A Further Study of the Analytical Properties of the Generalized Hurwitz-Lerch Zeta Function and Pathway Fractional Integral Operator

Savita Panwar, Rupakshi Mishra Pandey, Prakriti Rai, Pankaj Mathur

***Abstract*—This article introduces a novel generalization of the Hurwitz-Lerch Zeta function, which is precisely reducible to several remarkable extensions of the Hurwitz-Lerch Zeta function. Our new version of the Hurwitz-Lerch Zeta function is defined with the help of the generalized Beta function. Analytical characteristics such as differential formulas, generating functions, and multiple integral representations have been studied in further detail. We study the pathway fractional integral formulas for our newly generalized formation of the extended Hurwitz-Lerch Zeta functions. We attain several particular and limiting cases of our main results. We additionally look at some statistical uses of our defined Hurwitz-Lerch Zeta function in probability distribution theory.**

***Index Terms*—Extended Hurwitz-Lerch Zeta function, Mittag-Leffler function, Generating functions, Generalized beta function, Pathway fractional integral operator, Probability density function.**

I. Introduction

THE Hurwitz-Lerch Zeta function (HLZf) and all of its extended forms are substantially used in numerous branches of mathematics and physics. Răducanu and Srivastava [4], in their extensive study of numerous analytic functions Classes in the theory of geometric functions in complex analysis, utilized the HLZf $\Phi(z, s, a)$ to derive a special linear convolution operator. Gupta et al. [13] also discussed their research on statistical inference, reliability characteristics, and structural characteristics of the widely-known HLZ distribution. Recalling the HLZf [1]

$$\Phi(\xi, t, w) = \sum_{j=0}^{\infty} \frac{\xi^j}{(j+w)^t}, \tag{1}$$

$$(w \neq 0, -1, -2, ...;\ t \in \mathbb{C}\ when\ |\xi| < 1),$$

and its integral representation is given as

$$\Phi(\xi, t, w) \quad = \quad \frac{1}{\Gamma(t)} \int_0^{\infty} \frac{z^{t-1} e^{-wz}}{1 - \xi e^{-z}}\, dz \tag{2}$$

$$(\Re(t) > 0;\ \Re(w) > 0\ when\ |\xi| < 1).$$

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Goyal et al. [17] introduced the HLZf as follows:

$$\Phi^*_{\vartheta}(\xi, t, w) = \sum_{j=0}^{\infty} \frac{(\vartheta)_j}{j!} \frac{\xi^j}{(j+w)^t}, \tag{3}$$

$$(\vartheta \in \mathbb{C};\ w \neq 0, -1, -2, ...;\ t \in \mathbb{C}\ when\ |\xi| < 1),$$

in which the Pochhammer symbol $(\vartheta)_j$ characterises as [7]

$$(\vartheta)_j = \begin{cases} \vartheta(\vartheta+1)\dots(\vartheta+j-1) & \text{if } j \geq 1, \vartheta \in \mathbb{C}, \\ 1 & j = 0, \vartheta \in \mathbb{C} \setminus \{0\} \end{cases}$$

The following integral representation of (3) is given as:

$$\Phi^*_{\vartheta}(\xi, t, w) = \frac{1}{\Gamma(t)} \int_0^{\infty} \frac{x^{t-1} e^{-wx}}{(1 - \xi e^{-x})^{\vartheta}}\, dx \tag{4}$$

$$(\Re(w) > 0; \Re(t) > 0\ when\ |\xi| < 1).$$

Firstly, Parmar et al. [15] made use of beta function to generalize the HLZf. Further, Rahman et al. [6] define the HLZf's extension in the following way:

$$\Phi^{\zeta,\chi}_{\varrho,\vartheta,\alpha}(\xi, t, w) = \sum_{j=0}^{\infty} \frac{(\vartheta)_j}{j!} \frac{B_{\varrho,\alpha}(\zeta+j, \chi-\zeta)}{B(\zeta, \chi-\zeta)} \frac{\xi^j}{(j+w)^t}, \tag{5}$$

$$(\varrho \geq 0, \alpha > 0, \vartheta, \zeta \in \mathbb{C}, \chi, w \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ t \in \mathbb{C}\ when\ |\xi| < 1)$$

where $B_{\varrho,\alpha}(\chi_1, \chi_2)$ represents the extension of beta function established by Shadab et al. [9] and defined in the following way:

$$B_{\varrho,\alpha}(\chi_1, \chi_2) = \int_0^1 z^{\chi_1-1}(1-z)^{\chi_2-1} E_{\alpha}\left(\frac{-\varrho}{z(1-z)}\right) dz, \tag{6}$$

where $\Re(\chi_1) > 0, \Re(\chi_2) > 0, \Re(\varrho) \geq 0$. and In [5], the Mittag-Leffler function $E_{\alpha}(.)$ is described as

$$E_{\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}.$$

Further, Panwar et al. [18] define the generalized form of extended beta function in the following manner:

$$B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_1, \chi_2) = \int_0^1 z^{\chi_1-1}(1-z)^{\chi_2-1} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho}{z^{\lambda}(1-z)^{\lambda}}\right) dz, \tag{7}$$

where, $\alpha, \varkappa, \mu_l, \nu_l, \beta_n, \kappa_n \in \mathbb{C}$ with min $\Re(\alpha, \varkappa, \mu_l, \nu_l, \beta_n, \kappa_n) > 0$ for $l = 1, \dots r_1; n = 1, \dots, r_2$, $r_1 + r_2 = m - 2$, where $m \in \mathbb{Z}^+$ and $\Re(\chi_1) > 0, \Re(\chi_2) > 0, \Re(\varrho) \geq 0, \Re(\lambda) > 0$, and

$E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(z)$ is the m-parameter Mittag-Leffler function created by Agarwal et al. [14] as follows:

$$E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(z)=\sum_{j=0}^{\infty}\frac{(\mu_1)_{\nu_1 j}(\mu_2)_{\nu_2 j}\ldots(\mu_{r_1})_{\nu_{r_1} j}}{\Gamma(\alpha j+\varkappa)(\beta_1)_{\kappa_1 j}\ldots(\beta_{r_2})_{\kappa_{r_2} j}}z^j \quad (8)$$

where $\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n \in \mathbb{C}$ with min $\Re\{\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n\}>0$ for $l=1\ldots,r_1$ and $n=1\ldots,r_2$ with $r_1+r_2=m-2$, the complex variable z and any positive integer m.

In [18], they also defined the generalized hypergeometric function as follows:

$$F^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_1,\chi_2;\chi_3;w) = \sum_{j=0}^{\infty}(\chi_1)_j\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_2+j,\chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\frac{w^j}{j!}, \quad (9)$$

where $\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n \in \mathbb{C}$ with min $\Re(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n)>0$ for $l=1,\ldots r_1; n=1,\ldots,r_2; r_1+r_2=m-2$, where $m\in\mathbb{C}$ and $\Re(\chi_3)>\Re(\chi_2)>0, \Re(\varrho)\geq 0, \Re(\lambda)>0$ and $|w|<1$.

Recently, Nair et al. [19] introduced a novel fractional integral operator known as the Pathway fractional integral operator. This operator has numerous applications in various fields of science (for more details, see [8], [3]) and is defined as follows:

$$(P^{\eta,\Upsilon}_{0+}f)(y_1) = y_1^{\eta}\int_0^{\left[\frac{y_1}{a_1(1-\Upsilon)}\right]}\left(1-\frac{a_1(1-\Upsilon)x_1}{y_1}\right)^{\frac{\eta}{(1-\Upsilon)}}f(x_1)\,dx_1, \quad (10)$$

where f is a Lebesgue measurable function, $\eta\in\mathbb{C},\Re(\eta)>0, a_1>0$ and $\Upsilon<1$, represents a pathway parameter. The pathway model for scalar random variables and a real scalar Υ is shown by the probability density function [2] as follows:

$$f(y_1)=\frac{c}{|y_1|^{1-v}}[1-a_1(1-\Upsilon)|y_1|^{\vartheta}]^{\frac{a_2}{1-\Upsilon}}, \quad (11)$$

where $y_1\in(-\infty,\infty); \vartheta>0; a_2\geq 0; 1-a_1(1-\Upsilon)|y_1|^{\vartheta}>0; v>0$ and c, Υ stand for the normalizing constant, pathway parameter, respectively. For $\Upsilon\in\mathbb{R}$, the normalizing constant articulated in the following way:

$$c=\begin{cases}\frac{1}{2}\frac{\vartheta[a_1(1-\Upsilon)]^{\frac{v}{\vartheta}}\Gamma(\frac{v}{\vartheta}+\frac{a_2}{1-\Upsilon}+1)}{\Gamma(\frac{v}{\vartheta})\Gamma(1+\frac{a_2}{1-\Upsilon})}, & \text{if } (\Upsilon<1),\\ \frac{1}{2}\frac{\vartheta[a_1(\Upsilon-1)]^{\frac{v}{\vartheta}}\Gamma(\frac{a_2}{\Upsilon-1})}{\Gamma(\frac{v}{\vartheta})\Gamma(\frac{a_2}{\Upsilon-1}-\frac{v}{\vartheta})}, & \text{if } (\frac{1}{\Upsilon-1}-\frac{v}{\vartheta}>0,\Upsilon>1),\\ \frac{1}{2}\frac{\vartheta[a_1\,a_2]^{\frac{v}{\vartheta}}}{\Gamma(\frac{v}{\vartheta})}, & \text{if } (\Upsilon\to 1)\end{cases}$$

For $\Upsilon<1$ and $1-a_1(1-\Upsilon)|y_1|^{\vartheta}>0$, the pathway density function in (11), includes the uniform density, the triangular density, and lots of more p.d.f. For $\Upsilon>1$, and put $(1-\Upsilon)=-(\Upsilon-1)$ in (10) yields

$$(P^{\eta,\Upsilon}_{0+}f)(y_1)=y_1^{\eta} \times\int_0^{\left[\frac{y_1}{-a_1(\Upsilon-1)}\right]}\left(1+\frac{a_1(\Upsilon-1)x_1}{y_1}\right)^{\frac{\eta}{-(\Upsilon-1)}}f(x_1)\,dx_1,$$

and

$$f(y_1)=\frac{c}{|y_1|^{1-v}}[1+a_1(\Upsilon-1)|y_1|^{\vartheta}]^{\frac{a_2}{-(\Upsilon-1)}}, \quad (12)$$

where $y_1\in(-\infty,\infty); \vartheta>0; a_2\geq 0; \Upsilon>1$ describes the generalized type 2 beta model for real values of x_1. For $\Upsilon\to 1$, (10) reduces to the Laplace integral transform.

Also, $\Upsilon=0, a_1=1$ and for $\eta-1$ in place of η in (10), then we get the Riemann-Liouville fractional integral operator

$$\big(P^{\eta-1,0}_{0+}f\big)(y_1)=\Gamma(\eta)\big(I^{\eta}_{0+}f\big)(y_1), (\Re(\eta)>1). \quad (13)$$

The Riemann-Liouville fractional derivative operator D^h_z is defined by

$$D^h_z\{g(z)\}=\frac{1}{\Gamma(-h)}\int_0^z(z-t)^{-h-1}g(t)\,dt, \quad (14)$$
$$(\Re(h)<0)$$

and

$$D^h_z\{g(z)\}=\frac{d^s}{dz^s}\{D^{h-s}_z\{g(z)\}\}, \quad (15)$$
$$(s-1\leq\Re(h)<s, s\in\mathbb{N}).$$

Also, we have

$$D^h_z\{z^s\}=\frac{\Gamma(s+1)}{\Gamma(s-h+1)}z^{s-h}, (\Re(s)>-1). \quad (16)$$

We have the Fox-Wright function ${}_l\psi_r(z)$ $(l,r\in\mathbb{N}_0)$ with l upper and r lower parameters where $w_1,w_2,\ldots,w_l\in\mathbb{C}$ and $y_1,y_2,\ldots,y_r\in\mathbb{C}\setminus\mathbb{Z}_0^-$ (see [7])

$${}_l\psi_r\left[\begin{matrix}(w_1,x_1),\ldots,(w_l,x_l);\\(y_1,z_1),\ldots,(y_r,z_r);\end{matrix}z\right] = \sum_{m=0}^{\infty}\frac{\Gamma(w_1+x_1m)\cdots\Gamma(w_l+x_lm)}{\Gamma(y_1+z_1m)\cdots\Gamma(y_r+z_rm)}\frac{z^m}{m!}, \quad (17)$$

where the coefficients $x_1,\ldots,x_l,z_1,\ldots,z_r\in\mathbb{R}^+$ are such that

$$1+\sum_{n=1}^{r}z_n-\sum_{i=1}^{l}x_i\geq 0.$$

In this study, we employ the notion of the Hadamard product, which allows us to decompose a recently developed function into popular ones.

Definition I.1. *Let m(y) and n(y) represent two power series, where the radii of convergence of those series are denoted by r_m and r_n, respectively. Let*

$$m(y):=\sum_{k=0}^{\infty}a_ky^k \quad (|y|<r_m)$$

and

$$n(y):=\sum_{k=0}^{\infty}b_ky^k \quad (|y|<r_n)$$

Also, the Hadamard product involving two power series, which is again a power series, is described by(for details, see [20])

$$(m*n)(y):=\sum_{k=0}^{\infty}a_kb_ky^k \quad (|y|<r), \quad (18)$$

Its radius of convergence, r, fulfils

$$r_mr_n\leq r.$$

Numerous extensions of the HLZf's and their utilization have been shown in the literature (see references [4], [10], [13]). Recently, Pathan et al. [11] have presented and thoroughly examined numerous characteristics and results of their extension of the HLZf. Motivated by the work done above, we look into the HLZf's generalization, properties, and its utilization in statistical distribution theory.

This is the layout of our article: In Section II, we present the HLZf in generalized form, along with a couple of particular and limiting cases. In Section III, we discuss a few of the significant characteristics, like its integral representations, its derivatives, and its generating relations. In Section IV, we derive the Pathway fractional integral formulas involving the generalized HLZf. Finally, in Section V, we conclude by discussing an application to distribution theory.

II. A Generalized Hurwitz-Lerch Zeta function

This section defines a novel generalized form of the extended HLZf, making use of the generalized beta function. The generalized Hurwitz-Lerch Zeta function (GHLZf) is defined as

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) = \sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\xi^j}{(j+w)^t}, \tag{19}$$

where $\varrho \geq 0, \lambda > 0$; $\vartheta,\varsigma \in \mathbb{C}$; $w,\chi \in \mathbb{C}\setminus\mathbb{Z}_0^-$; $t \in \mathbb{C}$ and $\min(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n) > 0$ for $l = 1,\dots r_1; n = 1,\dots,r_2, r_1+r_2 = m-2$, where $m \in \mathbb{Z}^+$, when $|\xi| < 1$.

Remark: We outline the specific and limiting cases of our newly established GHLZf here:

1) For $r_1 = r_2 = 0, \varkappa = 1 = \lambda$, we get the HLZf given by Rahman et al. [6]

$$\Phi^{\varrho,\vartheta}_{\alpha,\varsigma,\chi}(\xi,t,w) = \sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B_{\varrho,\alpha}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\xi^j}{(j+w)^t}, \tag{20}$$

$$(\varrho \geq 0, \alpha > 0, \vartheta,\varsigma,\chi \in \mathbb{C}, w \in \mathbb{C}\setminus\mathbb{Z}_0^-,$$
$$t \in \mathbb{C}\ for\ |\xi| < 1).$$

2) For $r_1 = r_2 = 0, \alpha = \varkappa = \lambda = 1$, we get the HLZf given by Parmar et al. [15]

$$\Phi^{\varrho,\vartheta}_{\varsigma,\chi}(\xi,t,w) = \sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B_{\varrho}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)} \times\frac{\xi^j}{(j+w)^t}, \tag{21}$$

$(\varrho \geq 0; \vartheta,\varsigma \in \mathbb{C}; w,\chi \in \mathbb{C}\setminus; t \in \mathbb{C}\ \ when\ |\xi| < 1)$.

3) For $r_1 = r_2 = 0 = \varrho, \alpha = \varkappa = \lambda = 1$, then yield the GHLZf introduced by Garg et al. [10]

$$\Phi_{\vartheta,\varsigma,\chi}(\xi,t,w) = \sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{(\varsigma)_j}{(\chi)_j}\frac{\xi^j}{(j+w)^t}, \tag{22}$$

$$(\vartheta,\varsigma \in \mathbb{C}; \chi, w \in \mathbb{C}\setminus\mathbb{Z}_0^-; t \in \mathbb{C}\ when\ |\xi| < 1).$$

4) The limiting case of our novel GHLZf is given by

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},*}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) = \lim_{|\vartheta|\to\infty}\left\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},*}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}\left(\frac{\xi}{\vartheta},t,w\right)\right\} = \sum_{j=0}^{\infty}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)j!}\frac{\xi^j}{(j+w)^t}, \tag{23}$$

where $\varrho \geq 0, \lambda > 0$; $\vartheta,\varsigma \in \mathbb{C}$; $w,\chi \in \mathbb{C}\setminus\mathbb{Z}_0^-$; $t \in \mathbb{C}$ and $\min(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n) > 0$ for $l = 1,\dots r_1; n = 1,\dots,r_2, r_1+r_2 = m-2$, where $m \in \mathbb{Z}^+$, when $|\xi| < 1$.

III. Analytical Properties of the Generalized Hurwitz-Lerch Zeta function

A. Integral Representations

Theorem III.1. *For $\varrho \geq 0, \lambda > 0$, $\Re(\vartheta) > 0, \Re(t) > 0, \Re(w) > 0$ and min $(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n) > 0$ for $l = 1,\dots r_1; n = 1,\dots,r_2, r_1+r_2 = m-2$, where $m \in \mathbb{Z}^+$, when $|\xi| < 1$ Thus, the integral representation shown below is true.*

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) = \frac{1}{\Gamma(t)}\int_0^{\infty} x^{t-1}e^{-wx}F^{\varrho,\lambda,(\mu,\nu)_{r_1}}_{\alpha,\varkappa,(\beta,\kappa)_{r_2}}(\vartheta,\varsigma;\chi;\xi e^{-x})\,dx.$$

Proof: We notice that the Gamma function's Eulerian integral is

$$\frac{1}{(j+w)^t} = \frac{1}{\Gamma(t)}\int_0^{\infty} x^{t-1}e^{-(j+w)x}\,dx \tag{24}$$

$$(min\{\Re(t),\Re(w) > 0\}, j \in \mathbb{N}_0).$$

Using the results noted above in equation (19) and then inverting the summation and integration orders, under the conditions of Theorem III.1, we attain

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) = \frac{1}{\Gamma(t)}\int_0^{\infty} x^{t-1}e^{-wx} \times \left(\sum_{j=0}^{\infty}(\vartheta)_j\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{(\xi e^{-x})^j}{j!}\right)dx \tag{25}$$

utilizing the equation (9), we attained the intended result. ■

Remark III.2. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(3.1) in [6].*

Theorem III.3. *For $\varrho \geq 0, \lambda > 0$, $\Re(\vartheta) > 0, \Re(t) > 0, \Re(w) > 0$ and min $(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n) > 0$ for $l = 1,\dots r_1; n = 1,\dots,r_2, r_1+r_2 = m-2$, where $m \in \mathbb{Z}^+$, when $|\xi| < 1$ Thus, the integral representation shown below is true.*

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) := \frac{1}{\Gamma(\vartheta)}\int_0^{\infty}(x^{\vartheta-1}e^{-x} \times\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},*}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi x,t,w))\,dx \tag{26}$$

where $\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},*}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi x,t,w)$ is the limiting case in (23).

Proof: We have the following Pochhamer symbol's integral representation:

$$(\vartheta)_j = \frac{1}{\Gamma(\vartheta)}\int_0^{\infty} x^{\vartheta+j-1}e^{-x}\,dx. \tag{27}$$

Using this integral representation in (19) and then inverting the summation and integration orders, under the conditions of Theorem III.3, we attain

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) = \frac{1}{\Gamma(\vartheta)}\int_0^\infty x^{\vartheta-1}e^{-x} \times \sum_{j=0}^{\infty}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{(\xi x)^j}{j!(j+w)^t}\,dx \tag{28}$$

Now, using (23), we arrive at the intended result. ■

Remark III.4. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(3.3) in [6].*

Theorem III.5. *The following integral representations hold true:*

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) := \frac{\Gamma(\chi)}{\Gamma(\varsigma)\Gamma(\chi-\varsigma)} \times \int_0^\infty \frac{u^{\varsigma-1}}{(1+u)^{\chi}} E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(1+u)^{2\lambda}}{u^{\lambda}}\right) \times\Phi^*_{\vartheta}\left(\frac{\xi u}{1+u},t,w\right)du \tag{29}$$

and

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) := \frac{\Gamma(\chi)}{\Gamma(t)\Gamma(\varsigma)\Gamma(\chi-\varsigma)} \times \int_0^\infty\int_0^\infty \frac{u^{\varsigma-1}e^{-wx}x^{t-1}}{(1+u)^{\chi}} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(1+u)^{2\lambda}}{u^{\lambda}}\right)\left(1-\frac{\xi u e^{-x}}{1+u}\right)^{-\vartheta}dxdu \tag{30}$$

where $\varrho \geq 0, \lambda > 0$; $\Re(t) > 0, \Re(w) > 0, \Re(\chi) > \Re(\varsigma) > 0$, and $\min(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n) > 0$ for $l = 1,\ldots r_1; n = 1,\ldots,r_2, r_1+r_2 = m-2, m \in \mathbb{Z}^+$, when $|\xi| < 1$, assuming that (29) *and* (30)*'s integrals on their right-hand sides converge.*

Proof: *By taking $\chi_1 = \varsigma + j$ and $\chi_2 = \chi - \varsigma$ in the subsequent representation of $B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_1,\chi_2)$ (see [18])*

$$B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_1,\chi_2) = \int_0^\infty \frac{u^{\chi_1-1}}{(1+u)^{\chi_1+\chi_2}} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(1+u)^{2\lambda}}{u^{\lambda}}\right)du, \tag{31}$$

we get

$$B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma) = \int_0^\infty \frac{u^{\varsigma+j-1}}{(1+u)^{\chi+j}} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(1+u)^{2\lambda}}{u^{\lambda}}\right)du, \tag{32}$$

when considering the above relationship and applying the definition (19)*, equation* (3)*, clearly gives the first statement of Theorem. III.5*
Furthermore, we derive the requisite result (30) *by utilising the integral representation* (4) *in equation* (29). ■

Remark III.6. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(3.2) in [6].*

Theorem III.7. *The following integral representations hold true:*

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) := \frac{(b-a)^{1-\chi}\Gamma(\chi)}{\Gamma(\varsigma)\Gamma(\chi-\varsigma)} \int_a^b (u-a)^{\varsigma-1}(b-u)^{\chi-\varsigma-1} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(b-a)^{2\lambda}}{(u-a)^{\lambda}(b-u)^{\lambda}}\right) \times\Phi^*_{\vartheta}\left(\xi\left(\frac{u-a}{b-a}\right),t,w\right)du \tag{33}$$

and

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) := \frac{(b-a)^{1-\chi}\Gamma(\chi)}{\Gamma(\varsigma)\Gamma(\chi-\varsigma)\Gamma(t)} \int_a^b\int_0^\infty (u-a)^{\varsigma-1}(b-u)^{\chi-\varsigma-1}e^{-wx}x^{t-1} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(b-a)^{2\lambda}}{(u-a)^{\lambda}(b-u)^{\lambda}}\right) \times\left(1-\xi\left(\frac{u-a}{b-a}\right)e^{-x}\right)^{-\vartheta}dxdu \tag{34}$$

where $\varrho \geq 0, \lambda > 0$; $\Re(t) > 0, \Re(w) > 0, \Re(\chi) > \Re(\varsigma) > 0$, and $\min(\alpha,\varkappa,\mu_l,\nu_l,\beta_n,\kappa_n) > 0$ for $l = 1,\ldots r_1; n = 1,\ldots,r_2, r_1+r_2 = m-2, m \in \mathbb{Z}^+$, when $|\xi| < 1$, assuming that (33) *and* (34)*'s integrals on their right-hand sides converge.*

Proof: *By taking $\chi_1 = \varsigma + j$ and $\chi_2 = \chi - \varsigma$ in the subsequent representation of $B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_1,\chi_2)$ (see [18])*

$$B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\chi_1,\chi_2) = (b-a)^{1-\chi_1-\chi_2} \times \int_a^b (u-a)^{\chi_1-1}(b-u)^{\chi_2-1} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(b-a)^{2\lambda}}{(u-a)^{\lambda}(b-u)^{\lambda}}\right)du, \tag{35}$$

we get

$$B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma) = (b-a)^{1-j-\chi} \times \int_a^b (u-a)^{\varsigma+j-1}(b-u)^{\chi-\varsigma-1} \times E^{(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}\left(\frac{-\varrho(b-a)^{2\lambda}}{(u-a)^{\lambda}(b-u)^{\lambda}}\right)du, \tag{36}$$

when considering the above relationship and applying the definition (19)*, equation* (3)*, clearly gives the first statement of Theorem. III.7*
Furthermore, we derive the requisite result (34) *by utilising the integral representation* (4) *in equation* (33). ■

B. Derivative Formula

Theorem III.8. *The following differential formula for $\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)$ holds true:*

$$\frac{d^j}{d\xi^j}\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\} = \frac{(\vartheta)_j(\varsigma)_j}{(\chi)_j} \times \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta+j}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma+j,\chi+j}(\xi,t,w+j). \tag{37}$$

where $j \in \mathbb{N}$.

Proof: Taking into consideration the derivative of equation (19) with respect to ξ, we obtain

$$\frac{d}{d\xi}\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\} = \sum_{j=1}^{\infty}\frac{(\vartheta)_j}{(j-1)!}\times \frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\xi^{j-1}}{(j+w)^t}. \tag{38}$$

Replacing j by j+1 in equation (38) and utilizing the identities

$$\begin{aligned} B(\chi_1,\chi_2-\chi_1) &= \frac{\chi_2}{\chi_1}B(\chi_1+1,\chi_2-\chi_1),\\ (\vartheta)_{s+1} &= \vartheta(\vartheta+1)_s, \end{aligned} \tag{39}$$

we have

$$\frac{d}{d\xi}\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\} = \frac{\vartheta\varsigma}{\chi}\times \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta+1}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma+1,\chi+1}(\xi,t,w+1). \tag{40}$$

repeating the process j times, we attain the required result. ■

Remark III.9. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(3.4) in [6].*

C. Fractional Derivative of $\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)$.

For Riemann-Liouville fractional derivative operator $D^h_z\{z^s\}$ of order h, as stated above by (14). We now create the following relation.

Theorem III.10. *Let $\{(\varsigma-l),(\vartheta-l),(\chi-l)\} \neq \mathbb{Z}_0^-$, $l \in \mathbb{N}_0$. Then*

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w) = \frac{(-1)^l(1-\chi)_l}{(1-\vartheta)_l(1-\varsigma)_l} D^l_\xi\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta-l}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma-l,\chi-l}(\xi,t,w-l)\}. \tag{41}$$

Proof: Taking R.H.S and denote it by K . Then, using (16) *and* (19)*, we get*

$$K = \frac{(-1)^l(1-\chi)_l}{(1-\vartheta)_l(1-\varsigma)_l}\sum_{j=0}^{\infty}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j-l,\chi-\varsigma)}{B(\varsigma-l,\chi-\varsigma)} \frac{(\vartheta-l)_j}{(j-l)!}\frac{\xi^{j-l}}{(j+w-l)^t}$$

Now, by letting j = j + l and utilizing the identities

$$B(\chi_1-l,\chi_2-\chi_1) = \frac{(1-\chi_2)_l}{(1-\chi_1)_l}B(\chi_1,\chi_2-\chi_1) \tag{42}$$

and

$$(x)_{-r} = \frac{(-1)^r}{(1-x)_r} \tag{43}$$

we get

$$= \sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\xi^j}{(j+w)^t}, \tag{44}$$

We get the left side of (41). ■

Remark III.11. *It is easy to reduce the fractional relation mentioned above to the results found in [11], by selecting the appropriate parameters,.*

D. Generating Functions

Theorem III.12. *For $p \geq 0, \lambda > 0,\ \vartheta \in \mathbb{C}$ and $|x| < 1$, Then, the generating function given below is true:*

$$\sum_{j=0}^{\infty}(\vartheta)_j\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta+j}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\frac{x^j}{j!} = (1-x)^{-\vartheta}\times \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}\left(\frac{\xi}{1-x},t,w\right) \tag{45}$$

Proof: Taking into consideration the L.H.S of Theorem III.12 and denoted by X and using (19), we get

$$X = \sum_{j=0}^{\infty}(\vartheta)_j\left\{\sum_{l=0}^{\infty}(\vartheta+j)_l\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+l,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\times \frac{\xi^l}{(l+w)^t l!}\right\}\frac{x^j}{j!}. \tag{46}$$

Inverting the summation order's under the above conditions and after simplification, we have

$$\begin{aligned} &= \sum_{l=0}^{\infty}(\vartheta)_l\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+l,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\\ &\times \left\{\sum_{j=0}^{\infty}(\vartheta+l)_j\frac{x^j}{j!}\right\}\frac{\xi^l}{l!(l+w)^t}, \end{aligned} \tag{47}$$

Now, using the binomial expansion

$$(1-x)^{-(\vartheta+l)} = \sum_{j=0}^{\infty}(\vartheta+l)_j\frac{x^j}{j!}, |x| < 1. \tag{48}$$

and using (19), we attain the intended result. ■

Remark III.13. *For $\varkappa = 1$ and $r_1 = r_2 = 0$, we get the Theorem(4.2) in [6].*

Theorem III.14. *For $\varrho \geq 0, \lambda > 0,\ \vartheta \in \mathbb{C}$ and $|x| < |w|$; $t \neq 1$. Then, the generating function given below is true:*

$$\sum_{j=0}^{\infty}\frac{(t)_j}{j!}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t+j,w)x^j = \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w-x). \tag{49}$$

Proof: Using the (19) in the R.H.S of Theorem III.14, we get

$$\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w-x) = \sum_{l=0}^{\infty}(\vartheta)_l \times \frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+l,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\xi^l}{l!(l+w-x)^t}, \tag{50}$$

$$\begin{aligned} &\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w-x)\\ &= \sum_{l=0}^{\infty}(\vartheta)_l\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+l,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\xi^l}{l!(l+w)^t}\\ &\times \left(1-\frac{x}{l+w}\right)^{-t} \end{aligned} \tag{51}$$

using the binomial expansion (48) and after some simplification, we get the intended result. ■

Remark III.15. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(4.3) in [6].*

IV. Generalized Hurwitz-Lerch Zeta function and Pathway fractional integral operator

In this section, we obtain the Pathway fractional integral formulae involving the generalized HLZf.

Theorem IV.1. *For $\varrho \geq 0, \lambda > 0$; $\vartheta, \varsigma \in \mathbb{C}$; $w, \chi \in \mathbb{C}\setminus\mathbb{Z}_0^-$; , $\Re(\varpi) > 0, \Re(\eta) > 0; \Re(\frac{\eta}{1-\Upsilon}) > -1, \Upsilon < 1, \theta \in \mathbb{R}$; $t \in \mathbb{C}$ and min $(\alpha, \varkappa, \mu_l, \nu_l, \beta_n, \kappa_n) > 0$ for $l = 1, \ldots r_1; n = 1, \ldots, r_2, r_1 + r_2 = m - 2$, where $m \in \mathbb{Z}^+$, when $|\xi| < 1$. The following formula holds true:*

$$\begin{aligned} & P_{0+}^{\eta,\Upsilon}\big(x_1^{\varpi-1}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\theta x_1,t,w)\big)(y_1) \\ = & \frac{y_1^{\varpi+\eta}\Gamma\big(1+\frac{\eta}{1-\Upsilon}\big)}{[a_1(1-\Upsilon)]^{\varpi}} \\ \times & \ \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}\left\{\frac{\theta y_1}{a_1(1-\Upsilon)},t,w\right\} \\ \times & \ {}_2\psi_1\left[\begin{matrix}(\varpi,1),(1,1); \\ (\varpi+\frac{\eta}{1-\Upsilon}+1,1);\end{matrix}\frac{\theta y_1}{a_1(1-\Upsilon)}\right]. \end{aligned} \quad (52)$$

Proof: Using the definition (10) of Pathway fractional integral operator , we have

$$K = y_1^{\eta}\int_0^{[\frac{y_1}{a_1(1-\Upsilon)}]} x_1^{\varpi-1}\left[1-\frac{a_1(1-\Upsilon)x_1}{y_1}\right]^{\frac{\eta}{1-\Upsilon}} \times \sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{(\theta x_1)^j}{(j+w)^t}\,dx_1. \quad (53)$$

By inverting the summation and integration orders, we have

$$\begin{aligned} K = & \ y_1^{\eta}\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\theta^j}{(j+w)^t} \\ \times & \int_0^{[\frac{y_1}{a_1(1-\Upsilon)}]} x_1^{\varpi+j-1}[1-\frac{a_1(1-\Upsilon)x_1}{y_1}]^{\frac{\eta}{1-\Upsilon}}\,dx_1. \end{aligned}$$

Utilizing the substitution Z = $\frac{a_1(1-\Upsilon)x_1}{y_1}$, we get

$$\begin{aligned} K = & \ \frac{y_1^{\varpi+\eta}}{[a_1(1-\Upsilon)]^{\varpi}}\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)} \\ \times & \ \frac{\theta^j}{(j+w)^t}\left(\frac{y_1}{a_1(1-\Upsilon)}\right)^j \\ \times & \int_0^1 Z^{\varpi+j-1}[1-Z]^{\frac{\eta}{1-\Upsilon}}\,dZ. \end{aligned} \quad (54)$$

Using the Beta function [7], we have

$$\begin{aligned} K = & \ \frac{y_1^{\varpi+\eta}}{[a_1(1-\Upsilon)]^{\varpi}}\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)} \\ \times & \ \frac{\theta^j}{(j+w)^t}\left(\frac{y_1}{a_1(1-\Upsilon)}\right)^j \\ \times & \ \frac{\Gamma(\varpi+j)\Gamma(1+\frac{\eta}{\Upsilon-1})}{\Gamma(\varpi+j+\frac{\eta}{\Upsilon-1}+1)} \end{aligned} \quad (55)$$

Using equation (17) and the Hadamard product completes the theorem's proof. ■

Corollary IV.2. *If $a_1 = 1, \Upsilon = 0$, $\eta - 1$ in place of η in Theorem IV.1, we have*

$$\begin{aligned} & I_{0+}^{\eta}\big(x_1^{\varpi-1}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\theta x_1,t,w)\big)(y_1) \\ = & \ y_1^{\varpi+\eta-1}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}\{\theta y_1,t,w\} \\ \times & \ {}_2\psi_1\left[\begin{matrix}(\varpi,1),(1,1); \\ (\varpi+\eta,1);\end{matrix}\theta y_1\right]. \end{aligned} \quad (56)$$

Remark IV.3. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(2.1) in [12].*

Theorem IV.4. *For $\varrho \geq 0, \lambda > 0$; $\vartheta, \varsigma \in \mathbb{C}$; $w, \chi \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\Re(\varpi) > 0, \Re(\eta) > 0; \Re(\frac{\eta}{1-\Upsilon}) > -1, \Upsilon > 1, \theta \in \mathbb{R}$; ; $t \in \mathbb{C}$ and min $(\alpha, \varkappa, \mu_l, \nu_l, \beta_n, \kappa_n) > 0$ for $l = 1, \ldots r_1; n = 1, \ldots, r_2, r_1 + r_2 = m - 2$, where $m \in \mathbb{Z}^+$, when $|\xi| < 1$. The following formula holds true:*

$$\begin{aligned} & P_{0+}^{\eta,\Upsilon}\big(x_1^{\varpi-1}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\theta x_1,t,w)\big)(y_1) \\ = & \frac{y_1^{\varpi+\eta}\Gamma\big(1-\frac{\eta}{\Upsilon-1}\big)}{[-a_1(\Upsilon-1)]^{\varpi}} \\ \times & \ \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}\left\{\frac{\theta y_1}{-a_1(\Upsilon-1)},t,w\right\} \\ \times & \ {}_2\psi_1\left[\begin{matrix}(\varpi,1),(1,1); \\ (\varpi-\frac{\eta}{\Upsilon-1}+1,1);\end{matrix}\frac{\theta y_1}{-a_1(\Upsilon-1)}\right]. \end{aligned} \quad (57)$$

Proof: Using the equation (12) of Pathway fractional integral operator, we have

$$K = y_1^{\eta}\int_0^{[\frac{y_1}{-a_1(\Upsilon-1)}]} x_1^{\varpi-1}\left[1+\frac{a_1(\Upsilon-1)x_1}{y_1}\right]^{\frac{\eta}{-(\Upsilon-1)}} \times\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{(\theta x_1)^j}{(j+w)^t}\,dx_1. \quad (58)$$

By inverting the summation and integration orders, we have

$$\begin{aligned} K = & \ y_1^{\eta}\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)}\frac{\theta^j}{(j+w)^t} \\ \times & \int_0^{[\frac{y_1}{-a_1(\Upsilon-1)}]} x_1^{\varpi+j-1}[1+\frac{a_1(\Upsilon-1)x_1}{y_1}]^{\frac{\eta}{-(\Upsilon-1)}}\,dx_1, \end{aligned}$$

using the substitution W = $\frac{-a_1(\Upsilon-1)x_1}{y_1}$, we get

$$\begin{aligned} K = & \ \frac{y_1^{\varpi+\eta}}{[-a_1(\Upsilon-1)]^{\varpi}}\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{j!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)} \\ \times & \ \frac{\theta^j}{(j+w)^t}\left(\frac{y_1}{-a_1(\Upsilon-1)}\right)^j \\ \times & \int_0^1 W^{\varpi+j-1}[1-W]^{\frac{\eta}{-(\Upsilon-1)}}\,dW, \end{aligned} \quad (59)$$

using the Beta function [7], we have

$$\begin{aligned} K = & \ \frac{y_1^{\varpi+\eta}}{[-a_1(\Upsilon-1)]^{\varpi}}\sum_{j=0}^{\infty}\frac{(\vartheta)_j}{n!}\frac{B^{\varrho,\lambda;(\mu,\nu)_{r_1}}_{\alpha,\varkappa;(\beta,\kappa)_{r_2}}(\varsigma+j,\chi-\varsigma)}{B(\varsigma,\chi-\varsigma)} \\ \times & \ \frac{\theta^j}{(j+w)^t}\left(\frac{y_1}{-a_1(\Upsilon-1)}\right)^j \\ \times & \ \frac{\Gamma(\varpi+j)\Gamma(1-\frac{\eta}{\Upsilon-1})}{\Gamma(\varpi+j-\frac{\eta}{\Upsilon-1}+1)} \end{aligned} \quad (60)$$

Using equation (17) and the Hadamard product completes the theorem's proof. ■

Remark IV.5. *For $\varkappa = 1 = \lambda$ and $r_1 = r_2 = 0$, we get the Theorem(2.5) in [12].*

V. Applications to the Probability Distribution

In the following section, we investigate a general probability distribution using the generalized Hurwitz distribution, where the probability density function is described as

If a random variable Υ's probability density function is as follows, it is considered to be the generalized Hurwitz distribution

$$f_\Upsilon(w) = \begin{cases} \frac{t\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t+1,w)}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} & \text{if } w \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{61}$$

where it is implicitly presumed that the arguments ξ, t and parameters $\alpha, \varkappa, \chi, \varsigma, \varrho, \lambda, \mu_l, \nu_l, \beta_n, \kappa_n$ for $l = 1, \ldots r_1; n = 1, \ldots, r_2$, are fixed as well as appropriately restricted. Hence, the probability density function $f_\Upsilon(w)$ continues to be non-negative.

Theorem V.1. *The probability density function of the continuous random variable Υ is provided by (61). The moment generating function M(x) of Υ (random variable) is therefore given by*

$$M(x) = E_t[e^{\Upsilon x}] = \sum_{m=0}^{\infty} E_t[\Upsilon^m]\frac{x^m}{m!}, \tag{62}$$

where the moments $E_t[\Upsilon^m]$ of order m are given by

$$E_t[\Upsilon^m] = \sum_{l=0}^{m} \frac{m!}{(m-l)!}\frac{\Gamma(t-l)}{\Gamma(t)} \times \frac{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t-l,1)}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)}. \tag{63}$$

Proof: The assertion in (62) can be simply inferred via the series expansion of $e^{\Upsilon x}$. In order to prove (63), we note that

$$\frac{d}{dw}\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\} = -t\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t+1,w), \tag{64}$$

which simply follows from (19), and from the definition of $E_t[\Upsilon^m]$, we have

$$\begin{aligned} E_t[\Upsilon^m] &= \int_1^\infty w^m f_\Upsilon(w)\,dw \\ &= \frac{t}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} \\ &\times \int_1^\infty w^m \phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t+1,w)\,dw, \end{aligned}$$

$$= \frac{-1}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} \times \int_1^\infty w^m \frac{d}{dw}\{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\}\,dw, \tag{65}$$

$$\begin{aligned} &= \left[\frac{-w^m\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)}\right]_{w=1}^{\infty} \\ &+ \frac{m}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} \\ &\times \int_1^\infty w^{(m-1)}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\,dw, \end{aligned} \tag{66}$$

$$\begin{aligned} &= 1 - \lim_{w\to\infty}\left\{\frac{w^m\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)}\right\} \\ &+ \frac{m}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} \\ &\times \int_1^\infty w^{m-1}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\,dw, \end{aligned} \tag{67}$$

$$E_t[\Upsilon^m] = 1 + \frac{m}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} \times \int_1^\infty w^{m-1}\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\,dw \quad (m \in \mathbb{N}). \tag{68}$$

We have employed the following limit formula,

$$\begin{aligned} &\lim_{w\to\infty}\left\{w^m\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,w)\right\} \\ &= \lim_{w\to\infty}\left\{\frac{w^m}{\Gamma(t)}\int_1^\infty x^{t-1}e^{-wx} F^{\varrho,\lambda,(\mu,\nu)_{r_1}}_{\alpha,\varkappa,(\beta,\kappa)_{r_2}}(\vartheta,\varsigma;\chi;\xi e^{-x})\,dx\right\}, \\ &= \frac{1}{\Gamma(t)}\int_1^\infty \left(x^{t-1}\lim_{w\to\infty}\{w^m e^{-wx}\} \times F^{\varrho,\lambda,(\mu,\nu)_{r_1}}_{\alpha,\varkappa,(\beta,\kappa)_{r_2}}(\vartheta,\varsigma;\chi;\xi e^{-x})\,dx\right), \end{aligned} \tag{69}$$

= 0.

As a result, we have the following reduction formula for $E_t[\Upsilon^m]$

$$E_t[\Upsilon^m] = 1 + \frac{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t-1,1)}{\phi^{\varrho,\lambda,(\mu,\nu)_{r_1},\vartheta}_{\alpha,\varkappa,(\beta,\kappa)_{r_2},\varsigma,\chi}(\xi,t,1)} \times \frac{m}{(t-1)}E_{t-1}[\Upsilon^{m-1}] \tag{70}$$

by iterating the recurrence (68) , we get the intended result. ■

Remark V.2. *A particular case of Theorem V.1, when $r_1 = r_2 = 0, \alpha = \varkappa = \lambda = 1$, was taken into consideration by Parmar et al. [15].*

VI. Conclusion

Numerous researchers have examined the fractional calculus formulas of various special functions due to their extensive use in modelling and applied sciences. We look into the composition of our newly established HLZf and the pathway fractional integral operator that S.S. Nair presented. Further, in this work, we investigate the analytical properties of our newly interpreted HLZf, which involves the beta function, in its generalized form. A probability distribution application is also taken into account. The main conclusions are specific cases for a couple of previously published and novel findings. Further, we can also develop various series relations for our generalized HLZf, which involve other special functions.

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