The Principal Axis Theorem for Real Symmetric Interval Matrices with Applications on Spring - Mass System

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Abstract—**The existence of real symmetric interval matrices is a natural phenomenon that can be found in a variety of real-life scenarios. In this particular feature, we develop the principal axis/spectral theorem for real symmetric interval matrices. This theorem paves the way for exciting developments in interval matrix theory, particularly concerning issues such as approximations, dealing with inexact or vague data, coping with the unavailability of data, and managing errors in measurements. To demonstrate the practical utility of our approach, we delve into a concrete real-world example involving vibration analysis within an uncertain environment. This research opens up new horizons for the effective utilization of interval matrices in addressing challenges arising from uncertainty and imprecision in diverse fields.**

Index Terms—**Interval, interval eigenvalues, interval eigenvectors, principal axis theorem, diagonalization, spring-mass system.**

I. Introduction

INTERVAL analysis allows us to automatically account for the propagation of uncertainties through mathematical operations. For instance, when we calculate the natural frequency using interval parameters, the result will be an interval that captures the range of possible frequencies due to uncertain mass and stiffness. Remember that interval arithmetic and its applications can be quite intricate, and their effective use requires a solid understanding of the underlying concepts.

The application of interval arithmetic in matrix computations was initiated by Hansen and Smith [1]. Building on this foundation, numerous researchers have delved into the study of interval matrices, drawing motivation from their work.

Ganesan and Veeramani [2] introduced novel arithmetic operations on interval numbers, while Ganesan [3] discussed pivotal properties inherent in interval matrices.

Modal analysis, a method for identifying a system's dynamic properties, is fundamentally based on eigenvalues and eigenvectors. Identification of prominent vibration modes, their frequencies, and associated mode shapes are made possible through modal analysis. The optimization of designs, foretelling possible problems, and assuring the secure operation of buildings all depend on this information. Particularly in the context of mechanical and structural systems, interval eigenvalues and eigenvectors play a significant role in vibration analysis. Vibration analysis entails analysing systems' oscillatory activity, which can be found in a variety of engineering applications, including equipment, bridges, buildings, and aeroplanes. A system's inherent frequencies, mode shapes, and stability can all be better understood by comprehending the eigenvalues and eigenvectors of the system's dynamic model.

The natural frequencies of a vibrating system are represented by interval eigenvalues. These oscillation rates are the system's natural ones when no outside forces are present. In order to prevent resonance, which can result in excessive vibrations and even failure, natural frequencies must be considered while designing and operating mechanical systems. The mode forms of a vibrating system are represented by interval eigenvectors. The patterns of displacement or deformation that the system experiences during vibration are each illustrated by a different eigenvector. Understanding how a building or machine moves and deforms during its vibrational motion depends on these shapes. Engineers can find possible weak spots or places prone to fatigue by analysing mode shapes. Amir et al. [4] developed a simplified computational model of the microbeam for computing the natural frequencies and mode shapes of the microstructure. In their study, Huang et al. [5]presented novel estimations of the diagonally dominant degree on the Schur complement of I(II)-block diagonally dominant matrices.

Assem Deif [6] introduced a significant advancement by characterizing the set of eigenvalues for general interval matrices and deriving upper and lower bounds for these eigenvalues. Zhiping Qiu et al. [7] discussed eigenvalue bounds of structures with uncertain-but-bounded parameters. Nerantzis and Adjiman [8] contributed a branch-and-bound algorithm that efficiently computes bounds for individual eigenvalues of symmetric interval matrices. Gavalec et al. [9] explored various forms of interval eigenvectors within max-plus algebra, distinguishing strong, strongly universal, and weak interval eigenvectors. Abhirup Sit [10] extended the discussion on interval matrices by investigating their properties. In doing so, the author established theoretical outcomes concerning the regularity and singularity of interval matrices through the utilization of their eigenvalues. Niranjan et al. [11] presented a comprehensive framework for the spectral theory using the techniques of multiparameter eigenvalue problems (MEP) in matrix form.

Singh and Gupta [12]accomplished this by utilising the deviation amplitude of the interval matrix as a perturbation around its nominal value when solving the classic interval eigenvalue problem.

Dimarogonas [13] studied interval analysis of vibrating

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systems. Mingjun et al. [14] proposed a method to correct the parameters of the mass-spring model by means of using a generative adversarial network.

Jinwu Li et al. [15] discussed uncertain vibration analysis based on the conceptions of differential and integral of interval process. Liu and Rao [16] studied vibration analysis in the presence of uncertainties using universal grey system theory. Rao and Berke [17] analysis uncertain structural systems using interval analysis. Suroto et al. [18] determined the solution of linear balance systems using Cholesky decomposition of a matrix over the symmetrized max-plus algebra. Beyond the realm of interval matrices, Helffer [19] engaged in the study of spectral theory and its applications. Rousseau [20] extended the scope of the spectral decomposition theorem for real symmetric matrices through a topos-theoretic perspective and unveiled its practical implications. Similarly, Oliveira [21] delved into the applications stemming from the spectral theorem, broadening the understanding of its significance.

Symmetric interval matrices have applications in various fields, including optimization, control theory, and uncertainty quantification. They are used to model situations where the exact values of certain parameters are uncertain and lie within a given range. Operations involving symmetric interval matrices can be more complex than those involving regular matrices with fixed values, as the arithmetic operations need to consider the interval arithmetic rules to maintain the intervals' bounds properly.

The topic of interval matrix algebra in a sound algebraic environment was examined by Hema Surya et al. [22]–[25] They accomplished this by meticulously defining a field and a vector space that includes equivalence classes of intervals. The set of symmetric interval matrices on IR lacks an algebraic structure. Consequently, by establishing an equivalence relation on IR, we can obtain a field E. The transition from IR to E is bypassed, and we simply approach the field R, which is isomorphic to E. The researchers used the existing classical findings in R and reverted to IR / IR^n / $IR^{n\times n}$using the pairing technique.

This paper demonstrates the principal axis/spectral theorem for real symmetric interval matrices by the utilisation of the pairing technique as described in [22]. In this study, we examine a spring-mass system that is subject to uncertainty, which is characterised by a set of interval ordinary differential equations. The articlepresents numerical examples to demonstrate the theories it has established.

In Section 2, we will review some basic facts for interval matrix theory. In Section 3, we define symmetric interval matrices, inner product space involving intervals and needed definitions. In Section 4, we prove the principal axis/spectral theorem for real symmetric interval matrices. In Section 5, we provide an example for the verification of the principal axis theorem. In Section 6, we provide a real world application. Section 7 provides the conclusion of this article.

II. Interval Matrix Theory

The fundamental principles and ideas pertaining to intervals and interval matrices, as explained in the work of [22], are as follows:

We define the set IR as follows: $IR = \{\tilde{a} = [a^L, a^U] : a^L \leq a^U \text{ and } a^L, a^U \in R\}$. This set comprises all the closed intervals in R. If $a^L = a^U$,then $\tilde{a}$ is called a degenerate interval.

These intervals can be represented as ordered pairs $\langle m, w\rangle$, which are defined as follows: Given an interval $\tilde{a} = [a^L, a^U] \subseteq R$, we define $m(\tilde{a})$ as $\left(\dfrac{a^L + a^U}{2}\right)$ and $w(\tilde{a})$ as $\left(\dfrac{a^U - a^L}{2}\right)$. Consequently, $\tilde{a}$ can be uniquely expressed as $\langle m(\tilde{a}), w(\tilde{a})\rangle$.

Conversely, if you have $\langle m(\tilde{a}), w(\tilde{a})\rangle$, you can determine a^L and a^U as follows: $m(\tilde{a})-w(\tilde{a}) = a^L$ and $m(\tilde{a})+w(\tilde{a}) = a^U$ for the interval $[a^L, a^U]$. Therefore, given $\langle m(\tilde{a}), w(\tilde{a})\rangle$, you can uniquely recover the interval $[a^L, a^U]$.

An interval matrix is a matrix in which each entry is represented as an interval of real numbers rather than a single precise value. Intervals are sets of numbers that include all values within a specified range. In the context of interval matrices, these intervals are used to represent uncertainty or imprecision in the matrix's entries. An interval matrix is expressed as $\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1n} \\ \dots & \dots & \dots \\ \tilde{a}_{m1} & \dots & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})$, for every $\tilde{a}_{ij} \in IR$. Let $m(\tilde{\mathbf{A}})$ and $w(\tilde{\mathbf{A}})$ are defined as $m(\tilde{\mathbf{A}}) = \begin{pmatrix} m(\tilde{a}_{11}) & \dots & m(\tilde{a}_{1n}) \\ \dots & \dots & \dots \\ m(\tilde{a}_{m1}) & \dots & m(\tilde{a}_{mn}) \end{pmatrix}$ and $w(\tilde{\mathbf{A}}) = \begin{pmatrix} w(\tilde{a}_{11}) & \dots & w(\tilde{a}_{1n}) \\ \dots & \dots & \dots \\ w(\tilde{a}_{m1}) & \dots & w(\tilde{a}_{mn}) \end{pmatrix}$ which is always non-negative.

Remark 1: Let $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix}$ for every $\tilde{x}_i \in IR$ is an interval vector. Also midpoint and width of an interval vector are defined as $m(\tilde{\mathbf{x}}) = \begin{pmatrix} m(\tilde{x}_1) \\ m(\tilde{x}_2) \\ \vdots \\ m(\tilde{x}_n) \end{pmatrix}$ and $w(\tilde{\mathbf{x}}) = \begin{pmatrix} w(\tilde{x}_1) \\ w(\tilde{x}_2) \\ \vdots \\ w(\tilde{x}_n) \end{pmatrix}$. If every element in $\tilde{\mathbf{x}}$ is non-negative, then thevector is said to be non-negative.

III. Symmetric Interval Matrices

A symmetric interval matrix is a mathematical concept that involves matrices with symmetric entries that are intervals rather than fixed values. An interval is a range of values between a lower and an upper bound. In this context, a symmetric interval matrix is a matrix where each entry is an interval, and the matrix is symmetric, meaning that the values across the main diagonal are reflected across that diagonal. Mathematically, a symmetric interval matrix A can be represented as: each entry in the matrix is an interval

$\tilde{a} = [a_{ij}^L, a_{ij}^U]$. The matrix is symmetric, so $\tilde{a}_{ij} = \tilde{a}_{ji}, \forall i, j$.

Definition 1: Let R^n be equipped with an Inner product $(\ |\ )$. We now proceed to define the inner product of two interval vectors in IR^n corresponding to the inner product $(\ |\ )$ we started with in R^n.

Let $\tilde{\mathbf{v}} = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_n \end{pmatrix}$ and $\tilde{\mathbf{w}} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \tilde{w}_n \end{pmatrix}$ be interval vectors in IR^n. Consider $m(\tilde{\mathbf{v}}) = \begin{pmatrix} m(\tilde{v}_1) \\ m(\tilde{v}_2) \\ \vdots \\ m(\tilde{v}_n) \end{pmatrix}$ and $m(\tilde{\mathbf{w}}) = \begin{pmatrix} m(\tilde{w}_1) \\ m(\tilde{w}_2) \\ \vdots \\ m(\tilde{w}_n) \end{pmatrix}$, where $m(\tilde{v}_i)$ is the midpoint of the interval $\tilde{v}_i, 1 \le i \le n$ and similarly for $\tilde{w}_i$.

The inner product of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$ denoted by $(\tilde{\mathbf{v}}|\tilde{\mathbf{w}})$ is defined corresponding to an inner product in R^n (since $R^n \simeq E^n$) as follows:

$$(\tilde{\mathbf{v}}|\tilde{\mathbf{w}}) = \langle (m(\tilde{\mathbf{v}}) \mid m(\tilde{\mathbf{w}})), \max\left\{ \min_{w(\tilde{v}_i)\neq 0} w(\tilde{\mathbf{v}}), \min_{w(\tilde{w}_i)\neq 0} w(\tilde{\mathbf{w}}) \right\} \rangle .$$

The context will reveal whether $(\ |\ )$ denotes an inner product in IR^n or in R^n. For both inner products, we use the same notation $(\ |\ )$.

Definition 2: The inverse of $\tilde{\mathbf{A}}$ exists if the inverse of $m(\tilde{\mathbf{A}})$ exists.

Definition 3: $\tilde{\mathbf{A}}$ is said to be orthogonal if $\tilde{\mathbf{A}}\tilde{\mathbf{A}}^t \approx \tilde{\mathbf{A}}^t\tilde{\mathbf{A}} \approx \tilde{\mathbf{I}}$. In particular, an orthogonal interval matrix is always invertible, and $\tilde{\mathbf{A}}^{-1} \approx \tilde{\mathbf{A}}^t$.

Definition 4: A set of interval vectors $\{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, ..., \tilde{\mathbf{x}}_n\}$ are mutually orthogonal if every distinct pair of interval vectors is orthogonal, i.e. $\tilde{\mathbf{x}}_i \cdot \tilde{\mathbf{x}}_j \approx \tilde{0}$ for all $i \neq j$.

Definition 5: If every vector in an interval orthogonal set has length equivalent to $\tilde{1}$, i.e. if $\| \tilde{\mathbf{x}}_i \| \approx \tilde{1}$ for all i, then the set is called an interval Orthonormal set.

IV. Principal Axis/Spectral Theorem for Symmetric Interval Matrices

Theorem 1: Suppose that $\tilde{\mathbf{A}} \in IR^{n\times n}$ is symmetric. Then,
(i). Every interval eigen value $\tilde{\lambda}$ of $\tilde{\mathbf{A}}$ is a closed interval of R.
(ii). Interval eigenvectors corresponding to distinct interval eigenvalues are necessarily orthogonal.
(iii). There exists a diagonal interval matrix $\tilde{\mathbf{D}} \in IR^{n\times n}$ and an orthogonal interval matrix $\tilde{\mathbf{U}} \in IR^{n\times n}$ such that $\tilde{\mathbf{U}}^t\tilde{\mathbf{A}}\tilde{\mathbf{U}} \approx \tilde{\mathbf{D}}$.
(iv). There exists an orthonormal basis for E^n in which every basis interval vector is an interval eigen vector of $\tilde{\mathbf{A}}$.
Proof: (i). Consider a real symmetric interval matrix $\tilde{\mathbf{A}}$. Since $\tilde{\mathbf{A}}$ is real symmetric in IR, $m(\tilde{\mathbf{A}})$ is real symmetric in R. Real symmetric matrices are known to have real eigenvalues. Consequently, the eigenvalues of $m(\tilde{\mathbf{A}})$ are all real numbers. To obtain unique real interval eigenvalues for $\tilde{\mathbf{A}}$ from the real eigenvalues of $m(\tilde{\mathbf{A}})$, a technique known as the "method of pairing" is employed [22].

The pairing procedure involves determining a pairing number for each eigenvalue, which is done by utilizing the width matrix $w(\tilde{\mathbf{A}})$. The smallest positive eigenvalue of $w(\tilde{\mathbf{A}})$ corresponds to a unique pairing number, facilitating the generation of a unique interval eigenvalue for $\tilde{\mathbf{A}}$. Consequently, all the interval eigenvalues of $\tilde{\mathbf{A}}$ are closed intervals. If $w(\tilde{\mathbf{A}})$ is nilpotent, then all its eigenvalues are zero and this results in a degenerate solution. The pairing number is zero and hence all the interval eigenvalues of $\tilde{\mathbf{A}}$ are degenerate closed intervals whose width is zero.

In essence, this demonstrates that the well-established theorem in matrix theory, which asserts that "All eigenvalues of a real symmetric matrix are real," remains valid and applicable to interval matrices as well with analogues changes.

(ii). Let $\tilde{\mathbf{A}}\tilde{\mathbf{v}}_1 \approx \tilde{\lambda}_1\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{A}}\tilde{\mathbf{v}}_2 \approx \tilde{\lambda}_2\tilde{\mathbf{v}}_2$, where $\tilde{\mathbf{v}}_1 \not\approx \tilde{\mathbf{0}}, \tilde{\mathbf{v}}_2 \not\approx \tilde{\mathbf{0}}$. We assume that $\tilde{\lambda}_1 \not\approx \tilde{\lambda}_2$. Taking the dot product of $\tilde{\mathbf{A}}\tilde{\mathbf{v}}_1 \approx \tilde{\lambda}_1\tilde{\mathbf{v}}_1$ with $\tilde{\mathbf{v}}_2$, we obtain $(\tilde{\mathbf{A}}\tilde{\mathbf{v}}_1 \mid \tilde{\mathbf{v}}_2) \approx (\tilde{\lambda}_1\tilde{\mathbf{v}}_1 \mid \tilde{\mathbf{v}}_2) \Rightarrow (\tilde{\mathbf{A}}\tilde{\mathbf{v}}_1)^t.\tilde{\mathbf{v}}_2 \approx (\tilde{\lambda}_1\tilde{\mathbf{v}}_1)^t\tilde{\mathbf{v}}_2 \Rightarrow \tilde{\mathbf{v}}_1^t\tilde{\mathbf{A}}^t\tilde{\mathbf{v}}_2 \approx \tilde{\mathbf{v}}_1^t\tilde{\lambda}_1^t\tilde{\mathbf{v}}_2 \Rightarrow \tilde{\mathbf{v}}_1^t\tilde{\mathbf{A}}\tilde{\mathbf{v}}_2 \approx \tilde{\mathbf{v}}_1^t\tilde{\lambda}_1\tilde{\mathbf{v}}_2 \Rightarrow \tilde{\mathbf{v}}_1^t\tilde{\lambda}_2\tilde{\mathbf{v}}_2 \approx \tilde{\mathbf{v}}_1^t\tilde{\lambda}_1\tilde{\mathbf{v}}_2 \Rightarrow \tilde{\mathbf{v}}_1^t\tilde{\lambda}_2\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1^t\tilde{\lambda}_1\tilde{\mathbf{v}}_2 \approx \tilde{0} \Rightarrow \tilde{\mathbf{x}}^t(\tilde{\lambda}_2 - \tilde{\lambda}_1)\tilde{\mathbf{y}} \approx \tilde{0} \Rightarrow (\tilde{\lambda}_2 - \tilde{\lambda}_1)\tilde{\mathbf{v}}_1^t\tilde{\mathbf{v}}_2 \approx \tilde{0}$.

Since $(\tilde{\lambda}_2 - \tilde{\lambda}_1) \not\approx \tilde{0}$, we can conclude that $\tilde{\mathbf{v}}_1^t\tilde{\mathbf{v}}_2 \approx \tilde{0}$. Thus, the interval eigenvectors corresponding to distinct interval eigenvalues are orthogonal.

(iii). Through the spectral decomposition of $m(\tilde{\mathbf{A}})$, we derive an orthogonal matrix denoted by $\mathbf{P}$, which satisfies the equation:

$$\mathbf{D} = \mathbf{P}^t m(\tilde{\mathbf{A}})\mathbf{P} \tag{1}$$

The right-hand side of equation (1) involving the orthogonal matrix $\mathbf{P}$ is transformed into $\tilde{\mathbf{P}}$ as given below. This transformation leads to the arrival of $\tilde{\mathbf{D}}$ given by:

$$\tilde{\mathbf{D}} \approx \tilde{\mathbf{P}}^t\tilde{\mathbf{A}}\tilde{\mathbf{P}} \tag{2}$$

To obtain $\tilde{\mathbf{D}}$ from $\mathbf{D}$, the eigenvalues of $m(\tilde{\mathbf{A}})$ undergo conversion into interval eigenvalues. This conversion process relies on the pairing technique used to determine interval eigenvalues for an interval matrix $\tilde{\mathbf{A}}$.

Each column in matrix $\mathbf{P}$ represents an eigenvector of $m(\tilde{\mathbf{A}})$. These eigenvectors are then transformed into interval eigenvectors to form $\tilde{\mathbf{P}}$ through the pairing procedure. Subsequently, we compute $\tilde{\mathbf{P}}^t\tilde{\mathbf{A}}\tilde{\mathbf{P}}$, which results in a diagonal interval matrix equivalent to $\tilde{\mathbf{D}}$. Therefore, the equation $\tilde{\mathbf{D}} \approx \tilde{\mathbf{P}}^t\tilde{\mathbf{A}}\tilde{\mathbf{P}}$ represents the spectral decomposition for $\tilde{\mathbf{A}}$.

(iv). According to the spectral decomposition theorem for real symmetric matrices, we can establish the existence of an orthonormal basis denoted as $\mathcal{B} = \{b_1, b_2, \cdots, b_n\}$, where each $b_i, 1 \le i \le n$ is an eigenvector of $m(\tilde{\mathbf{A}})$.

By utilizing the technique of pairing eigenvectors, we transform the original basis $\mathcal{B}$ into a new interval orthonormal set denoted as $\tilde{\mathcal{B}} = \{\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n\}$, which exists within IR^n. This transformation corresponds to the establishment of an orthonormal basis, denoted as $\bar{\mathcal{B}}$, within the vector space E^n.

Hence, the validity of the theorem holds for the interval symmetric matrix $\tilde{\mathbf{A}}$ in IR^n. This spectral decomposition enables us to approximate $\tilde{\mathbf{D}}$ through $\tilde{\mathbf{P}}^t\tilde{\mathbf{A}}\tilde{\mathbf{P}}$, where $\tilde{\mathbf{D}}$

represents a diagonal interval matrix encompassing the interval eigenvalues, and $\tilde{\mathbf{P}}$ constitutes an orthogonal matrix composed of interval eigenvectors.

V. Numerical Example

For the real symmetric interval matrix $\tilde{\mathbf{A}} = \begin{pmatrix} [1,9] & [-3,-1] & [-6,-2] \\ [-3,-1] & [1,3] & [1,3] \\ [-6,-2] & [1,3] & [1,9] \end{pmatrix}$, show that there exist a diagonal interval matrix $\tilde{\mathbf{D}} \in IR^{n\times n}$ and an orthogonal interval matrix $\tilde{\mathbf{U}} \in IR^{n\times n}$ such that $\tilde{\mathbf{U}}^t\tilde{\mathbf{A}}\tilde{\mathbf{U}} \approx \tilde{\mathbf{D}}$.

Solution: By applying the pairing technique and arithmetic operations [22], the interval eigenvalues of the given interval matrix $\tilde{\mathbf{A}}$ are $\tilde{\lambda}_1 = \langle 10, 0.63\rangle = [9.37, 10.67]$, $\tilde{\lambda}_2 = \langle 1, 0.63\rangle = [0.37, 1.63]$ and $\tilde{\lambda}_3 = \langle 1, 0.63\rangle = [0.37, 1.63]$ and the corresponding normalised interval eigenvectors are $\tilde{\mathbf{v}}_1 = \begin{pmatrix} [-0.85, -0.49] \\ [0.15, 0.51] \\ [0.49, 0.85] \end{pmatrix}$, $\tilde{\mathbf{v}}_2 = \begin{pmatrix} [0.53, 0.89] \\ [-0.18, 0.18] \\ [0.53, 0.89] \end{pmatrix}$ and $\tilde{\mathbf{v}}_3 = \begin{pmatrix} [0.06, 0.42] \\ [0.76, 1.12] \\ [-0.42, -0.06] \end{pmatrix}$ respectively.

An orthogonal interval matrix $\tilde{\mathbf{U}}$ is obtained by writing the normalised interval eigenvectors of $\tilde{\mathbf{A}}$ as columns. Hence

$$\begin{aligned}\tilde{\mathbf{U}} &= (\tilde{\mathbf{v}}_1 \ \ \tilde{\mathbf{v}}_2 \ \ \tilde{\mathbf{v}}_3)_{3\times 3} \\ &= \begin{pmatrix} [-0.85, -0.49] & [0.53, 0.89] & [0.06, 0.42] \\ [0.15, 0.51] & [-0.18, 0.18] & [0.76, 1.12] \\ [0.49, 0.85] & [0.53, 0.89] & [-0.42, -0.06] \end{pmatrix}.\end{aligned}$$

Also $m(\tilde{\mathbf{U}}) = \begin{pmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \end{pmatrix}$ and $w(\tilde{\mathbf{U}}) = \begin{pmatrix} 0.18 & 0.18 & 0.18 \\ 0.18 & 0.18 & 0.18 \\ 0.18 & 0.18 & 0.18 \end{pmatrix} = w(\tilde{\mathbf{U}}^t)$. Now we compute $m(\tilde{\mathbf{U}}^t).m(\tilde{\mathbf{A}}).m(\tilde{\mathbf{U}}) = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and hence

$$\begin{aligned}\alpha &= \max\left\{\min_{w(\tilde{u}_{ij})\neq 0} w(\tilde{\mathbf{U}}^t), \min_{w(\tilde{a}_{ij})\neq 0} w(\tilde{\mathbf{A}}), \min_{w(\tilde{u}_{ij})\neq 0} w(\tilde{\mathbf{U}})\right\} \\ &= \max\{0.18, 1, 0.18\} = 1.\end{aligned}$$

$$\begin{aligned}\text{Now } \tilde{\mathbf{U}}^t\tilde{\mathbf{A}}\tilde{\mathbf{U}} &= \left\langle m(\tilde{\mathbf{U}}^t).m(\tilde{\mathbf{A}}).m(\tilde{\mathbf{U}}), \alpha\right\rangle \\ &\approx \begin{pmatrix} [9, 11] & \tilde{0} & \tilde{0} \\ \tilde{0} & [0, 2] & \tilde{0} \\ \tilde{0} & \tilde{0} & [0, 2] \end{pmatrix} = \tilde{\mathbf{D}}.\end{aligned}$$

Hence $\tilde{\mathbf{U}}^t\tilde{\mathbf{A}}\tilde{\mathbf{U}} \approx \tilde{\mathbf{D}}$. By applying the Spectral theorem, we are able to diagonalize the given real symmetric interval matrix orthogonally into a diagonal form.

VI. An Application on Spring-Mass System

Interval uncertainty, in the context of free vibration of a second degree of freedom undamped system, refers to the uncertainty in the natural frequencies of the system within a certain range. In structural and mechanical engineering, it's common to encounter uncertainties due to manufacturing tolerances, material properties variations, and other factors that can affect the behavior of a system. It allows for the design of more robust systems. By considering a range of possible parameter values (such as mass and stiffness), we design for a worst-case scenario. This can lead to a system that is less sensitive to manufacturing variations or material property uncertainties, resulting in better performance in real-world conditions. Engineers can be confident that even if parameters are at the extreme ends of their ranges, the system will still meet performance requirements. This is especially critical in safety-critical applications, like aerospace or structural engineering. It provides a more realistic expectation of system performance. In practice, parameters like mass and stiffness can rarely be precisely determined. Accounting for this uncertainty helps avoid over-optimistic performance predictions that may not hold in real-world conditions.

Consider a spring mass system with three springs and two masses For a two degree of freedom undamped system,

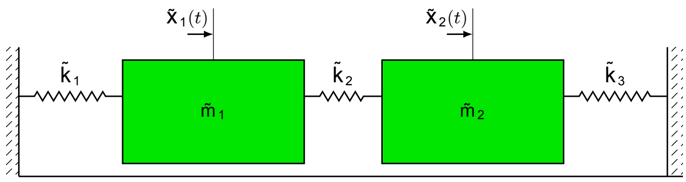

Fig. 1. Spring Mass System under interval uncertainty

the equations of motion can be represented as a system of second-order ordinary differential equations. The general form of the interval differential equations is:

$$\begin{aligned}\tilde{m}_1\ddot{\tilde{x}}_1 + (\tilde{k}_1 + \tilde{k}_2)\tilde{x}_1 - \tilde{k}_2\tilde{x}_2 = \tilde{0} \\ \tilde{m}_2\ddot{\tilde{x}}_2 - \tilde{k}_2\tilde{x}_1 + (\tilde{k}_2 + \tilde{k}_3)\tilde{x}_2 = \tilde{0},\end{aligned} \tag{3}$$

where $\tilde{m}_1$ and $\tilde{m}_2$ are the masses, $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$ are the stiffness coefficients of the springs, $\tilde{x}$ is the displacement of its equilibrium position and $\ddot{\tilde{x}}$ represents the second derivative of displacement with respect to time. The matrix form of the above system (3) is

$$\begin{pmatrix} \tilde{m}_1 & \tilde{0} \\ \tilde{0} & \tilde{m}_2 \end{pmatrix}\ddot{\tilde{\mathbf{x}}} + \begin{pmatrix} (\tilde{k}_1 + \tilde{k}_2) & -\tilde{k}_2 \\ -\tilde{k}_2 & (\tilde{k}_2 + \tilde{k}_3) \end{pmatrix}\tilde{\mathbf{x}} = \tilde{0}$$

$$\Rightarrow \tilde{\mathbf{M}}\ddot{\tilde{\mathbf{x}}} + \tilde{\mathbf{K}}\tilde{\mathbf{x}} = \tilde{0}.$$

The interval parameters are considered to be for numerical computation as, $\tilde{m}_1 = [9.6500, 10.3500]kg$, $\tilde{m}_2 = [0.8500, 1.1500]kg$, $\tilde{k}_1 = [29.9000, 30.1000]N/m$, $\tilde{k}_2 = [4.9300, 5.0700]N/m$ and $\tilde{k}_3 = [0, 0]N/m$ and initial conditions are $\tilde{x}_1(0) = [1, 1]$, $\tilde{x}_2(0) = \dot{\tilde{x}}_1(0) = \dot{\tilde{x}}_2(0) = [0, 0]$.

Therefore, the interval mass matrix

$$\tilde{\mathbf{M}} = \begin{pmatrix} [9.6500,\ 10.3500] & [0,\ 0] \\ [0,\ 0] & [0.8500,\ 1.1500] \end{pmatrix}$$

and the interval stiffness matrix is,

$$\tilde{\mathbf{K}} = \begin{pmatrix} [34.9000,\ 35.1000] & [-5.07,\ -4.9300] \\ [-5.07,\ -4.9300] & [4.9300,\ 5.0700] \end{pmatrix}.$$

The solution of the system (3) is of the form

$$\tilde{x}_1 = \tilde{\mathbf{v}}_1 cos(\tilde{w}t + \theta) \quad \text{and} \quad \tilde{x}_2 = \tilde{\mathbf{v}}_2 cos(\tilde{w}t + \theta) \qquad (4)$$

. In such a solution, $\tilde{m}_1$ and $\tilde{m}_2$ oscillate harmonically with same frequency and phase but with possibly different amplitudes $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$. Such a solution is called a "Normal modes".

In some situations where uncertainty plays a role, the natural frequency might be represented as an interval rather than a single fixed value. This reflects the fact that due to uncertainties, the natural frequency of the system can vary within a certain range. Here, the interval eigenvalues are represents the natural frequencies of a vibrating system. These frequencies are the system's inherent oscillation rates when no external forces are applied. Natural frequencies are essential in designing and operating mechanical systems to avoid resonance, which can lead to excessive vibrations and even failure. To find natural frequency, we have to solve

$$| - \tilde{w}^2\tilde{\mathbf{M}} + \tilde{\mathbf{K}}| \approx \tilde{0} \qquad (5)$$

The interval eigenvalues (natural frequencies) and interval eigenvectors (mode shapes) are computed by applying the pairing technique [22]. The midpoint of interval mass matrix and interval stiffness matrix are $m(\tilde{\mathbf{M}}) = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$ and $m(\tilde{\mathbf{K}}) = \begin{pmatrix} 35 & -5 \\ -5 & 5 \end{pmatrix}$ respectively. Similarly, width of interval mass matrix and interval stiffness matrix are $w(\tilde{\mathbf{M}}) = \begin{pmatrix} 0.350 & 0 \\ 0 & 0.150 \end{pmatrix}$ and $w(\tilde{\mathbf{K}}) = \begin{pmatrix} 0.10 & 0.07 \\ 0.07 & 0.07 \end{pmatrix}$ respectively. From the equation (5), the natural frequencies (eigenvalues) of $|m(-\tilde{w}^2\tilde{\mathbf{M}} + \tilde{\mathbf{K}})| \approx \tilde{0}$ are 1.5811, 2.4495 and the corresponding mode shapes (eigenvectors) are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ respectively. And the natural frequencies of $|w(-\tilde{w}^2\tilde{M} + \tilde{K})| \approx \tilde{0}$ are 0.2398 and 0.8336. So, for the minimum positive value 0.2398, the corresponding mode shape is $\begin{pmatrix} 0.8766 \\ 1 \end{pmatrix}$. Here 0.8766 is an eigenvector pairing number.

Therefore, the natural frequencies of the interval systems are $\tilde{w}_1 = \langle 1.5811, 0.2398\rangle$ and $\tilde{w}_2 = \langle 2.4495, 0.2398\rangle$ and the corresponding normal modes are $\tilde{\mathbf{v}}_1 = \begin{pmatrix} \langle 1, 0.8766\rangle \\ \langle 2, 0.8766\rangle \end{pmatrix}$ and $\tilde{\mathbf{v}}_2 = \begin{pmatrix} \langle 1, 0.8766\rangle \\ \langle -5, 0.8766\rangle \end{pmatrix}$. Then, solution for each mode will be:

For Mode 1:

$$\begin{aligned} \tilde{x}_1 =& \tilde{\mathbf{v}}_1 \cos(\tilde{w}_1 t + \theta_1) \\ =& \begin{pmatrix} \langle 1, 0.8766\rangle \\ \langle 2, 0.8766\rangle \end{pmatrix} \cos(\langle 1.5811, 0.2398\rangle t + \theta_1) \end{aligned}$$

For Mode 2:

$$\begin{aligned} \tilde{x}_2 =& \tilde{\mathbf{v}}_2 \cos(\tilde{w}_2 t + \theta_2) \\ =& \begin{pmatrix} \langle 1, 0.8766\rangle \\ \langle -5, 0.8766\rangle \end{pmatrix} \cos(\langle 2.4495, 0.2398\rangle t + \theta_2). \end{aligned}$$

The general solution can be obtained by a linear superposition of the two interval normal modes as

$$\begin{aligned} \tilde{\mathbf{x}}(t) =& c_1\tilde{x}_1 + c_2\tilde{x}_2 \\ =& c_1\ \tilde{\mathbf{v}}_1 cos(\tilde{w}_1 t + \theta_1) + c_2\ \tilde{\mathbf{v}}_2 cos(\tilde{w}_2 t + \theta_2) \end{aligned}$$

which implies that

$$\begin{aligned} \tilde{x}_1(t) &= c_1 \langle 1, 0.8766\rangle \cos(\langle 1.5811t, 0.2398\rangle + \theta_1) \\ &\quad + c_2 \langle 1, 0.8766\rangle \cos(\langle 2.4495t, 0.2398\rangle + \theta_2) \\ \tilde{x}_2(t) &= c_1 \langle 2, 0.8766\rangle \cos(\langle 1.5811t, 0.2398\rangle + \theta_1) \\ &\quad + c_2 \langle -5, 0.8766\rangle \cos(\langle 2.4495t, 0.2398\rangle + \theta_2). \end{aligned} \qquad (6)$$

Substituting the given initial conditions $\tilde{x}_1(0) = \langle 1, 0\rangle$, $\tilde{x}_2(0) = \dot{\tilde{x}}_1(0) = \dot{\tilde{x}}_2(0) = \langle 0, 0\rangle$ and $\theta_1 = \theta_2 = 0$ in equation (6) and solve resulting system, we get $c_1 = \left\langle \frac{5}{7}, 0 \right\rangle$, $c_2 = \left\langle \frac{2}{7}, 0 \right\rangle$. The General solution (6) becomes

$$\begin{aligned} \tilde{x}_1(t) &= \left\langle \frac{5}{7}, 0 \right\rangle \langle 1, 0.8766\rangle \cos(\langle 1.5811t, 0.2398\rangle) \\ &\quad + \left\langle \frac{2}{7}, 0 \right\rangle \langle 1, 0.8766\rangle \cos(\langle 2.4495t, 0.2398\rangle) \\ &= \left\langle \frac{5}{7}\cos(1.5811)t, \max\{0, 0.8766, \cos(0.2398)\} \right\rangle \\ &\quad + \left\langle \frac{2}{7}\cos(2.4495)t, \max\{0, 0.8766, \cos(0.2398)\} \right\rangle \\ &= \left\langle \frac{5}{7}\cos(1.5811)t, \max\{0, 0.8766, 0.9714\} \right\rangle \\ &\quad + \left\langle \frac{2}{7}\cos(2.4495)t, \max\{0, 0.8766, 0.9714\} \right\rangle \\ &= \left\langle \frac{5}{7}\cos(1.5811)t, 0.9714 \right\rangle \\ &\quad + \left\langle \frac{2}{7}\cos(2.4495)t, 0.9714 \right\rangle. \end{aligned}$$

$$\begin{aligned} \tilde{x}_2(t) &= \left\langle \frac{5}{7}, 0 \right\rangle \langle 2, 0.8766\rangle \cos(\langle 1.5811t, 0.2398\rangle) \\ &\quad + \left\langle \frac{2}{7}, 0 \right\rangle \langle -5, 0.8766\rangle \cos(\langle 2.4495t, 0.2398\rangle) \\ &= \left\langle \frac{10}{7}\cos(1.5811)t, \max\{0, 0.8766, \cos(0.2398)\} \right\rangle \\ &\quad + \left\langle \frac{-10}{7}\cos(2.4495)t, \max\{0, 0.8766, \cos(0.2398)\} \right\rangle \\ &= \left\langle \frac{10}{7}\cos(1.5811)t, \max\{0, 0.8766, 0.9714\} \right\rangle \\ &\quad + \left\langle \frac{-10}{7}\cos(2.4495)t, \max\{0, 0.8766, 0.9714\} \right\rangle \\ &= \left\langle \frac{10}{7}\cos(1.5811)t, 0.9714 \right\rangle \\ &\quad + \left\langle \frac{-10}{7}\cos(2.4495)t, 0.9714 \right\rangle. \end{aligned}$$

Hence the general solution is

$$\begin{aligned}\tilde{x}_1(t) &= \left\langle \frac{5}{7}\cos(1.5811)t, 0.9714 \right\rangle \\ &\quad + \left\langle \frac{2}{7}\cos(2.4495)t, 0.9714 \right\rangle . \\ \tilde{x}_2(t) &= \left\langle \frac{10}{7}\cos(1.5811)t, 0.9714 \right\rangle \\ &\quad + \left\langle \frac{-10}{7}\cos(2.4495)t, 0.9714 \right\rangle .\end{aligned} \tag{7}$$

That is the interval solution of the spring mass system is given by

$$\begin{aligned}\tilde{x}_1(t) &= \left[\frac{5}{7}\cos(1.5811)t + \frac{2}{7}\cos(2.4495)t - 0.9714, \right. \\ &\qquad \left. \frac{5}{7}\cos(1.5811)t + \frac{2}{7}\cos(2.4495)t + 0.9714\right] . \\ \tilde{x}_2(t) &= \left[\frac{10}{7}\cos(1.5811)t + \frac{-10}{7}\cos(2.4495)t - 0.9714, \right. \\ &\qquad \left. \frac{10}{7}\cos(1.5811)t + \frac{-10}{7}\cos(2.4495)t + 0.9714\right] .\end{aligned} \tag{8}$$

A. Results and Discussion

Figure 2 represented with different colour strips for Mass 1 and Mass 2, conveys the impact of interval uncertainty in the masses of the system. It provides insights into how variations in mass values within a specified range can affect the behavior of the system, specifically the oscillatory motion of the masses over time. These amplitudes determine how far Mass 1 and Mass 2 will move away from their respective equilibrium positions when they start oscillating. it means that Mass 1 will oscillate with a maximum displacement of 1 meter in either direction from its equilibrium position. The same applies for Mass 2.

Figure 3 depicts mass uncertainty with different amplitudes. That is, these amplitudes determine how far Mass 1 and Mass 2 will move away from their respective equilibrium positions when they start oscillating. For Mass 1, an amplitude of 2 meters signifies that it will oscillate, reaching a maximum displacement of 2 meters in either direction from its equilibrium position. Similarly, for Mass 2, an amplitude of -5 meters indicates that it will oscillate, reaching a maximum displacement of 5 meters in the opposite direction from its equilibrium position.

Figure 4 and Figure 5 illustrates the dynamic response of a two-mass, three-spring system in the presence of interval uncertainty. Interval uncertainty refers to variations or uncertainty in system parameters such as mass or stiffness, resulting in a range of possible system behaviors.

For graphical representation of the interval solution (8), it is represented as three crisp solutions as:

$\tilde{x}_1(t) = (U, M, L)$, where

U : Upper : $\frac{5}{7}\cos(1.5811)t + \frac{2}{7}\cos(2.4495)t + 0.9714.$

M : Middle : $\frac{5}{7}\cos(1.5811)t + \frac{2}{7}\cos(2.4495)t.$

L : Lower : $\frac{5}{7}\cos(1.5811)t + \frac{2}{7}\cos(2.4495)t - 0.9714.$

and $\tilde{x}_2(t) = (U, M, L)$, where

U : Upper : $\frac{10}{7}\cos(1.5811)t + \frac{-10}{7}\cos(2.4495)t + 0.9714.$

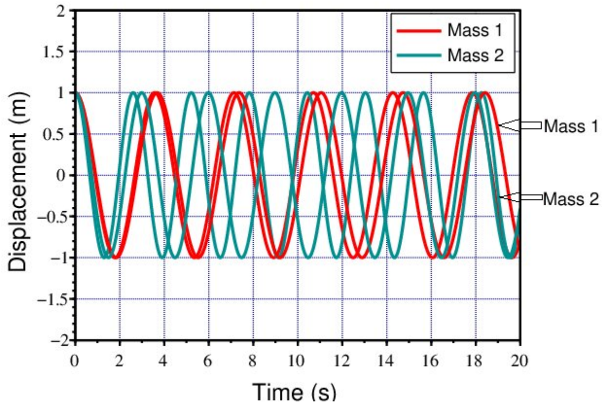

Fig. 2. Simple Hormonic Motion for Spring - Mass System under interval uncertainty with same amplitude

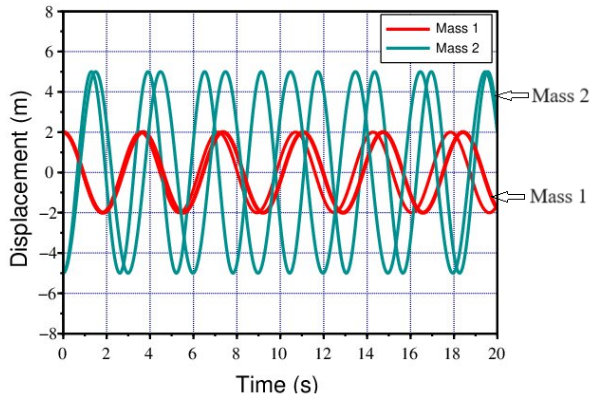

Fig. 3. Simple Hormonic Motion for Spring - Mass System under interval uncertainty with different amplitude

M : Middle : $\frac{10}{7}\cos(1.5811)t + \frac{-10}{7}\cos(2.4495)t.$

L : Lower : $\frac{10}{7}\cos(1.5811)t + \frac{-10}{7}\cos(2.4495)t + 0.9714.$

The "middle" solution curve (red curves) represents the nominal or expected response of the system, assuming typical parameter values. By plotting these three solutions together, we gain insight into the range of possible behaviors that the system can exhibit due to parameter uncertainty. It helps in assessing the system's robustness and understanding the sensitivity of the response to parameter variations. If we have precise knowledge of the mass and spring constants, the solution provided above can be directly converted into a solution for a deterministic mass-spring system. This conversion involves replacing the intervals in the solution with their respective midpoint values.

VII. Conclusion

A significant accomplishment of the article is the establishment of the renowned principal axis theorem for real symmetric interval matrices. This breakthrough not only broadens the horizons of interval matrix theory but also introduces valuable insights concerning approximations, uncertainty in data, data unavailability, measurement errors, and

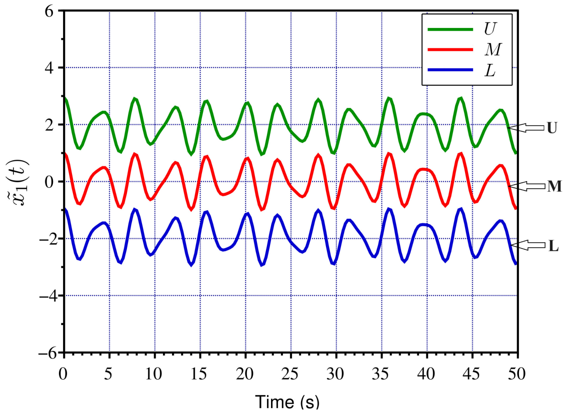

Fig. 4. Response of $\tilde{x}_1(t)$ in Spring - Mass System with interval uncertainty

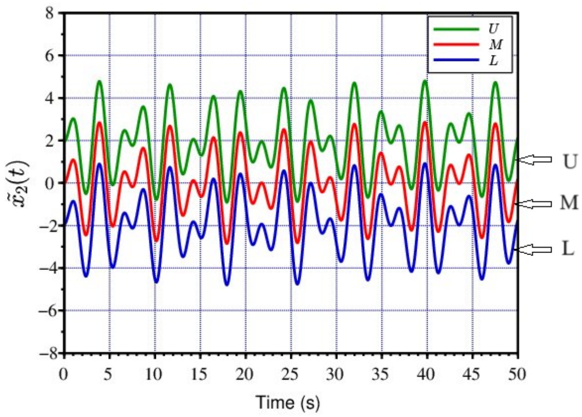

Fig. 5. Response of $\tilde{x}_2(t)$ in Spring - Mass System with interval uncertainty

related areas. Leveraging the pairing technique and arithmetic operations on interval matrices, the article delves into a real-world case study centered on spring mass system within an uncertain context. We considered a two-mass, three-spring system in the presence of interval uncertainty. Interval uncertainty refers to variations or uncertainty in system parameters such as mass or stiffness, resulting in a range of possible system behaviors. This scenario is represented as a system of interval linear differential equations and the interval solution is obtained effectively through the introduced technique. From the graphical representation of the interval solutions, we gained insight into the range of possible behaviors that the system can exhibit due to parameter uncertainty. It helps in assessing the system's robustness and understanding the sensitivity of the response to parameter variations. If we have precise knowledge of the mass and spring constants, the interval solution provided can be directly converted into a crisp solution for a deterministic mass-spring system. This conversion involves replacing the intervals in the solution with their respective midpoint values.

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