Cordiality and Magicness of the Cartesian Product of Digraphs

R Thamizharasi, R Rajeswari and R Suresh

Abstract— **We take into account a few combinatorial issues brought on by the requirement to provide security on a communications network. One of the key factors in the design of a security system is key management. We go through key distribution patterns, a way to condense the number of keys stored in a big network, and secret sharing systems that can be used to safeguard keys from theft or unauthorized access. We outline some combinatorial patterns with cordiality, referred to as authentication schemes, that could be utilized to create encodings that can spot such changes.**

Index Terms— **Key Management, Cayley digraph, Cordiality, harmonious, Edge product Cordial**

I. INTRODUCTION

Cordial labeling is a variation of both graceful and harmonious labelings and was introduced by Cahit [1]. Cahit and Yilmaz [2] introduced H cordial labeling and in the same paper he proved that K_n is H - Cordial if and only if n > 2 and n is even; Further Ghebleh and Khoeilar [3] proved that K_n is H- Cordial if and only if $n \equiv 0$ or 3 mod 4 and $n \neq 3$ and also showed that every wheel has H_2 cordial labeling. Sundaram et al. [4] have introduced product cordial labeling in which the absolute difference of vertex labels in cordial labeling is replaced by product of the vertex labels and in the same paper they investigated it on some standard graphs. Vaidya and Barasara [5] introduced a variant of product cordial labeling and named it as edge product cordial labeling and proved it for some undirected graphs. Unlike in product cordial labeling the roles of vertices and edges are interchanged. In that paper they proved edge product cordial labeling on many graphs such as Cycles with odd order, trees of order ≥ 3, and crown.

Thamizharasi and Rajeswari [6],[7].[8],[9] studied various labeling techniques for directed Cayley graphs. Li wang etc al [10] explained in detail about adjacent vertex total labeling of distinct graphs. Salat etc al [11] deliberated the Palindromic labeling of H graphs. They discussed the importance of Palindromic labeling in H graphs.

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R Thamizharasi is Assistant Professor in the Department of Mathematics, Saveetha Engineering College, Chennai, India 602105
(e-mail: thamizhvisu@gmail.com)
R Rajeswari is Professor in the Department of Mathematics,
Sathyabama Institute of Science and Technology, Chennai, India 600115
(e-mail: rajeswarivel1998@gmail.com)
R Suresh is Assistant Professor in the Department of Mathematics,
SRM Institute of Science and Technology, Tiruchirappalli, India 621105
(e-mail: sureshnational@gmail.com

II. PRELIMINARIES

A. *Edge Product Cordial labeling*

A digraph G is said to have edge product cordial labeling if there exists a mapping, $f : E(G)\rightarrow\{0,1\}$ and induced vertex labeling function $f^* : V(G)\rightarrow\{0,1\}$ is such that for any vertex $v_i \in V(G)$, $f^*(v_i)$ is the product of the labels of outgoing edges provided the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ is hold where $v_f(i)$ is the number of vertices of G having label i under f* and $e_f(i)$ is the number of edges of G having label i under f for i = 0,1

B. *H_n cordial labeling*

A digraph G(n,q) is H_n cordial if it is possible to label the outgoing edges with the numbers from the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that, at each vertex v the sum of the labels of the outgoing edges of v is in the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ and the inequalities $|v(i) - v(-i)| \leq 1$ and $|e(i) - e(-i)| \leq 1$ are also satisfied for each i with $1 \leq i \leq n$, where $v(i), e(i): i \in \{\pm 1, \pm 2, \ldots, \pm n\}$ are the number of vertices and edges labeled with i respectively.

III. CORDIALITY OF THE CARTESIAN PRODUCT OF CAYLEY DIGRAPHS

A. *Edge Product Cordial labeling*

The Cartesian product Cay (G,S) $\times \overrightarrow{K_2}$ admits edge product cordial labelling.

Proof:

Consider the Cayley digraph Cay (G,S) with p_l vertices and m generators. Every vertex has m indegree and m outdegree. Totally the Cayley digraph Cay (G,S) has m p_l = q arcs. Now multiply Cay (G,S) with $\overrightarrow{K_2}$ by the Cartesian product. From the definition of Cartesian product we have resultant digraph Cay (G,S) $\times \overrightarrow{K_2}$ has 2 p_l vertices. At first p_l vertices, we have m+1 outgoing and m incoming arcs and the remaining p vertices have m outgoing and m+1 incoming arcs. Hence Cay (G,S) $\times \overrightarrow{K_2}$ has 2p vertices and mp + (m+1) p_l = 2q + p_l arcs. Let r = 2q + p_l and n = 2 p_l. Let V and E are the vertex set and edge set of Cay (G,S) $\times \overrightarrow{K_2}$ respectively and e_{ij} is j^{th} outgoing arc of i^{th} vertex.

To prove Cay (G,S) $\times \overrightarrow{K_2}$ admits edge product cordial labeling we have to show that there exists a function f from the edge set of Cay (G,S) $\times \overrightarrow{K_2}$ to {0,1}and induced vertex labeled function f* from the vertex set of Cay (G,S)

$\times \overrightarrow{K_2}$ to $\{0,1\}$ such that for every vertex v_i, $1 \le i \le n$, $f^*(v_i) = \prod f(e_{ij})$ the product is taken over the labels of the outgoing arcs of v_i and the condition $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$ is hold. Where $v_f(i)$ is the number of vertices of Cay (G,S) $\times \overrightarrow{K_2}$ having label i under f^* and $e_f(i)$ is the number of edges of Cay (G,S) $\times \overrightarrow{K_2}$ having label i under f for i = 0,1.

Define $f : E \rightarrow \{0,1\}$ as

For $1 \le i \le 2p_l$ vertices with $m+1$ outgoing arcs and $1 \le j \le m+1$

$$f(e_{ij}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

Then the induced function

$f^*(v_i) = \prod_{j=1}^{m} f(e_{ij})$ for every vertex v_i, $1 \le i \le n$ is as follows

$$f^*(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

Here i varies from 1 to 2 p_l and number of vertices is always even. The number of vertices labeled with 1 is p_l and the number of vertices labeled with 0 is also p_l.
Therefore $|v_f(0) - v_f(1)| = |p_l - p_l| = 0 \le 1$.

The condition $|v_f(0) - v_f(1)| \le 1$ holds.

If p_l **is even**, then the number of edges labeled with 1 is $\frac{(m+1)p_l}{2} + \frac{mp_l}{2}$ and in the same way the number of edges labeled with 0 is $\frac{(m+1)p_l}{2} + \frac{mp_l}{2}$.

$$|e_f(0) - e_f(1)| = \left| \frac{(m+1)p_l}{2} + \frac{mp_l}{2} - \frac{(m+1)p_l}{2} + \frac{mp_l}{2} \right| = 0 \le 1.$$

If p_l **is odd**, then the number of edges labeled with 1 is $\frac{(m+1)(p_l+1)}{2} + \frac{m(p_l-1)}{2}$ and in the number of edges labeled with 0 is $\frac{(m+1)(p_l-1)}{2} + \frac{m(p_l+1)}{2}$.

$$|e_f(0) - e_f(1)| = \left| \frac{(m+1)(p_l+1)}{2} + \frac{m(p_l-1)}{2} - \frac{(m+1)(p_l-1)}{2} + \frac{m(p_l+1)}{2} \right| = 1.$$

Therefore, in both cases the condition $|e_f(0) - e_f(1)| \le 1$ holds.
Hence the digraph Cay (G,S) $\times \overrightarrow{K_2}$ admits edge product cordial labeling.

Example

The following figure shows the Cartesian product of Cayley digraph of alternating group A_4 and $\overrightarrow{K_2}$ with its edge product cordial labeling.

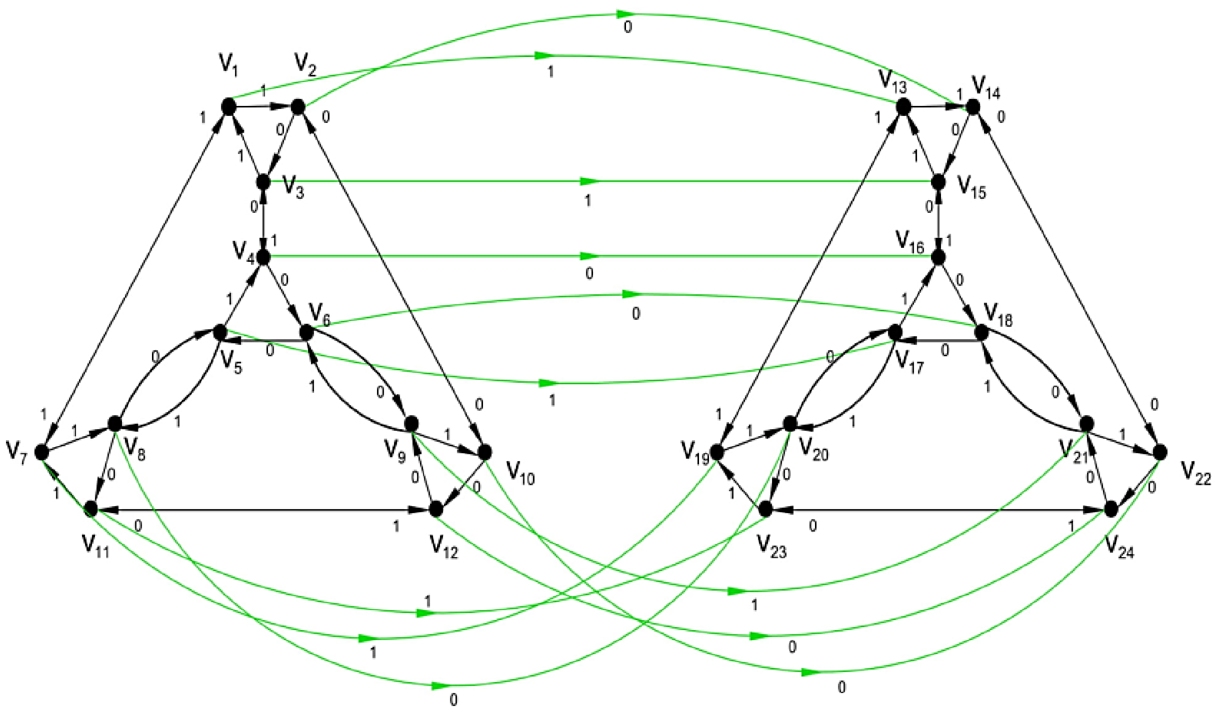

Fig 1 Edge product cordial labeling

Algorithm

Input:

The digraph product Cay(G,S) X K_2 with $2p_l$ vertices

Step: 1

Assume first p_l vertices with m+1 outgoing arcs and another p_l vertices with m outgoing arcs

Step: 2

Denote the vertex set $V = \{v_1, v_2, \ldots v_{2p_l}\}$

Step: 3

Define f such that

$$f(e_{ij}) = \begin{cases} 1 & \text{if i is odd} \\ 0 & \text{if i is even} \end{cases}$$

Step: 4

Define f* such that $f^*(v_i) = \prod_{j=1}^{m} f(e_{ij})$

Step: 5

$$f^*(v_i) = \begin{cases} 1 & \text{if i is odd} \\ 0 & \text{if i is even} \end{cases}$$

with the condition $|e_f(0) - e_f(1)| \leq 1$ holds.

Output: Cay (G,S) $\times \overrightarrow{K_2}$ with edge product cordial labelling.

B. H_n cordial labeling

The digraph Cay (G,S) $\times \overrightarrow{K_2}$ admits H_n cordial labeling.

Proof:

Consider the Cayley digraph Cay (G,S) with p_l vertices and m generators. Every vertex has m indegree and m outdegree. Totally the Cayley digraph Cay (G,S) has mp_l = q arcs. Now multiply Cay (G,S) with $\overrightarrow{K_2}$ by the Cartesian product. By the concept of Cartesian product, we have resultant digraph Cay (G,S) $\times \overrightarrow{K_2}$ has $2p_l$ vertices. At first p_l vertices, we have m+1 outgoing and m incoming arcs. Another p_l vertices have m outgoing and m+1 incoming arcs. Hence Cay (G,S) $\times \overrightarrow{K_2}$ has $2p_l$ vertices and m + (m+1) p_l = 2q + p_l arcs. Let r = 2q + p_l and n = $2p_l$.

To prove Cay(G,S) $\times \overrightarrow{K_2}$ admits H_n cordial labeling, we have to show that there exists a function f:E→$\{\pm 1, \pm 2, \ldots, \pm n\}$ and induced vertex labeled function f*:V→$\{\pm 1, \pm 2, \ldots \pm n\}$ such that for every vertex v_i, $1 \leq i \leq n$, $f^*(v_i) = f(e_{i1}) + f(e_{i2}) + \ldots + f(e_{im})$ and the condition $|v(i) - v(-i)| \leq 1$ and $|e(i) - e(-i)| \leq 1$ is hold where v(i) is the number of vertices of Cay(G,S) $\times \overrightarrow{K_2}$ having label i under f* and e(i) is the number of edges of Cay(G,S) $\times \overrightarrow{K_2}$ having label i under f for i = $\{\pm 1, \pm 2, \ldots, \pm n\}$ and e_{ij} is j^{th} outgoing arc from i^{th} vertex. We prove this theorem in four cases according to the number of vertices and number of generators of the Cayley digraph.

(i) m is odd and p_l is even

Now define f from the edge set of Cay (G,S) $\times \overrightarrow{K_2}$ (E) to the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ as follows.

For $1 \leq i \leq p_l$ and $1 \leq j \leq m + 1$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3, \ldots, m-2, m \\ -i & \text{for } j = 2,4, \ldots, m-1 \\ i - p_l - 1 & \text{for } j = m+1 \end{cases}$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3, \ldots, m-2 \\ -i & \text{for } j = 2,4, \ldots, m-1 \end{cases}$$

For j = m,

$$f(e_{ij}) = \begin{cases} i & \text{for } p_l + 1 \leq i \leq \frac{3p_l}{2} \\ i - 3p_l - 1 & \text{for } \frac{3p_l}{2} + 1 \leq i \leq 2p_l \end{cases}$$

Then the induced function for $1 \leq i \leq p_l$ and $1 \leq j \leq m + 1$

$$\begin{aligned} f^*(v_i) &= \textstyle\sum_{j=1}^{m+1} f(e_{ij}) \\ &= i - i + i - i + \cdots + i + i - p_l - 1 \\ &= 2i - p_l - 1 \end{aligned}$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f^*(v_i) = \textstyle\sum_{j=1}^{m} f(e_{ij})$$

For $p_l + 1 \leq i \leq \frac{3p_l}{2}$ and $1 \leq j \leq m$

$$\begin{aligned} f^*(v_i) &= i - i + i - i + \cdots + i - i + i \\ &= i \end{aligned}$$

For $\frac{3p_l}{2} + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$\begin{aligned} f^*(v_i) &= i - i + i - i + \cdots + i - i + i - 3p_l - 1 \\ &= i - 3p_l - 1 \end{aligned}$$

Therefore, the induced function $f^*(v_i)$ for $1 \leq i \leq 2p_l$ is

$$f^*(v_i) = \begin{cases} 2i - p_l - 1 & \text{for } 1 \leq i \leq p_l \\ i & \text{for } p_l + 1 \leq i \leq \frac{3p_l}{2} \\ i - 3p_l - 1 & \text{for } \frac{3p_l}{2} + 1 \leq i \leq 2p_l \end{cases}$$

For $1 \leq i \leq p_l$, $e(i) = \frac{m+1}{2} = e(-i)$. For $p_l + 1 \leq i \leq \frac{3p_l}{2}$, $e(i) = \frac{m+1}{2} = e(-i)$. For $\frac{3p_l}{2} + 1 \leq i \leq 2p_l$, $e(i) = \frac{m-1}{2} =$

$e(-i)$. Therefore $|e(i) - e(-i)| = 0 \leq 1$ for all i $1 \leq i \leq 2p_l$. We have $|v(i) - v(-i)| \leq 1$ for all i $1 \leq i \leq 2p_l$.

So the digraph Cay (G,S) $\times \overrightarrow{K_2}$ admits H_n cordial labeling when m is odd and p_l is even.

(ii) m and p_l are even

Now define f from the edge set of Cay (G,S) $\times \overrightarrow{K_2}$ (E) to the set $\{\pm 1, \pm 2, \dots \pm n\}$ as follows.

For $1 \leq i \leq p_l$ and $1 \leq j \leq m+1$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3 \dots m-1 \\ -i & \text{for } j = 2,4 \dots m \end{cases}$$

For j = m + 1,

$$f(e_{ij}) = \begin{cases} i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\ i - p_l - 1 & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l \end{cases}$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3 \dots m-1 \\ -i & \text{for } j = 2,4 \dots m-2 \\ i - 3p_l - 1 & \text{for } j = m \end{cases}$$

Then the induced function for $1 \leq i \leq p_l$ and $1 \leq j \leq m+1$

$$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$$

For $1 \leq i \leq \frac{p_l}{2}$ and $1 \leq j \leq m+1$

$$f^*(v_i) = i - i + i - i + \cdots + i - i + i$$
$$= i$$

For $\frac{p_l}{2} + 1 \leq i \leq p_l$ and $1 \leq j \leq m+1$

$$f^*(v_i) = i - i + i - i + \cdots + i - i + i - p_l - 1$$
$$= i - p_l - 1$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$$

$$= i - i + i - i + \cdots + i + i - 3p_l - 1$$

$$= 2i - 3p_l - 1$$

Therefore the induced function $f^*(v_i)$ is for $1 \leq i \leq 2p_l$ is

$$f^*(v_i) = \begin{cases} i & \text{for } 1 \leq i \leq \frac{p_l}{2} \\ i - p_l - 1 & \text{for } \frac{p_l}{2} + 1 \leq i \leq p_l \\ 2i - 3p_l - 1 & \text{for } p_l + 1 \leq i \leq 2p_l \end{cases}$$

For $1 \leq i \leq \frac{p_l}{2}$, $e(i) = \frac{m}{2} + 1 = e(-i)$. For $\frac{p_l}{2} + 1 \leq i \leq 2p_l$, $e(i) = \frac{m}{2} = e(-i)$. Therefore $|e(i) - e(-i)| = 0 \leq 1$ for all i $1 \leq i \leq 2p_l$. We have $|v(i) - v(-i)| \leq 1$ for all i $1 \leq i \leq 2p_l$.

So the digraph Cay (G,S) $\times \overrightarrow{K_2}$ admits H_n cordial labeling when m and p_l are even.

(iii) m and p_l are odd

Now define f from the edge set of Cay (G,S) $\times \overrightarrow{K_2}$ (E) to the set $\{\pm 1, \pm 2, \dots \pm n\}$ as follows.

For $1 \leq i \leq p_l$ and $1 \leq j \leq m+1$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3 \dots m-2, m \\ -i & \text{for } j = 2,4 \dots m-1 \\ i - p_l - 1 & \text{for } j = m+1 \text{ and } i \neq \frac{p_l+1}{2} \\ -\frac{[p_l-1]}{2} & \text{for } j = m+1 \text{ and } i = \frac{p_l+1}{2} \end{cases}$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3 \dots m-2 \\ -i & \text{for } j = 2,4 \dots m-1 \end{cases}$$

For j = m,

$$f(e_{ij}) = \begin{cases} i & \text{for } p_l + 1 \leq i \leq \frac{3p_l+1}{2} \\ i - 3p_l - 2 & \text{for } \frac{3p_l+1}{2} + 1 \leq i \leq 2p_l \end{cases}$$

Then the induced function for $1 \leq i \leq p_l$ and $1 \leq j \leq m+1$

$$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$$

For $1 \leq i \leq p_l$ and $i \neq \frac{p_l+1}{2}$

$$f^*(v_i) = i - i + i - i + \cdots + i + i - p_l - 1$$
$$= 2i - p_l - 1$$

For i $= \frac{p_l+1}{2}$

$$f^*(v_i) = i - i + i - i + \cdots + i - \frac{p_l-1}{2}$$
$$= \frac{p_l+1}{2} - \frac{p_l-1}{2} = 1$$

For $p_l + 1 \leq i \leq 2p_l$ and $1 \leq j \leq m$

$$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$$

For $p_l + 1 \leq i \leq \frac{3p_l+1}{2}$ and $1 \leq j \leq m$

$$f^*(v_i) = i - i + i - i + \cdots + i - i + i$$
$$= i$$

For $\frac{3p_l+1}{2}+1 \le i \le 2p_l$ and $1 \le j \le m$
$f^*(v_i) = i - i + i - i + \cdots + i - i + i - 3p_l - 2$
$= i - 3p_l - 2$

Therefore the induced function $f^*(v_i)$ is for $1 \le i \le 2p_l$ is

$$f^*(v_i) = \begin{cases} 2i - p_l - 1 & \text{for } 1 \le i \le p_l \text{ and } i \ne \frac{p_l+1}{2} \\ 1 & \text{for } i = \frac{p_l+1}{2} \\ i & \text{for } p_l + 1 \le i \le \frac{3p_l+1}{2} \\ i - 3p_l - 1 & \text{for } \frac{3p_l+1}{2} + 1 \le i \le 2p_l \end{cases}$$

For $1 \le i \le p_l$ and $i \ne \frac{p_l+1}{2}$, $e(i) = \frac{m+1}{2} = e(-i)$. For $i = \frac{p_l+1}{2}$, $e(i) = \frac{m+1}{2}$ and $e(-i) = \frac{m-1}{2}$. For $p_l + 1 \le i \le \frac{3p_l+1}{2}$, $e(i) = \frac{m+1}{2} = e(-i)$.

For $\frac{3p_l+1}{2} + 1 \le i \le 2p_l$, $e(i) = \frac{m-1}{2} = e(-i)$. Therefore $|e(i) - e(-i)| \le 1$for all i $1 \le i \le 2p_l$. We have $|v(i) - v(-i)| \le 1$ for all i $1 \le i \le 2p_l$.

So the digraph Cay (G,S) $\times \overrightarrow{K_2}$ admits H_n cordial labeling when m and p_l are odd.

(iv) m is even and p_l is odd

Now define f from the edge set of Cay (G,S) $\times \overrightarrow{K_2}$ (E) to the set $\{\pm 1, \pm 2, \ldots \pm n\}$ as follows.

For $1 \le i \le p_l$ and $1 \le j \le m + 1$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3 \ldots m - 1 \\ -i & \text{for } j = 2,4 \ldots m \end{cases}$$

For j = m + 1,

$$f(e_{ij}) = \begin{cases} i & \text{for } 1 \le i \le \frac{p_l + 1}{2} \\ i - p_l - 1 & \text{for } \frac{p_l + 3}{2} \le i \le p_l \end{cases}$$

For $p_l + 1 \le i \le 2p_l$ and $1 \le j \le m$

$$f(e_{ij}) = \begin{cases} i & \text{for } j = 1,3 \ldots m - 1 \\ -i & \text{for } j = 2,4 \ldots m - 2 \end{cases}$$

For j = m

$$f(e_{ij}) = \begin{cases} i - 3p_l - 2 & \text{for } p_l + 1 \le i \le 2p_l \text{ and } i \ne \frac{3p_l + 1}{2} \\ -\left[\frac{3p_l - 1}{2}\right] & \text{for } i = \frac{3p_l + 1}{2} \end{cases}$$

Then the induced function for $1 \le i \le p_l$

and $1 \le j \le m + 1$

$f^*(v_i) = \sum_{j=1}^{m+1} f(e_{ij})$
For $1 \le i \le \frac{p_l+1}{2}$ and $1 \le j \le m + 1$
$f^*(v_i) = i - i + i - i + \cdots + i - i + i = i$
For $\frac{p_l+1}{2} + 1 \le i \le p_l$ and $1 \le j \le m + 1$
$f^*(v_i) = i - i + i - i + \cdots + i - i + i - p_l - 1$
$= i - p_l - 1$
For $p_l + 1 \le i \le 2p_l$ and $1 \le j \le m$
$f^*(v_i) = \sum_{j=1}^{m} f(e_{ij})$
For $p_l + 1 \le i \le 2p_l$ and $i \ne \frac{3p_l+1}{2}$
$f^*(v_i) = i - i + i - i + \cdots + i + i - 3p_l - 1$
$= 2i - 3p_l - 1$

For $i = \frac{3p_l+1}{2}$

$f^*(v_i) = i - i + i - i + \cdots + i - \frac{3p_l-1}{2} = \frac{3p_l+1}{2} - \frac{3p_l-1}{2} = 1$

Therefore the induced function $f^*(v_i)$ is for $1 \le i \le 2p_l$ is

$$f^*(v_i) = \begin{cases} i & \text{for } 1 \le i \le \frac{p_l+1}{2} \\ i - p_l - 1 & \text{for } \frac{p_l+3}{2} \le i \le p_l \\ 2i - 3p_l - 1 & \text{for } p_l + 1 \le i \le 2p_l \text{ and } i \ne \frac{3p_l+1}{2} \\ 1 & \text{for } i = \frac{3p_l+1}{2} \end{cases}$$

For $1 \le i \le \frac{p_l-1}{2}$, $e(i) = \frac{m}{2} + 1 = e(-i)$. For $\frac{p_l+3}{2} \le i \le 2p_l$ and $i \ne \frac{p_l+1}{2}, \frac{3p_l+1}{2}$ $e(i) = \frac{m}{2} = e(-i)$. For $i = \frac{p_l+1}{2}, \frac{3p_l+1}{2}$ $|e(i) - e(-i)| = 1$. Therefore $|e(i) - e(-i)| \le$ 1for all i $1 \le i \le 2p_l$. We have $|v(i) - v(-i)| \le$ 1 for all i $1 \le i \le 2p_l$.

So the digraph Cay (G,S) $\times \overrightarrow{K_2}$ admits H_n cordial labeling when m is even and p is odd.

Hence the digraph Cay (G,S) $\times \overrightarrow{K_2}$ is H_n cordial.

C. Bimagic Labeling

The Cayley digraph Cay (G,S) with p vertices and $|S| \equiv 0 (\text{mod } 2)$ admits total bimagic labeling.

Proof:

From the construction of the Cayley digraph, we have p vertices and mp arcs where m is the number of generators. Let us denote the vertex set of Cay (G,S) as $V = \{v_1, v_2, \ldots, v_p\}$ and the edge set of Cay(G,S) as $E = E_{S_1} \cup E_{S_2} \ldots \cup E_{S_m} = \{e_{11}, e_{12}, \ldots, e_{1m}, e_{21}, e_{22}, \ldots, e_{p1}, \ldots, e_{pm}\}$ where
E_{S_1} = set of all outgoing arcs from v_i generated by S_1

E_{S_2}= set of all outgoing arcs from v_i generated by S_2

...

E_{S_m}= set of all outgoing arcs from v_i generated by S_m and e_{ij} is the outgoing arc of vertex v_i generated by S_j. To prove the Cayley digraph Cay (G,S) is total bimagic, we have to show that there exists a bijection f: $V \cup E \rightarrow \{1,2,\ldots,|V \cup E|\}$ such that for any vertex vi the sum of the labels of outgoing edges of vi together with the label of itself is equal to either of constants k1 and k2. We prove this theorem in two cases as for odd number of vertices and even number of vertices.

Case (i) when p is even

Define f : V → {1,2,…,p} as
$f(v_i) = i$ for $1 \le i \le p$ and

Define f : E → $\{p+1, p+2, \ldots, mp\}$ as

For j ≠ m and $1 \le i \le p$

$$f(e_{ij}) = \begin{cases} (m+1)p - i + 1 & \text{for } j = 1,3,\ldots,m-3 \\ mp + i & \text{for } j = 2,4,\ldots,m-2, m-1 \end{cases}$$

For j = m

$$f(e_{ij}) = \begin{cases} (m+1)p - 2i + 2 & \text{for } 1 \le i \le \frac{p}{2} \\ (m+2)p - 2i + 1 & \text{for } \left(\frac{p}{2}\right) + 1 \le i \le p \end{cases}$$

For any arbitrary vertex vi, $1 \le i \le \frac{p}{2}$

The sum

$$S(v_i) = i + 2p - i + 1 + 2p + i + 4p - i + 1 + 4p + i + \cdots + (m-2)p - i + 1 + (m-2)p + i + (m-1)p + i + (m+1)p - 2i + 2$$

$$= i(0) + 2(2p + 4p + \cdots + (m-2)p) + (m-1)p + (m+1)p + \frac{1(m-2)}{2} + 2$$

$$= \frac{m+2}{2}(1 + mp)$$

$= k_1$(Say) for all m & p.

For any arbitrary vertex vi, $\left(\frac{p}{2}\right) + 1 \le i \le p$

The sum

$$S(v_i) = i + 2p - i + 1 + 2p + i + 4p - i + 1 + 4p + i + \cdots + (m-2)p - i + 1 + (m-2)p + i + (m-1)p + i + (m+2)p - 2i + 1$$

$$= i(0) + 2(2p + 4p + \cdots + (m-2)p) + (m-1)p + (m+2)p + \frac{1(m-2)}{2} + 1$$

$$= \frac{1}{2}(p(m^2 + 2m + 2) + m)$$

$= k_2$(Say) for all m & p.

Now we have k1 ≠ k2 for all m & p.

Therefore the Cayley digraph admits total bimagic labeling when it has even number of vertices.

Case (ii) when p is odd

Define f : V → {1,2,…,p} as
$f(v_i) = i$, for $1 \le i \le p$

Define f : E → $\{p+1, p+2, \ldots, mp\}$ as

For j ≠ m and $1 \le i \le p$

$$f(e_{ij}) = \begin{cases} (m+1)p - i + 1 & \text{for } j = 1,3,\ldots,m-3 \\ mp + i & \text{for } j = 2,4,\ldots,m-2, m-1 \end{cases}$$

For j = m

$$f(e_{ij}) = \begin{cases} (m+1)p - 2i + 2 & \text{for } 1 \le i \le \frac{p+1}{2} \\ (m+2)p - 2i + 2 & \text{for } \frac{p+3}{2} \le i \le p \end{cases}$$

For any arbitrary vertex vi, $1 \le i \le \frac{p+1}{2}$

The sum

$$S(v_i) = i + 2p - i + 1 + 2p + i + 4p - i + 1 + 4p + i + \cdots + (m-2)p - i + 1 + (m-2)p + i + (m-1)p + i + (m+1)p - 2i + 2$$

$$= i(0) + 2(2p + 4p + \cdots + (m-2)p) + (m-1)p + (m+1)p + \frac{1(m-2)}{2} + 2$$

$$= \frac{m+2}{2}(1 + mp)$$

$= k_1$(Say) for all m & p.

For any arbitrary vertex vi, $\frac{p+3}{2} \le i \le p$

The sum

$$S(v_i) = i + 2p - i + 1 + 2p + i + 4p - i + 1 + 4p + i + \cdots + (m-2)p - i + 1 + (m-2)p + i + (m-1)p + i + (m+2)p - 2i + 2$$

$$= i(0) + 2(2p + 4p + \cdots + (m-2)p) + (m-1)p + (m+2)p + \frac{1(m-2)}{2} + 2$$

$$= \frac{1}{2}\left(p(m^2 + 2m + 2) + (m+2)\right)$$

$= k_2$(Say) for all m & p.

Now we have k1 ≠ k2 for all m & p.

Therefore, the Cayley digraph admits total bimagic labeling when it has odd number of vertices.

IV. CONCLUSION

We extended H_n cordial labeling for digraphs and proved that the Cartesian product Cay (G,S) $\times \overrightarrow{K_2}$ is H_n cordial. Subsequently we defined edge product cordial labeling for digraphs and proved that the Cartesian product Cay (G,S) $\times \overrightarrow{K_2}$ admit edge product cordial labeling and also proved the Cayley digraph admits total bimagic labeling when it has odd number of vertices.

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