Italian Domination Number of Strong Product of Cycles

Liyang Wei and Feng Li

***Abstract*—The investigation into the domination problem and the associated subset problem of graphs is a focal point in graph theory research, sparking widespread interest and extensive exploration. This paper mainly studies the Italian domination of the strong product of two cycles. By constructing recursive Italian dominating functions, a well-defined bound for the Italian domination number in $C_n \otimes C_m$ is obtained. Furthermore, through mathematical derivation and proof, the precise Italian domination number of $C_3 \otimes C_m$ is determined.**

***Index Terms*—Roman domination, Italian domination, Cycle, Strong product.**

I. Introduction

CONSIDERING that graph $G = (V, E)$ is a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a given vertex v in G, the open neighborhood $N(v)$ comprises all vertices adjacent to v in G. The closed neighborhood of v is $N[v] = v \cup N(v)$. The degree of v is $deg(v) = |N(v)|$. The minimum and maximum values of vertex degrees in G are denoted as the minimum and maximum degrees of G, represented by $\delta(G)$ and $\Delta(G)$, respectively. We denote by P_m the path graph of order m, and by C_n the cycle graph of order n.

The cartesian product of G_1 and G_2 is denoted as $G_1 \cdot G_2$, whose the vertex set is $V(G_1) \times V(G_2)$. In $G_1 \cdot G_2$, any two vertices (u, v) and (u', v') are adjacent if $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$. The strong product of G_1 and G_2 is denoted as $G_1 \otimes G_2$, whose the vertex set is $V(G_1) \times V(G_2)$. In $G_1 \otimes G_2$, any two vertices (u, v) and (u', v') are adjacent if $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$ or $uu' \in E(G_1)$ and $vv' \in E(G_2)$.

Considering a subset D of vertices in V. If each vertex outside of D is connected to at least one vertex within D, then D forms a dominating set for G. The smallest number of vertices in such a dominating set is referred to as the domination number of G, denoted by $\gamma(G)$. Considering practical needs, various other formulations of domination can be explored and extended. A Roman dominating function of G is a function $f\colon V(G) \to \{0, 1, 2\}$ defined on the vertex set of G. This function f has the property that if every vertex v with a property of $f(v) = 0$ is connected to at least one vertex u satisfying $f(u) = 2$, such a function is termed a Roman dominating function or RDF. The weight of f is $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of an RDF on G is called the Roman domination number of G, denoted by $\gamma_R(G)$.

The origin of Roman domination can be traced back to defense issues in the ancient Roman army. Subsequently, numerous scholars have delved into its study, and extensive research on the Roman domination number can be found in the literature [1]–[5]. In 2016, Chellali et al. [6] introduced the concept of the Roman $\{2\}$-domination number. Specifically, it is defined as a function $f\colon V(G) \to \{0, 1, 2\}$ on the vertex set of G, satisfying that for each vertex $f(v) = 0$ in G, $\sum_{u \in N(v)} f(u) \geq 2$. Such a function is called an Italian dominating function or IDF. The weight of f is $w(f) = \sum_{v \in V(G)} f(v)$, and the minimum weight of among all IDFs on G is called the Italian domination number of G, denoted by $\gamma_I(G)$. Until 2017, Henning et al. [7] referred to it as the Italian domination number.

Italian domination, being an increasingly active subject of research in graph theory, has attracted the attention of many scholars, particularly within large graph classes like cartesian product graphs and lexicographic product graphs [8,9]. Gao Hong et al. [10] provide bounds for the Italian domination number in $C_n \cdot P_m$. Gao Hong et al. [11] also determine the exact Italian domination number of the generalized petersen graph $P(n, 3)$. Li Zepeng et al. [12] establish the lower bound of $\gamma_I(C_n \cdot C_5)$, and find the exact values of $\gamma_I(C_n \cdot C_3)$ and $\gamma_I(C_n \cdot C_4)$. Martinez et al. [13] study the Italian domination number of lexicographic product graphs and their relationship with the traditional domination number. Both literature [14] and literature [15] study the Italian domination number of some product graphs for directed cycles. Volkmann [16] delves into the Italian domination number of directed graphs, providing specific values for Italian domination numbers within various directed graph classes. In addition, scholars have investigated various other domination numbers, including double Italian domination [17], double Roman domination [18], global Italian domination [19], perfect Italian domination [20,21], independent Italian domination [22], out-independent Italian domination [23], total Italian domination [24] and so on.

This paper primarily establishes the exact Italian domination number of $C_3 \otimes C_m$, along with determining the bounds for $\gamma_I(C_n \otimes C_m)$.

II. Main results

Let $G = C_n \otimes C_m$ be the strong product of two cycles of order n and m. The vertex set of G is defined as: $V = \{v_{i,j} | 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$. An edge exists between vertices $v_{a,x}$ and $v_{b,y}$ precisely if one of the following conditions applies: $|b - a| = 1, n-1$ and $|y - x| = 1, m-1$ or $b = a$ and $|y - x| = 1, m-1$ or $y = x$ and $|b - a| = 1, n-1$. An Italian dominating function for a vertex $v_{i,j}$ is denoted as $f(v_{i,j})$ in G.

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The graphical structure of $C_n \otimes C_m$ is illustrated in Figure 1:

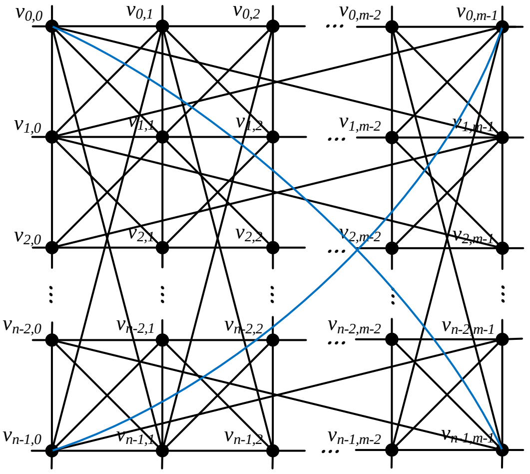


Fig. 1. The graphical structure of $C_n \otimes C_m$

Lemma 2.1. ([6]) For any graph G, $\Delta(G)$ represents the maximum degree of G and $V(G)$ represents the vertex set of G, then:

$$\gamma_I(G) \geq \frac{2|V(G)|}{\Delta(G)+2}.$$

The following result directly follows from Lemma 2.1.

Corollary 2.2. For $G = C_n \otimes C_m$, where $n \geq 3$ and $m \geq 3$, we have $\gamma_I(G) \geq \lceil \frac{mn}{5} \rceil$.

Now we give the upper bound of $\gamma_I(C_n \otimes C_m)$ by constructing Italian dominating functions for $C_n \otimes C_m$.

Theorem 2.3. Let G be the strong product of the cycles C_n and C_m, where $n \geq 3$ and $m \geq 3$. If $n \equiv 0(\text{mod } 3)$, then:

$$\gamma_I(G) \leq \begin{cases} \frac{2mn}{9}, & m \equiv 0(\text{mod } 3), \\ \frac{2mn+n}{9}, & m \equiv 1(\text{mod } 3), \\ \frac{2mn+2n}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

Proof: When $n \equiv 0(\text{mod } 3)$, for arbitrary i and $j (0 \leq i \leq n-1, 0 \leq j \leq m-1)$, we define the dominating function f as follows:

$$f(v_{i,j}) = \begin{cases} & i \equiv 0,1(\text{mod } 3) \\ & \quad \cap j \equiv 1(\text{mod } 3) \cap m \equiv 0,2(\text{mod } 3) \\ 1, & \cup i \equiv 1(\text{mod } 3) \\ & \quad \cap j \equiv 0,1(\text{mod } 3) \cap m \equiv 1(\text{mod } 3), \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 2, we enumerate the IDFs of $C_3 \otimes C_3$, $C_3 \otimes C_4$ and $C_3 \otimes C_5$, where R_n denotes repeating the top three rows as n increases, and R_m denotes repeating the top three columns as m increases. Observing all vertices in the figure below, and ensuring that $\sum_{u \in N(v_{i,j})} f(u) \geq 2$ for all vertices with $f(v_{i,j}) = 0$. We conclude that, according to the definition of Italian domination, the function f is an Italian dominating function.

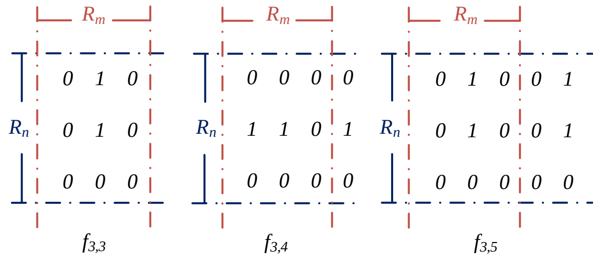


Fig. 2. Some dominating functions of $C_3 \otimes C_m$

The weight of f is:

$$w(f) = \begin{cases} 2 \times \frac{n}{3} \times \frac{m}{3} & \\ = \frac{2mn}{9}, & m \equiv 0(\text{mod } 3), \\ 2 \times \frac{n}{3} \times \frac{m-1}{3} + 1 \times \frac{n}{3} & \\ = \frac{2mn+n}{9}, & m \equiv 1(\text{mod } 3), \\ 2 \times \frac{n}{3} \times \frac{m-2}{3} + 2 \times \frac{n}{3} & \\ = \frac{2mn+2n}{9} & m \equiv 2(\text{mod } 3). \end{cases}$$

then,

$$\gamma_I(G) \leq \begin{cases} \frac{2mn}{9}, & m \equiv 0(\text{mod } 3), \\ \frac{2mn+n}{9}, & m \equiv 1(\text{mod } 3), \\ \frac{2mn+2n}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

Theorem 2.4. Let G be the strong product of the cycles C_n and C_m, where $n \geq 3$ and $m \geq 3$. If $n \equiv 1(\text{mod } 3)$, then:

$$\gamma_I(G) \leq \begin{cases} \frac{2mn+m}{9}, & m \equiv 0(\text{mod } 3), \\ \frac{2mn+4n+m+2}{9}, & m \equiv 1(\text{mod } 3), \\ \frac{2mn+2n+m+1}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

Proof: When $n \equiv 1(\text{mod } 3)$, for arbitrary i and $j(0 \leq i \leq n-1, 0 \leq j \leq m-1)$, we define the dominating function f as follows:

$$f(v_{i,j}) = \begin{cases} & i \equiv 0,1(\text{mod } 3) \\ & \quad \cap j \equiv 1(\text{mod } 3) \cap m \equiv 0,2(\text{mod } 3) \\ 1, & \cup i \equiv 0,1(\text{mod } 3) \\ & \quad \cap j \equiv 1(\text{mod } 3) \cap m \equiv 1(\text{mod} 3) \\ & \cup i \equiv 0,1(\text{mod } 3) \\ & \quad \cap j = m-1 \cap m \equiv 1(\text{mod } 3), \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 3, we enumerate the IDFs of $C_4 \otimes C_3$, $C_4 \otimes C_4$ and $C_4 \otimes C_5$, where R_n denotes repeating the top three rows as n increases, and R_m denotes repeating the top three columns as m increases. Observing all vertices in the figure below, and ensuring that $\sum_{u \in N(v_{i,j})} f(u) \geq 2$ for all vertices with $f(v_{i,j}) = 0$. We conclude that, according to the definition of Italian domination, the function f is an Italian dominating function.

The weight of f is:

$$w(f) = \begin{cases} 2 \times \frac{n-1}{3} \times \frac{m}{3} + 1 \times \frac{m}{3} & \\ = \frac{2mn+m}{9}, & m \equiv 0(\text{mod } 3), \\ 2 \times \frac{n-1}{3} \times \frac{m-1}{3} + 1 \times \frac{m-1}{3} + 2 \times \frac{n-1}{3} + 1 & \\ = \frac{2mn+4n+m+2}{9}, & m \equiv 1(\text{mod } 3), \\ 2 \times \frac{n-1}{3} \times \frac{m-2}{3} + 1 \times \frac{m-2}{3} + 2 \times \frac{n-1}{3} + 1 & \\ = \frac{2mn+2n+m+1}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

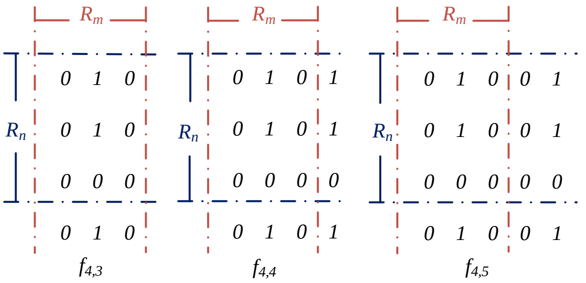


Fig. 3. Some dominating functions of $C_4 \otimes C_m$

then,

$$\gamma_I(G) \le \begin{cases} \frac{2mn+m}{9}, & m \equiv 0(\text{mod } 3), \\ \frac{2mn+4n+m+2}{9}, & m \equiv 1(\text{mod } 3), \\ \frac{2mn+2n+m+1}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

Theorem 2.5. Let G be the strong product of the cycles C_n and C_m, $n \ge 3$ and $m \ge 3$. If $n \equiv 2(\text{mod } 3)$, then:

$$\gamma_I(G) \le \begin{cases} \frac{2mn+2m}{9}, & m \equiv 0(\text{mod } 3), \\ \frac{2mn+n+2m+1}{9}, & m \equiv 1(\text{mod } 3), \\ \frac{2mn+2n+2m+2}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

Proof: When $n \equiv 2(\text{mod } 3)$, for arbitrary i and $j(0 \le i \le n-1, 0 \le j \le m-1)$, we define the dominating function f as follows:

$$f(v_{i,j}) = \begin{cases} & i \equiv 0,1(\text{mod } 3) \\ & \quad \cap\, j \equiv 1(\text{mod } 3) \cap m \equiv 0,2(\text{mod } 3) \\ 1, & \cup i \equiv 1(\text{mod } 3) \\ & \quad \cap\, j \equiv 0,1(\text{mod } 3) \cap m \equiv 1(\text{mod } 3), \\ 0, & \text{otherwise.} \end{cases}$$

In Figure 4, we enumerate the IDFs of $C_5 \otimes C_3$, $C_5 \otimes C_4$ and $C_5 \otimes C_5$, where R_n denotes repeating the top three rows as n increases, and R_m denotes repeating the top three columns as m increases. Observing all vertices in the figure below, and ensuring that $\sum_{u \in N(v_{i,j})} f(u) \ge 2$ for all vertices with $f(v_{i,j}) = 0$. We conclude that, according to the definition of Italian domination, the function f is an Italian dominating function.

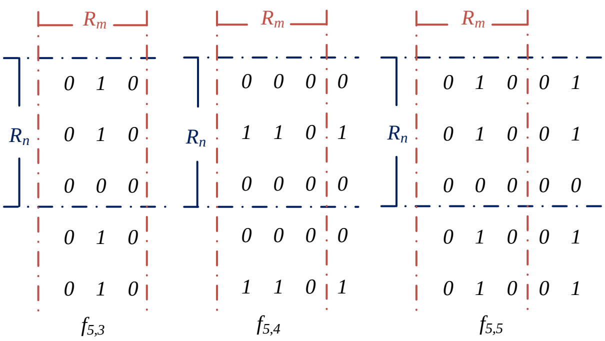


Fig. 4. Some dominating functions of $C_5 \otimes C_m$

The weight of f is:

$$w(f) = \begin{cases} 2 \times \frac{n-2}{3} \times \frac{m}{3} + 2 \times \frac{m}{3} & \\ = \frac{2mn+2m}{9}, & m \equiv 0(\text{mod } 3), \\ 2 \times \frac{n-2}{3} \times \frac{m-1}{3} + 2 \times \frac{m-1}{3} + 1 \times \frac{n-2}{3} + 1 & \\ = \frac{2mn+n+2m+1}{9}, & m \equiv 1(\text{mod } 3), \\ 2 \times \frac{n-2}{3} \times \frac{m-2}{3} + 2 \times \frac{m-2}{3} + 2 \times \frac{n-2}{3} + 2 & \\ = \frac{2mn+2n+2m+2}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

then,

$$\gamma_I(G) \le \begin{cases} \frac{2mn+2m}{9}, & m \equiv 0(\text{mod } 3), \\ \frac{2mn+n+2m+1}{9}, & m \equiv 1(\text{mod } 3), \\ \frac{2mn+2n+2m+2}{9}, & m \equiv 2(\text{mod } 3). \end{cases}$$

Through the proofs of the above theorems, we can establish the bounds for the Italian domination number of $C_n \otimes C_m$, as presented in Theorem 2.6.

Theorem 2.6. For $n, m \ge 3$, we have:

$$\left\lceil \frac{mn}{5} \right\rceil \le \gamma_I(C_n \otimes C_m) \le \begin{cases} \frac{2mn+4n+m+2}{9}, & 2n \ge m, \\ \frac{2mn+2n+2m+2}{9}, & 2n < m. \end{cases}$$

Above we provided bounds for the Italian domination number of the strong product for any two cycles. Next, we will concretize the graphs, derive the upper bound of $\gamma_I(C_3 \otimes C_m)$, and prove the lower bound of $\gamma_I(C_3 \otimes C_m)$.

Firstly, by applying Theorems 2.3, 2.4 and 2.5, we deduce the upper bound of $\gamma_I(C_3 \otimes C_m)$ as Corollary 2.7.

Corollary 2.7. Let $G = C_3 \otimes C_m$, $m \ge 3$, then:

$$\gamma_I(G) \le \begin{cases} \frac{2m}{3}, & m \equiv 0(\text{mod } 3), \\ \frac{2m+1}{3}, & m \equiv 1(\text{mod } 3), \\ \frac{2m+2}{3}, & m \equiv 2(\text{mod } 3). \end{cases}$$

In the next process, we will prove the lower bound of $\gamma_I(C_3 \otimes C_m)$. For the graph $G = C_3 \otimes C_m$, the vertex set is denoted as $V = \{v_{i,j} | 0 \le i \le 2, 0 \le j \le m-1\}$. We define $V^j = \{v_{i,j} | 0 \le i \le 2\}$ for $0 \le j \le m-1$. Let f be an Italian dominating function of G, and $f_j = f(V^j) = \sum_{v_{i,j} \in V^j} f(v_{i,j})$.

Theorem 2.8. In $G = C_3 \otimes C_m$, if f is an Italian dominating function of G, the following conclusions can be drawn:

(1) If $f_j = 0 (0 \le j \le m-1)$, then $f_{j-1} + f_{j+1} \ge 2$.

(2) For $m \equiv 0(\text{mod } 3)$, the following conclusions hold true: if $f_0 = 0$, then $f_1 \ge 0$; if $f_0 = 1$, then $f_1 \ge 0$. Moreover $f_0 + f_1 + f_2 \ge 2$, and $f_{m-1} + f_{m-2} + f_{m-3} \ge 2$.

(3) For $m \equiv 1(\text{mod } 3)$, the following conclusions hold true: if $f_0 = 0$, then $f_1 \ge 1$; if $f_0 = 1$, then $f_1 \ge 0$. When $f_0 + f_1 \ge 1$, $f_{m-1} + f_{m-2} \ge 2$.

(4) For $m \equiv 2(\text{mod } 3)$, the following conclusions hold true: if $f_0 = 0$, then $f_1 \ge 1$; if $f_0 = 1$, then $f_1 \ge 0$. When $f_0 + f_1 + f_2 \ge 2$, $f_{m-1} + f_{m-2} \ge 2$.

(5) $f_{j-1} + f_j + f_{j+1} \ge 2 (2 \le j \le m-3)$.

Proof: (1) If $f_j = 0 (0 \le j \le m-1)$, then $f(v_{0,j}) = f(v_{1,j}) = f(v_{2,j}) = 0$, then

$$\begin{aligned} f_{j-1} + f_{j+1} &= \sum_{i=0}^{2} f(v_{i,j-1}) + \sum_{i=0}^{2} f(v_{i,j+1}) \\ &= (f(v_{0,j-1}) + f(v_{2,j-1}) + f(v_{0,j+1}) \\ &\quad + f(v_{2,j+1})) + (f(v_{1,j-1}) + f(v_{1,j+1})) \\ &\ge 1 + 1 \\ &\ge 2. \end{aligned}$$

or

$$\begin{aligned}f_{j-1}+f_{j+1}&=\sum_{i=0}^{2}f(v_{i,j-1})+\sum_{i=0}^{2}f(v_{i,j+1})\\&=(f(v_{0,j-1})+f(v_{2,j-1})+f(v_{0,j+1})\\&\quad+f(v_{2,j+1}))+(f(v_{1,j-1})+f(v_{1,j+1}))\\&\geq 0+2\\&\geq 2.\end{aligned}$$

In summary, there is $f_{j-1}+f_{j+1}\geq 2$.

(2) If $f_0=0$, according to this theorem (1), it follows that: $f_{m-1}+f_1\geq 2$. If $f_1\geq 0$, then $f_{m-1}\geq 2$. Assuming $f_1=0$, by this theorem (1), we have: $f_0+f_2\geq 2$, implying that $f_2\geq 2$;

If $f_0=1$, then at least one of $f(v_{0,0})$ and $f(v_{1,0})$ is equal to 0. Now, if $f_{m-1}\geq 1$, then $f_1\geq 0$. Suppose that $f_1=0$, according to this theorem (1), we have $f_0+f_2\geq 2$, implying that $f_2\geq 1$;

In summary, we can conclude that $f_0+f_1+f_2\geq 2$. Based on the same principle, we can infer that $f_{m-1}+f_{m-2}+f_{m-3}\geq 2$.

(3) If $f_0=0$, according to this theorem (1), we have: $f_{m-1}+f_1\geq 2$. If $f_1\geq 1$, then $f_{m-1}\geq 1$, implying that at least one of $f(v_{0,m-1})$ and $f(v_{1,m-1})$ is equal to 0, we can deduce that $f_{m-2}\geq 1$;

If $f_0=1$, then at least one of $f(v_{0,0})$ and $f(v_{1,0})$ is equal to 0. Now, if $f_{m-1}\geq 1$, it implies that $f_1\geq 0$, and at least one of $f(v_{0,m-1})$ and $f(v_{1,m-1})$ is equal to 0, we can deduce that $f_{m-2}\geq 1$;

In summary, when $f_0+f_1\geq 1$, there is $f_{m-1}+f_{m-2}\geq 2$.

(4) If $f_0=0$, according to this theorem (1), we have: $f_{m-1}+f_1\geq 2$. If $f_1\geq 1$, then it implies that $f_{m-1}\geq 1$. Let's consider the case when $f_1=1$. In this scenario, there exist vertices in V^1 with a function value of 0. As a result, we can conclude that $f_2\geq 1$. Now, let's suppose that $f_{m-1}=1$. In this situation, there are vertices in V^{m-1} with a function value of 0. Hence, $f_{m-2}\geq 1$;

If $f_0=1$, it implies that at least one of $f(v_{0,0})$ and $f(v_{1,0})$ is equal to 0. Now, if $f_{m-1}\geq 1$, we can deduce that $f_1\geq 0$. Let's consider the case when $f_1=0$. According to this theorem (1), we have $f_0+f_2\geq 2$, which means $f_2\geq 1$. Now, let's assume that $f_{m-1}=1$. In this scenario, there exist vertices in V^{m-1} with a function value of 0, thus $f_{m-2}\geq 1$;

In summary, when $f_0+f_1+f_2\geq 2$, there is $f_{m-1}+f_{m-2}\geq 2$.

(5) If $f_j=0$, it is known from this theorem (1) that $f_{j-1}+f_{j+1}\geq 2$, so $f_{j-1}+f_j+f_{j+1}\geq 2$; If $f_j=1$, suppose that $f(v_{1,j})=1$. In this case, at least one of $f(v_{0,j})$ and $f(v_{2,j})$ is equal to 0. Thus, we can deduce that $f(v_{1,j-1})+f(v_{1,j+1})\geq 1$. In summary, $f_{j-1}+f_j+f_{j+1}\geq 2$.

Theorem 2.9. Let G be the strong product of the cycles C_3 and C_m, where $m\geq 3$, then $\gamma_I(G)\geq\lceil\frac{2m}{3}\rceil$.

Proof: The following are categorical discussions of the cases when $m=3$ and $m\geq 4$.

Case 1. $m\geq 4$;

From Theorem 2.8, for $m\equiv 0(\mathrm{mod}\ 3)$, $f_0+f_1+f_2\geq 2$, $f_{m-1}+f_{m-2}+f_{m-3}\geq 2$; for $m\equiv 1(\mathrm{mod}\ 3)$, when $f_0+f_1\geq 1$, $f_{m-1}+f_{m-2}\geq 2$; for $m\equiv 2(\mathrm{mod}\ 3)$, when $f_0+f_1+f_2\geq 2$, $f_{m-1}+f_{m-2}\geq 2$. $f_{j-1}+f_j+f_{j+1}\geq 2(2\leq j\leq m-3)$.

When $m\equiv 0(\mathrm{mod}\ 3)$, we have

$$\begin{aligned}w(f)=\sum_{j=0}^{m-1}f_j&=(f_0+f_1+f_2)+\sum_{j=4}^{m-5}(f_{j-1}+f_j+f_{j+1})\\&\quad+(f_{m-3}+f_{m-2}+f_{m-1})\\&\geq 2+\frac{2(m-6)}{3}+2=\frac{2m}{3}.\end{aligned}$$

When $m\equiv 1(\mathrm{mod}\ 3)$, we have

$$\begin{aligned}w(f)=\sum_{j=0}^{m-1}f_j&=(f_0+f_1)+\sum_{j=3}^{m-4}(f_{j-1}+f_j+f_{j+1})\\&\quad+(f_{m-2}+f_{m-1})\\&\geq 1+\frac{2(m-4)}{3}+2=\frac{2m+1}{3}.\end{aligned}$$

When $m\equiv 2(\mathrm{mod}\ 3)$, we have

$$\begin{aligned}w(f)=\sum_{j=0}^{m-1}f_j&=(f_0+f_1+f_2)+\sum_{j=4}^{m-4}(f_{j-1}+f_j+f_{j+1})\\&\quad+(f_{m-2}+f_{m-1})\\&\geq 2+\frac{2(m-5)}{3}+2=\frac{2m+2}{3}.\end{aligned}$$

Case 2. $m=3$;

According to Theorem 2.8, we have $w(f)=f_0+f_1+f_2\geq 2$. Therefore $\gamma_I(C_3\otimes C_3)\geq 2=\lceil\frac{2m}{3}\rceil$.

In summary, $\gamma_I(G)\geq\lceil\frac{2m}{3}\rceil$.

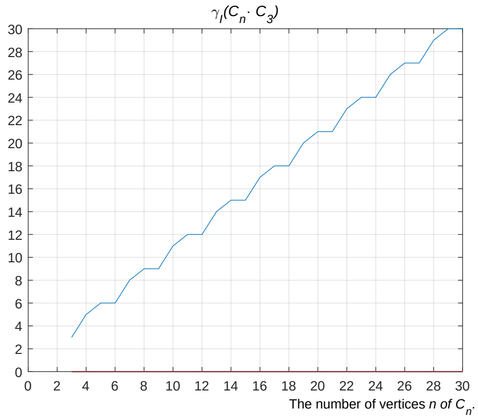
Corollary 2.7 establishes $\gamma_I(C_3\otimes C_m)\leq\lceil\frac{2m}{3}\rceil$, and Theorem 2.9 proves $\gamma_I(C_3\otimes C_m)\geq\lceil\frac{2m}{3}\rceil$. Therefore, the definite value of $\gamma_I(C_3\otimes C_m)$ that can be deduced is as follows:

Theorem 2.10. Let $G=C_3\otimes C_m$, $m\geq 3$, then $\gamma_I(G)=\lceil\frac{2m}{3}\rceil$.

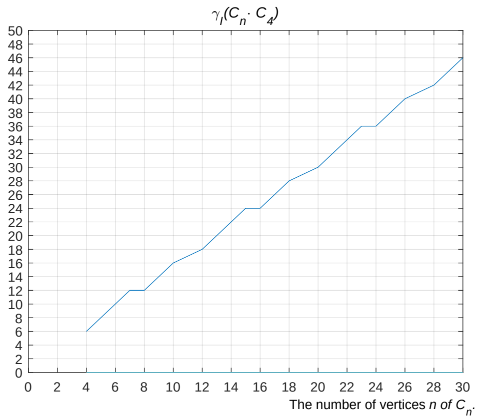
III. Simulation experiment

The cartesian product graph of cycles, known for its widespread applications in mathematics and computer science, constitutes a network structure composed of smaller graphs. Information regarding the Italian domination number of this network is provided in reference [12]. Specifically, for $n\geq 5$, $\gamma_I(C_n\cdot C_5)\geq 2n$; for $n\geq 4$, when $m\equiv 2,6,7(\mathrm{mod}\ 8)$, $\gamma_I(C_n\cdot C_4)=\lceil\frac{3n}{2}\rceil+1$; otherwise, $\gamma_I(C_n\cdot C_4)=\lceil\frac{3n}{2}\rceil$; for $n\geq 3$, when $m\equiv 0(\mathrm{mod}\ 3)$, $\gamma_I(C_n\cdot C_3)=n$; otherwise, $\gamma_I(C_n\cdot C_3)=n+1$.

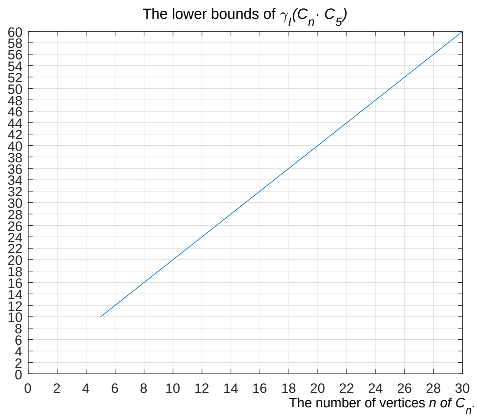
Based on existing conclusions, the strong product of cycles is regarded as an extension of its cartesian product, has stronger connectivity and smaller diameter. This implies that, compared to the cartesian product of cycles, the strong product of cycles manifests a more efficient topological structure. To compare the Italian domination numbers of the two graphs, we conduct simulation experiments to investigate the Italian domination numbers of cartesian and strong product graphs of cycles for various orders. The specific comparative results are illustrated in Figures 5 and 6, revealing the relative magnitudes of their Italian domination numbers. This further emphasizes the superiority of the strong product of cycles over its cartesian product in terms of topological structure.



(a)



(b)



(c)

Fig. 5. The variation of Italian domination numbers for $C_n \cdot C_3$ (a), $C_n \cdot C_4$ (b) and $C_n \cdot C_5$ (c) with respect to the order of factor graphs

Figure 5(a) corresponds to the scale of orders $3 \leq n \leq 30$ and $m = 3$, Figure 5(b) to the scale of orders $4 \leq n \leq 30$ and $m = 4$, Figure 5(c) to the scale of orders $5 \leq n \leq 30$ and $m = 5$, and Figure 6 to the scale of orders $3 \leq n \leq 30$ and $3 \leq m \leq 30$. Through comparison, it is observed that when the order is fixed, the upper bounds of the Italian domination number of $C_n \otimes C_m$ is consistently smaller than the lower bound or determined value of the Italian domination number of its cartesian product. As the order increases, the difference between the two also gradually enlarges. This indicates that, under the conditions of ensuring network stability and information propagation, the strong product network of cycles requires fewer controlled nodes. With the expansion of the network scale, this advantage becomes more pronounced.

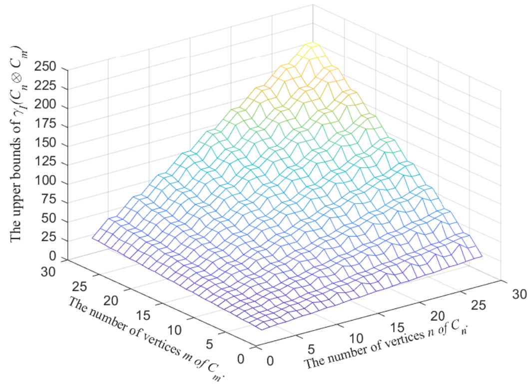


Fig. 6. The variation of Italian domination number for $C_n \otimes C_m$ with respect to the order of factor graphs

IV. Conclusion

This paper primarily delves into the Italian domination number of the strong product of two cycles. It establishes a precise bound for the Italian domination number of $C_n \otimes C_m$ by devising recursive Italian dominating functions. Furthermore, it rigorously determines the exact value of the Italian domination number of $C_3 \otimes C_m$ through a mathematical derivation, yielding $\gamma_I(C_3 \otimes C_m) = \lceil \frac{2m}{3} \rceil$.

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