$$

Setting $G = Ch^s \left(|\mathcal{S}_{\mathcal{N}} \mathcal{S}_{\mathcal{L}} \bar{u}^m|_s + |\bar{u}^m|_s \right) + C\tau^3$, we have

$$\left\| U^{m+1} - I^h \bar{u}^{m+1} \right\| \le \left\| U^m - I^h \bar{u}^m \right\| + G.$$

Using the Gronwall inequality and the fact $\left\| U^0 - I^h \bar{u}^0 \right\| = 0$, it holds that

$$\begin{aligned} \left\| U^{m+1} - I^h \bar{u}^{m+1} \right\| &\le \left\| U^0 - I^h \bar{u}^0 \right\| + (m+1)G \\ &\le CT \frac{h^s}{\tau} + CT\tau^2. \end{aligned} \tag{18}$$

Combination of Eqs. (12), (13) and (18) yields

$$\left\| U^{m+1} - I^h u^{m+1} \right\| \le C\left(\tau^2 + \frac{h^s}{\tau}\right). \tag{19}$$

Thus, we have completed the proof. □

IV. Numerical experiments

In this section, we validate the feasibility and accuracy of our proposed second-order numerical scheme through a plethora of 1D and 2D numerical examples with periodic boundary conditions. To calculate the convergence rates, we take the numerical solution with $\tau = 2^{-12}$ as the reference solution in the following numerical experiments.

A. 1D perturbed FHN neuron model

For the 1D perturbed FHN neuron model, we consider the initial condition given by

$$u_0(x) = e^{-\frac{(x-0.5)^2}{\varepsilon}}.$$

The perturbed FHN model (1) is defined on the domain $\Omega = [0, 1]$. We select a grid size of $h = \frac{1}{128}$ and vary the time step size $\tau = 2^{-1}, 2^{-2}, ..., 2^{-5}$ to calculate numerical solution by the proposed scheme (11). The L^2-norm errors and temporal convergence rates with $\alpha = -0.01, -0.5, -0.99$ for $\varepsilon = 10^{-3}$ are given in Table 1. As anticipated, the error rate in time of our proposed scheme (11) is observed to be approximately 2, corroborating the second-order accuracy of our mathod.

Furthermore, we examine the evolution of numerical solutions generated by the second-order Strang splitting scheme (11). With $\tau = 0.01$, the progression of numerical solutions for $\alpha = -0.01, \alpha = -0.5$ and $\alpha = -0.99$ at $T = 1, 5, 10$ is clearly depicted in Figure 1, respectively.

Table I. L^2-errors and convergence rates in time for 1D perturbed FHN neuron model with $h = \frac{1}{128}$ at $T = 1$.

	α=-0.01		α=-0.5		α=-0.99	
τ	Err_{L^2}	Rate	Err_{L^2}	Rate	Err_{L^2}	Rate
2^{-1}	2.9205E-04	-	1.2396E-03	-	3.4353E-03	-
2^{-2}	6.6265E-05	2.1399	2.7366E-04	2.1795	7.5560E-04	2.1847
2^{-3}	1.5848E-05	2.0640	6.5131E-05	2.0710	1.8293E-04	2.0464
2^{-4}	3.8794E-06	2.0304	1.5942E-05	2.0305	4.5356E-07	2.0119
2^{-5}	9.5993E-07	2.0148	3.9467E-06	2.0141	1.1312E-07	2.0034

Table II. L^2-errors and convergence rates in time for 2D perturbed FHN neuron model with $h_x = h_y = \frac{\pi}{128}$ at $T = 1$.

	α=-0.01		α=-0.5		α=-0.99	
τ	Err_{L^2}	Rate	Err_{L^2}	Rate	Err_{L^2}	Rate
2^{-3}	9.0946E-08	-	8.0749E-05	-	9.2842E-04	-
2^{-4}	2.2783E-08	1.9970	2.0678E-05	1.9654	2.4323E-04	1.9325
2^{-5}	5.7014E-09	1.9986	5.2315E-06	1.9828	6.2239E-05	1.9664
2^{-6}	1.4259E-09	1.9995	1.3155E-06	1.9916	1.5739E-05	1.9835
2^{-7}	3.5633E-10	2.0005	3.2962E-07	1.9967	3.9547E-06	1.9927

B. 2D perturbed FHN neuron model

In this subsection, we demonstrate of the simulation of the 2D perturbed FHN neuron model using our proposed scheme (11). The computational domain is set to $\Omega = [0, 2\pi]\times[0, 2\pi]$. The initial condition is specified by

$$u_0(x, y) = 0.05 \sin(x) \sin(y).$$

For this simulation, we take $\tau = 2^{-3}, 2^{-4}, ..., 2^{-7}$ and $h_x = h_y = \frac{\pi}{128}$. The remaining parameters are chosen as $\varepsilon = 0.1$ and $\alpha = -0.01, -0.5, -0.99$. The results of the L^2-norm errors and temporal convergence rates with different α are compiled in Table 2. It is evident from the results that the convergence rate adheres to a second-order accuracy. Figure 2 illustrates the temporal development of the numerical solution with $\tau = 0.01$ and $\alpha = -0.5$ at $T = 1, 5, 10$, showcasing the dynamic behavior of the model at these specific time frames.

V. Conclusion

In this work, a numerical scheme for the perturbed FHN equation with a small parameter is developed and analyzed. The FHN model is a nonlinear partial differential equation that involves second-order spatial derivative. To develop an efficient time-stepping numerical scheme, we use operator s-plitting method to realize numerical scheme based on Fourier spectral method and SSP-RK2 method. The convergence of the proposed scheme is proved rigorously. All kinds of 1D and 2D numerical experiments are performed to validate the accuracy and efficiency.

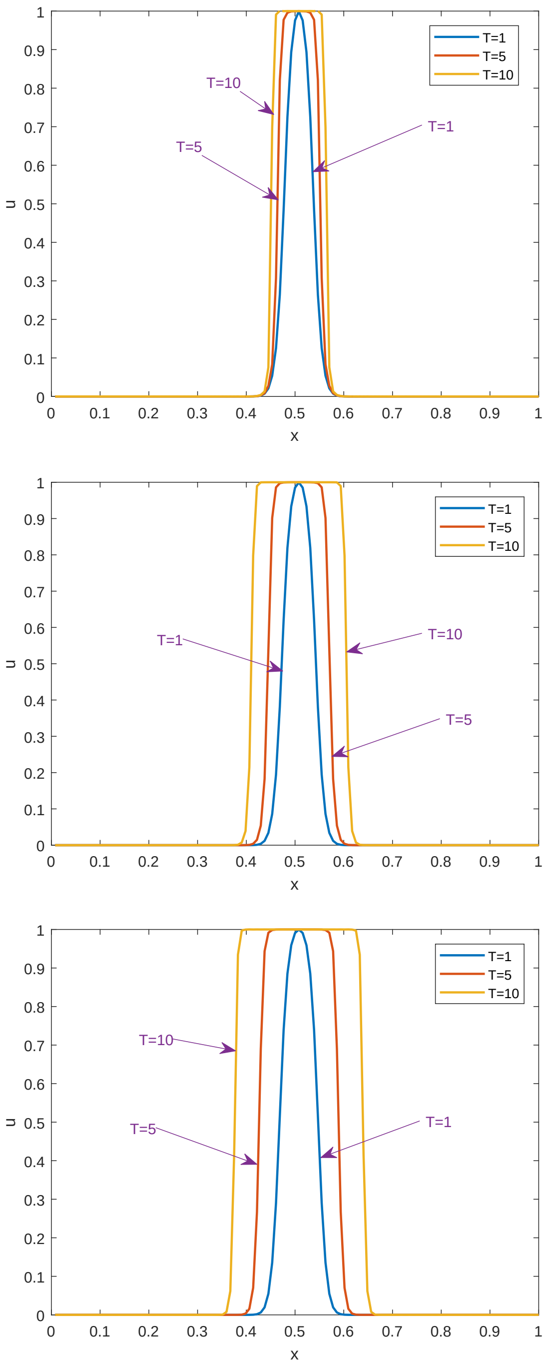

Fig. 1: The numerical solutions for 1D perturbed FHN neuron model with $\alpha = -0.01, \alpha = -0.5$ and $\alpha = -0.99$ at $T = 1, 5, 10$, respectively.

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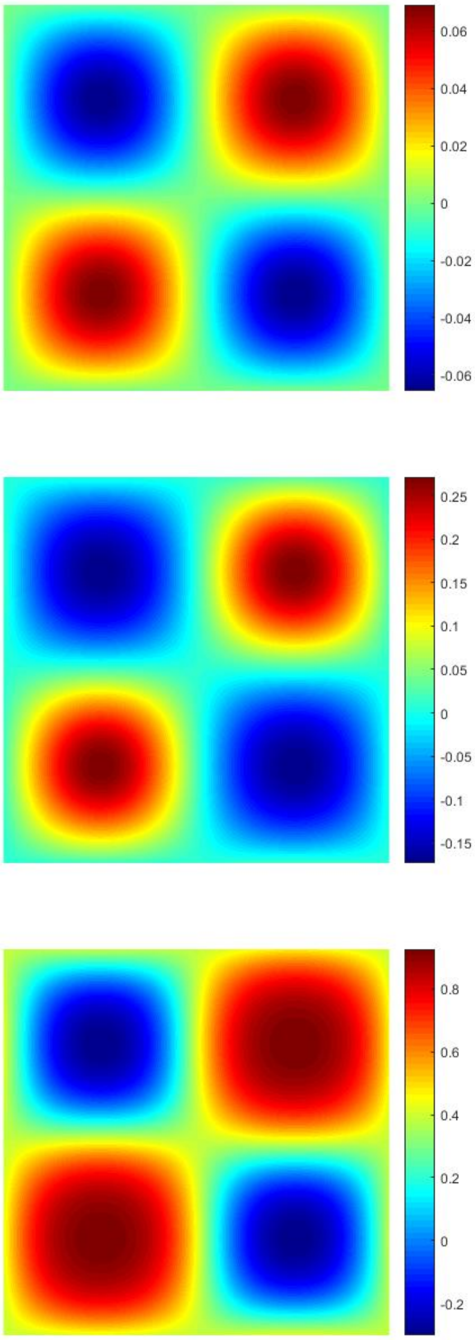

Fig. 2: The evolution of the numerical solution for 2D perturbed FHN neuron model with $\alpha = -0.5$ at $T = 1, 5, 10$, respectively.

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