Cordiality and Magicness of the Cartesian Product of Digraphs

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Abstract— We take into account a few combinatorial issues brought on by the requirement to provide security on a communications network. One of the key factors in the design of a security system is key management. We go through key distribution patterns, a way to condense the number of keys stored in a big network, and secret sharing systems that can be used to safeguard keys from theft or unauthorized access. We outline some combinatorial patterns with cordiality, referred to as authentication schemes, that could be utilized to create encodings that can spot such changes.

Index Terms— Key Management, Cayley digraph, Cordiality, harmonious, Edge product Cordial

I. INTRODUCTION

Cordial labeling is a variation of both graceful and harmonious labelings and was introduced by Cahit [1]. Cahit and Yılmaz [2] introduced H cordial labeling and in the same paper he proved that K_n is H - Cordial if and only if n > 2 and n is even; Further Ghebleh and Khoeilar [3] proved that K_n is H - Cordial if and only if n = 0 or 3 mod 4 and n ≠ 3 and also showed that every wheel has H_2 cordial labeling. Sundaram et al. [4] have introduced product cordial labeling in which the absolute difference of vertex labels in cordial labeling is replaced by product of the vertex labels and in the same paper they investigated it on some standard graphs. Vaidya and Barasara [5] introduced a variant of product cordial labeling and named it as edge product cordial labeling and proved it for some undirected graphs. Unlike in product cordial labeling the roles of vertices and edges are interchanged. In that paper they proved edge product cordial labeling on many graphs such as Cycles with odd order, trees of order ≥ 3, and crown.


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II. PRELIMINARIES

A. Edge Product Cordial labeling

A digraph G is said to have edge product cordial labeling if there exists a mapping, \( f : E(G) \rightarrow \{0,1\} \) and induced vertex labeling function \( f^* : V(G) \rightarrow \{0,1\} \) is such that for any vertex \( vi \in V(G) \), \( f^* (vi) \) is the product of the labels of outgoing edges provided the condition \( \left| v_f(0) - v_f(1) \right| \leq 1 \) and \( \left| e_f(0) - e_f(1) \right| \leq 1 \) is hold where \( v_f(i) \) is the number of vertices of G having label i under \( f^* \) and \( e_f(i) \) is the number of edges of G having label i under f for i = 0,1

B. H_n cordial labeling

A digraph G(n,q) is H_n cordial if it is possible to label the outgoing edges with the numbers from the set {± 1, ± 2, ..., ± n} in such a way that, at each vertex v the sum of the labels of the outgoing edges of v is in the set {± 1, ± 2, ..., ± n} and the inequalities \( \left| v(i) - v(-i) \right| \leq 1 \) and \( \left| e(i) - e(-i) \right| \leq 1 \) are also satisfied for each i with 1 ≤ i ≤ n, where v(i),e(i): i ∈ {± 1, ± 2,..., ± n} are the number of vertices and edges labeled with i respectively.

III. CORDIALITY OF THE CARTESIAN PRODUCT OF CAYLEY DIGRAPHS

A. Edge Product Cordial labeling

The Cartesian product Cay (G,S) × \( \overline{K}_2 \) admits edge product cordial labeling.

Proof:

Consider the Cayley digraph Cay (G,S) with p vertices and m generators. Every vertex has m indegree and m outdegree. Totally the Cayley digraph Cay (G,S) has m p = q arcs. Now multiply Cay (G,S) with \( \overline{K}_2 \) by the Cartesian product. From the definition of Cartesian product we have resultant digraph Cay (G,S) × \( \overline{K}_2 \) has 2 p vertices. At first \( p_i \) vertices, we have m+1 outgoing and m incoming arcs and the remaining \( p \) vertices have m outgoing and m+1 incoming arcs. Hence Cay (G,S) × \( \overline{K}_2 \) has 2p vertices and mp + (m+1) \( p_i = 2q + p_i \) and \( n = 2 + p_i \). Let V and E are the vertex set and edge set of Cay (G,S) × \( \overline{K}_2 \) respectively and \( e_{ij} \) is \( j^{th} \) outgoing arc of \( i^{th} \) vertex.

To prove Cay (G,S) × \( \overline{K}_2 \) admits edge product cordial labeling we have to show that there exists a function f from the edge set of Cay (G,S) × \( \overline{K}_2 \) to \{0,1\} and induced vertex labeled function \( f^* \) from the vertex set of Cay (G,S)

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× \overline{K}_2 \times \{0,1\} such that for every vertex \( v_i, 1 \leq i \leq n, f^* (v_i) = \prod f(e_{ij}) \) the product is taken over the labels of the outgoing arcs of \( v_i \) and the condition \( |v_l(0) - v_l(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \) is hold. Where \( v_l(i) \) is the number of vertices of Cay (G,S) × \overline{K}_2 having label \( i \) under \( f^* \) and \( e_l(i) \) is the number of edges of Cay (G,S) × \overline{K}_2 having label \( i \) under \( f \) for \( i = 0,1 \).

Define \( f : E \rightarrow \{0,1\} \) as

For \( 1 \leq i \leq 2p_l \) vertices with \( m+1 \) outgoing arcs and \( 1 \leq j \leq m+1 \)

\[ f(e_{ij}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \]

Then the induced function \( f^* (v_i) = \prod_{l=1}^{m} f(e_{ij}) \) for every vertex \( v_i, 1 \leq i \leq n \) is as follows

\[ f^* (v_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \]

Here \( i \) varies from 1 to 2 \( p_l \) and number of vertices is always even. The number of vertices labeled with 1 is \( p_l \) and the number of vertices labeled with 0 is also \( p_l \).

Therefore \( |v_l(0) - v_l(1)| = |p_l - p_l| = 0 \leq 1 \).

The condition \( |v_l(0) - v_l(1)| \leq 1 \) holds.

If \( p_l \) is even, then the number of edges labeled with 1 is \( \frac{(m+1)p_l}{2} + \frac{mp_l}{2} \) and in the same way the number of edges labeled with 0 is \( \frac{(m+1)p_l}{2} + \frac{mp_l}{2} \).

\[ |e_f(0) - e_f(1)| = \left| \frac{(m+1)p_l}{2} + \frac{mp_l}{2} - \frac{(m+1)p_l}{2} + \frac{mp_l}{2} \right| = 0 \leq 1. \]

If \( p_l \) is odd, then the number of edges labeled with 1 is \( \frac{(m+1)(p_l+1)}{2} + \frac{m(p_l-1)}{2} \) and in the number of edges labeled with 0 is \( \frac{(m+1)(p_l+1)}{2} + \frac{m(p_l+1)}{2} \).

\[ |e_f(0) - e_f(1)| = \left| \frac{(m+1)(p_l+1)}{2} + \frac{m(p_l-1)}{2} - \frac{(m+1)(p_l+1)}{2} + \frac{m(p_l+1)}{2} \right| = 1. \]

Therefore, in both cases the condition \( |e_f(0) - e_f(1)| \leq 1 \) holds.

Hence the digraph Cay (G,S) × \overline{K}_2 admits edge product cordial labeling.

\textbf{Example}

The following figure shows the Cartesian product of Cayley digraph of alternating group \( A_4 \) and \( \overline{K}_2 \) with its edge product cordial labeling.

![Fig 1 Edge product cordial labeling](image)

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Algorithm

Input:
The digraph product Cay(G,S) X K₂ with 2p₁ vertices

Step: 1
Assume first p₁ vertices with m+1 outgoing arcs and another p₁ vertices with m outgoing arcs.

Step: 2
Denote the vertex set V = {v₁, v₂, ..., v₂p₁}

Step: 3
Define f such that
\[ f(e_{ij}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \]

Step: 4
Define f* such that f* (v₁) = \prod_{i=1}^{m} f(e_{ij})

Step: 5
f* (v₁) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}

with the condition |eᵢ(0) - eᵢ(1)| ≤ 1 holds.

Output: Cay (G,S) × K₂ with edge product cordial labelling.

B. Hₜ cordial labeling
The digraph Cay (G,S) × K₂ admits Hₜ cordial labeling.

Proof:
Consider the Cayley digraph Cay (G,S) with p₁ vertices and m generators. Every vertex has m indegree and m outdegree. Totally the Cayley digraph Cay (G,S) has mp₁ = q arcs. Now multiply Cay (G,S) with K₂ by the Cartesian product. By the concept of Cartesian product, we have resultant digraph Cay (G,S) × K₂ has 2p₁ vertices. At first p₁ vertices, we have m+1 outgoing and m incoming arcs. Another p₁ vertices have m outgoing and m+1 incoming arcs. Hence Cay (G,S) × K₂ has 2p₁ vertices and m + (m+1)p₁ = 2q + p₁ arcs. Let r = 2q + p₁ and n = 2p₁.

To prove Cay(G,S) × K₂ admits Hₜ cordial labeling, we have to show that there exists a function f:E→{± 1, ± 2,..., ± n} and induced vertex labeled function f*:V→{± 1, ± 2,..., ± n} such that for every vertex vᵢ, 1 ≤ i ≤ n, f* (vᵢ) = f(eᵢ₁) + f(eᵢ₂) + ... + f(eᵢₙ) and the condition |v(i) - v(-i)| ≤ 1 and |eᵢ(0) - eᵢ(1)| ≤ 1 is hold where v(i) is the number of vertices of Cay(G,S) × K₂ having label i under f* and e(i) is the number of edges of Cay(G,S) × K₂ having label i under f for i = {± 1, ± 2,..., ± n} and eᵢ is jth outgoing arc from i th vertex. We prove this theorem in four cases according to the number of vertices and number of generators of the Cayley digraph.

(i) m is odd and p₁ is even

Now define f from the edge set of Cay (G,S) × K₂ (E) to the set {± 1, ± 2,..., ± n} as follows.

For 1 ≤ i ≤ p₁ and 1 ≤ j ≤ m + 1
\[ f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3,..., m - 2, m \\ -i & \text{for } j = 2,4,..., m - 1 \\ i - p₁ - 1 & \text{for } j = m + 1 \end{cases} \]

For p₁ + 1 ≤ i ≤ 2p₁ and 1 ≤ j ≤ m
\[ f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3,..., m - 2 \\ -i & \text{for } j = 2,4,..., m - 1 \end{cases} \]

For j = m,
\[ f(e_{ij}) = \begin{cases} i & \text{for } i = 1,3,..., m - 2 \\ -i & \text{for } i = 2,4,..., m - 1 \end{cases} \]

Then the induced function for 1 ≤ i ≤ p₁ and 1 ≤ j ≤ m + 1
\[ f*(vᵢ) = ∑_{j=1}^{m+1} f(e_{ij}) \]
\[ = i - i + i - i + ... + i + i - p₁ - 1 \]
\[ = 2i - p₁ - 1 \]
For p₁ + 1 ≤ i ≤ 2p₁ and 1 ≤ j ≤ m
\[ f*(vᵢ) = ∑_{j=1}^{m} f(e_{ij}) \]
\[ = i - i + i - i + ... + i - i + i - p₁ - 1 \]
\[ = 2i - p₁ - 1 \]

Therefore, the induced function f*(vᵢ) for 1 ≤ i ≤ 2p₁ is
\[ f*(vᵢ) = \begin{cases} 2i - p₁ - 1 & \text{for } 1 ≤ i ≤ p₁ \\ i & \text{for } p₁ + 1 ≤ i ≤ \frac{3p₁}{2} \\ i - 3p₁ - 1 & \text{for } \frac{3p₁}{2} + 1 ≤ i ≤ 2p₁ \end{cases} \]
e(−i). Therefore \[ |e(i) - e(−i)| = 0 \leq 1 \] for all \( i \leq 2p_l \). We have \[ |v(i) - v(−i)| \leq 1 \] for all \( i \leq 2p_l \).

So the digraph \( Cay(G,S) \times \overrightarrow{K}_2 \) admits \( H_0 \) cordial labeling when \( m \) is odd and \( p_l \) is even.

(ii) \( m \) and \( p_l \) are even

Now define \( f \) from the edge set of \( Cay(G,S) \times \overrightarrow{K}_2 \) (E) to the set \{ \pm 1, \pm 2, ..., \pm n \} as follows.

For \( 1 \leq i \leq p_l \) and \( 1 \leq j \leq m + 1 \)

\[
f(e_{ij}) = \begin{cases} 
  i & \text{for } j = 1,3 \ldots m - 1 \\
  -i & \text{for } j = 2,4 \ldots m
\end{cases}
\]

For \( j = m + 1 \),

\[
f(e_{ij}) = \begin{cases} 
  i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\
  -i - p_l - 1 & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l
\end{cases}
\]

For \( p_l + 1 \leq i \leq 2p_l \) and \( 1 \leq j \leq m \)

\[
f(e_{ij}) = \begin{cases} 
  i & \text{for } j = 1,3 \ldots m - 1 \\
  -i & \text{for } j = 2,4 \ldots m - 2 \\
  -i - 3p_l - 1 & \text{for } j = m
\end{cases}
\]

Then the induced function for \( 1 \leq i \leq p_l \) and \( 1 \leq j \leq m + 1 \)

\[
f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})
\]

For \( 1 \leq i \leq \frac{p_l}{2} \) and \( 1 \leq j \leq m + 1 \)

\[
f^*(v_i) = i - i - i + \ldots + i - i + i
\]

For \( \frac{p_l}{2} + 1 \leq i \leq p_l \) and \( 1 \leq j \leq m + 1 \)

\[
f^*(v_i) = i - i + i - i + \ldots + i - i + i - p_l - 1 = i - p_l - 1
\]

For \( p_l + 1 \leq i \leq 2p_l \) and \( 1 \leq j \leq m \)

\[
f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})
\]

\[
= i - i + i - i + \ldots + i + i - 3p_l - 1
= 2i - 3p_l - 1
\]

Therefore the induced function \( f^*(v_i) \) is for \( 1 \leq i \leq 2p_l \) is

\[
f^*(v_i) = \begin{cases} 
  i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\
  i - p_l - 1 & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l \\
  2i - 3p_l - 1 & \text{for } p_l + 1 \leq i \leq 2p_l
\end{cases}
\]

For \( 1 \leq i \leq \frac{p_l}{2} \), \( e(i) = \frac{m}{2} + 1 = e(−i) \). For \( \frac{p_l}{2} + 1 \leq i \leq 2p_l \), \( e(i) = \frac{m}{2} = e(−i) \). Therefore \( |e(i) - e(−i)| = 0 \leq 1 \) for all \( 1 \leq i \leq 2p_l \). We have \( |v(i) - v(−i)| \leq 1 \) for all \( 1 \leq i \leq 2p_l \).

So the digraph \( Cay(G,S) \times \overrightarrow{K}_2 \) admits \( H_0 \) cordial labeling when \( m \) and \( p_l \) are even.

(iii) \( m \) and \( p_l \) are odd

Now define \( f \) from the edge set of \( Cay(G,S) \times \overrightarrow{K}_2 \) (E) to the set \{ \pm 1, \pm 2, ..., \pm n \} as follows.

For \( 1 \leq i \leq p_l \) and \( 1 \leq j \leq m + 1 \)

\[
f(e_{ij}) = \begin{cases} 
  i & \text{for } j = 1,3 \ldots m - 2 \\
  -i & \text{for } j = 2,4 \ldots m - 1 \\
  i - p_l - 1 & \text{for } j = m + 1 \text{ and } i \neq \frac{p_l + 1}{2} \\
  -i - \frac{p_l - 1}{2} & \text{for } j = m + 1 \text{ and } i = \frac{p_l + 1}{2}
\end{cases}
\]

For \( p_l + 1 \leq i \leq 2p_l \) and \( 1 \leq j \leq m \)

\[
f(e_{ij}) = \begin{cases} 
  i & \text{for } j = 1,3 \ldots m - 2 \\
  -i & \text{for } j = 2,4 \ldots m - 1
\end{cases}
\]

For \( j = m \),

\[
f(e_{ij}) = \begin{cases} 
  i & \text{for } p_l + 1 \leq i \leq \frac{2p_l + 1}{2} \\
  i - 3p_l - 2 & \text{for } \frac{3p_l + 1}{2} \leq i \leq 2p_l
\end{cases}
\]

Then the induced function for \( 1 \leq i \leq p_l \) and \( 1 \leq j \leq m + 1 \)

\[
f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})
\]

For \( 1 \leq i \leq \frac{p_l}{2} \) and \( i \neq \frac{p_l + 1}{2} \)

\[
f^*(v_i) = i - i + i - i + \ldots + i + i - p_l - 1 = i - p_l - 1
\]

For \( i = \frac{p_l + 1}{2} \)

\[
f^*(v_i) = i - i + i - i + \ldots + i + i - p_l - 1 = \frac{p_l + 1 - p_l - 1}{2} = 1
\]

For \( p_l + 1 \leq i \leq 2p_l \) and \( 1 \leq j \leq m \)

\[
f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})
\]

\[
= i - i + i - i + \ldots + i - i + i = i
\]

Therefore the induced function \( f^*(v_i) \) is for \( 1 \leq i \leq 2p_l \) is

\[
f^*(v_i) = \begin{cases} 
  i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\
  i - p_l - 1 & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l \\
  2i - 3p_l - 1 & \text{for } p_l + 1 \leq i \leq 2p_l
\end{cases}
\]
For $\frac{3p_{l+1}}{2} + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$
\[ f^*(v_i) = i - i + i + \cdots + i - i + i - 3p_l - 2 = i - 3p_l - 2 \]

Therefore the induced function $f^*(v_i)$ is for $1 \leq i \leq 2p_l$
\[ f^*(v_i) = \begin{cases} 
2i - p_l - 1 & \text{for } 1 \leq i \leq p_l \text{ and } i \neq \frac{p_{l+1}}{2} \\
1 & \text{for } i = \frac{p_{l+1}}{2} \\
i & \text{for } p_l + 1 \leq i \leq \frac{3p_{l+1}}{2} \\
i - 3p_l - 1 & \text{for } \frac{3p_{l+1}}{2} + 1 \leq i \leq 2p_l
\end{cases} \]

For $1 \leq i \leq p_l$ and $i \neq \frac{p_{l+1}}{2}$, $e(i) = \frac{m+1}{2} = e(-i)$. For $i = \frac{p_{l+1}}{2}$, $e(i) = \frac{m+1}{2}$ and $e(-i) = \frac{m-1}{2}$. For $p_l + 1 \leq i \leq \frac{3p_{l+1}}{2}$, $e(i) = \frac{m+1}{2} = e(-i)$.

For $\frac{3p_{l+1}}{2} + 1 \leq i \leq 2p_l$, $e(i) = \frac{m-1}{2} = e(-i)$. Therefore $|e(i) - e(-i)| = 1$ for all $i \leq 2p_l$. We have $|v(i) - v(-i)| = 1$ for all $1 \leq i \leq 2p_l$.

So the digraph $\text{Cay}(G, S) \times \overrightarrow{K}_2$ admits $H_n$ cordial labeling when $m$ and $p_l$ are odd.

(iv) $m$ is even and $p_l$ is odd

Now define $f$ from the edge set of $\text{Cay}(G, S) \times \overrightarrow{K}_2$ (E) to the set $\{ \pm 1, \pm 2, \ldots, \pm n \}$ as follows.

For $1 \leq i \leq p_l$ and $1 \leq j \leq m + 1$
\[ f(e_{ij}) = \begin{cases} 
i & \text{for } j = 1, 3, \ldots, m - 1 \\
- i & \text{for } j = 2, 4, \ldots, m
\end{cases} \]

For $j = m + 1$,
\[ f(e_{ij}) = \begin{cases} 
i & \text{for } 1 \leq i \leq \frac{p_l + 1}{2} \\
i - p_l - 1 & \text{for } \frac{p_l + 3}{2} \leq i \leq p_l
\end{cases} \]

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$
\[ f(e_{ij}) = \begin{cases} 
i & \text{for } j = 1, 3, \ldots, m - 1 \\
- i & \text{for } j = 2, 4, \ldots, m - 2
\end{cases} \]

For $j = m$
\[ f(e_{ij}) = \begin{cases} 
i - 3p_l - 2 & \text{for } p_l + 1 \leq i \leq 2p_l \text{ and } i \neq \frac{3p_l + 1}{2} \\
i - 3p_l - 1 & \text{for } i = \frac{3p_l + 1}{2}
\end{cases} \]

Then the induced function for $1 \leq i \leq p_l$
\[ f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij}) \]

For $1 \leq i \leq \frac{p_{l+1}}{2}$ and $1 \leq j \leq m + 1$
\[ f^*(v_i) = i - i + i - i + \cdots + i - i + i = i \]

For $\frac{p_{l+1}}{2} + 1 \leq i \leq p_l$ and $1 \leq j \leq m + 1$
\[ f^*(v_i) = i - i + i - i + \cdots + i + i - p_l - 1 = i - p_l - 1 \]

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$
\[ f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij}) \]

For $p_l + 1 \leq i \leq 2p_l$ and $i \neq \frac{3p_{l+1}}{2}$
\[ f^*(v_i) = i - i + i - i + \cdots + i + i - 3p_l - 1 = 2i - 3p_l - 1 \]

For $i = \frac{3p_{l+1}}{2}$
\[ f^*(v_i) = i - i + i - i + \cdots + i \frac{3p_{l+1}}{2} - \frac{3p_{l+1}}{2} = 1 \]

Therefore the induced function $f^*(v_i)$ is for $1 \leq i \leq 2p_l$
\[ f^*(v_i) = \begin{cases} 
i & \text{for } 1 \leq i \leq \frac{p_{l+1}}{2} \\
i - p_l - 1 & \text{for } \frac{p_{l+1}}{2} \leq i \leq p_l \\
2i - 3p_l - 1 & \text{for } p_l + 1 \leq i \leq 2p_l \text{ and } i \neq \frac{3p_{l+1}}{2} \\
1 & \text{for } i = \frac{3p_{l+1}}{2}
\end{cases} \]

For $1 \leq i \leq \frac{p_{l+1}}{2}$, $e(i) = \frac{m}{2} + 1 = e(-i)$. For $\frac{p_{l+1}}{2} \leq i \leq 2p_l$

For $i = \frac{p_{l+1}}{2}$, $e(i) = \frac{m}{2} = e(-i)$. For $i = \frac{p_{l+1}}{2}$, $e(i) = \frac{m}{2} = e(-i)$.

Therefore $|e(i) - e(-i)| = 1$. Therefore $|v(i) - v(-i)| \leq 1$ for all $1 \leq i \leq 2p_l$. We have $|v(i) - v(-i)| \leq 1$ for all $1 \leq i \leq 2p_l$.

So the digraph $\text{Cay}(G, S) \times \overrightarrow{K}_2$ admits $H_n$ cordial labeling when $m$ is even and $p$ is odd.

Hence the digraph $\text{Cay}(G, S) \times \overrightarrow{K}_2$ is $H_n$ cordial.

C. Bimagic Labeling

The Cayley digraph $\text{Cay}(G, S)$ with $p$ vertices and $|S| \equiv 0 \pmod{2}$ admits total bimagic labeling.

Proof:

From the construction of the Cayley digraph, we have $p$ vertices and $mp$ arcs where $m$ is the number of generators. Let us denote the vertex set of $\text{Cay}(G, S)$ as $V = \{v_1, v_2, \ldots, v_p\}$ and the edge set of $\text{Cay}(G, S)$ as $E = E_S \cup E_{S_2} \cup \cdots \cup E_{S_m} = \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}, e_{22}, \ldots, e_{p1}, \ldots, e_{pm}\}$ where

$E_{S_1} = \text{set of all outgoing arcs from } v_1 \text{ generated by } S_1$
E_{S_2} = \text{set of all outgoing arcs from } v_i \text{ generated by } S_2 \\
\vdots \\
E_{S_m} = \text{set of all outgoing arcs from } v_i \text{ generated by } S_m \text{ and } e_{ij} \text{ is the outgoing arc of vertex } v_i \text{ generated by } S_j. \text{ To prove the Cayley digraph } Cay(G,S) \text{ is total bimagic, we have to show that there exists a bijection } f: V \cup E \rightarrow \{1,2,\ldots,[V \cup E]\} \text{ such that for any vertex } v_i \text{ the sum of the labels of outgoing edges of } v_i \text{ together with the label of itself is equal to either of constants } k_1 \text{ and } k_2. \text{ We prove this theorem in two cases as for odd number of vertices and even number of vertices.}

Case (i) when p is even

Define } f: V \rightarrow \{1,2,\ldots,p\} \text{ as } f(v_i) = i \text{ for } 1 \leq i \leq p \text{ and } f(e_{ij}) = ((m+1)p - i + 1) \text{ for } j = 1,3,\ldots,m - 3 \\
\text{ or } (mp + i) \text{ for } j = 2,4,\ldots,m - 2, m - 1 \text{ for } j = m \\

For any arbitrary vertex } v_i, 1 \leq i \leq p \text{ of digraph } Cay(G,S) \text{ and proved that } Cay(G,S) \text{ labelings of graphs may admit edge product cordial labeling when it has odd number of vertices.}

Theorem

For any arbitrary vertex } v_i, 1 \leq i \leq \frac{p+1}{2} \text{ of digraph } Cay(G,S) \text{ the sum of the labels of outgoing edges of } v_i \text{ together with the label of itself is equal to either of constants } k_1 \text{ and } k_2. \text{ We prove this theorem in two cases as for odd number of vertices and even number of vertices.}

Case (ii) when p is odd

Define } f: V \rightarrow \{1,2,\ldots,p\} \text{ as } f(v_i) = i \text{ for } 1 \leq i \leq p \text{ and } f(e_{ij}) = \begin{cases} (m+1)p - i + 1 & \text{ for } j = 1,3,\ldots,m - 3 \\
mp + i & \text{ for } j = 2,4,\ldots,m - 2, m - 1 \end{cases} \text{ for } j = m \\

For any arbitrary vertex } v_i, 1 \leq i \leq \frac{p+1}{2} \text{ of digraph } Cay(G,S) \text{ the sum of the labels of outgoing edges of } v_i \text{ together with the label of itself is equal to either of constants } k_1 \text{ and } k_2. \text{ We prove this theorem in two cases as for odd number of vertices and even number of vertices.}

REFERENCES


