Optimization of High-speed Railway Train Timetabling Based on Lagrange Relaxation and Fuzzy Subgradient Optimization

Lihui An, Xuelei Meng, Ruhu Gao, Zheng Han, Yanxin Fu, Dongzhi Li, Ruidong Wang

***Abstract*—To address the formidable challenge of optimizing and accurately solving the extensive scale of train timetables for high-speed railways, this study adopts a directed space-time network to depict the train timetable. By introducing incompatible arc sets, constraints such as minimum headway time and train overtaking are consolidated into mutually exclusive arc segment constraints, forming an integer programming model. The model is processed using the Lagrangian relaxation method, coupled with fuzzy theory, and an enhancement is made to the key iterative algorithm—the subgradient optimization algorithm—within the Lagrangian relaxation algorithm. The aim is to eliminate potential conflicts in the allocation of transportation resources among different train operation lines. The improved fuzzy subgradient optimization algorithm effectively leverages historical subgradient information and updates the subgradient reasonably. Finally, using the Beijing-Shanghai high-speed railway as a case study, experiments are conducted to optimize and compile the train timetables of 82 train lines in the segment. The computational performance of the standard subgradient algorithm and the fuzzy subgradient algorithm is compared. The results demonstrate that, while ensuring computational accuracy, the Lagrangian relaxation algorithm based on fuzzy subgradient optimization significantly enhances the quality of the optimal solution, reducing the dual gap value from 8.51% to 7.18%. This refined Lagrangian relaxation algorithm serves as an effective approach to obtain a higher-quality train timetable for high-speed railway trains.**

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***Index Terms*——High-speed railway, Lagrange relaxation, Fuzzy, Train timetabling**

I. INTRODUCTION

Compared with the general speed railway, the high speed railway has the characteristics of high speed, large capacity, high departure density and short departure time. The train timetabling prescribes the sequence of the occupied area and the arrival, departure and passage time of the train at each station, which is the precondition to ensure the normal completion of the high-speed train transportation. As the core problem of transport organization theory and method, train timetabling can be optimized in coordination with operation plan up or integrated optimization with EMU operation plan down, is a very meaningful research topic. At the same time, the optimization of train timetabling is a problem with clear practical background and huge theoretical challenge, which requires us to consider the model's comprehensiveness and universality when we study the problem, the model should be simplified as much as possible to facilitate the solution. The optimization of train timetabling is a very large-scale multi-objective optimization problem, which is difficult to solve. How to design an efficient and reliable solution is always the key to the optimization of train timetabling [1]-[2].

When solving the optimization problem of train timetables, there are generally two approaches. One is to use commercial optimization software such as CPLEX, Gurobi, etc., for problem-solving. The other is to design heuristic algorithms for solving. Among various heuristic algorithms, the classical Lagrangian relaxation algorithm is prominent. It decomposes the complex problem into several easily solvable sub-problems, solves each sub-problem separately, performs iterative calculations, and ultimately obtains the optimal solution to the problem [3]. The convergence and optimality of the problem solution, which can be evaluated by analyzing the duality gap to assess the solution quality, can be guaranteed. It is an efficient method for solving train timetabling optimization problems. ZHOU Wenliang et al. [4] established a multi-path search model based on the construction of a weighted directed graph, taking the shortest total train travel time as the objective function, decomposing the model by introducing Lagrange multipliers, and design a multi-path search sub-algorithm to optimize the feasible solution and the dual solution. TIAN Xiaopeng et al. [5] established a linear integer programming model based on space-time arc variables to achieve the goal of minimizing the total operational cost of trains. Using the Lagrangian relaxation method, the original problem is decomposed into space-time path sub-problems and train type sub-problems. Based on lower bound dual information, a two-stage heuristic method is designed for solving the problem. GE Xin et al. [6] developed an energy-saving operation graph model to optimize the allocation of time points for multi-train interval operation. They utilized the Lagrangian relaxation algorithm to simplify complex constraints into the objective function. This decomposition enabled solving independent sub-problems within each interval using the sub-gradient optimization method, ensuring accurate solutions and synchronous allocation of time and minutes for multi-train interval operation. Alberto Caprara et al. [7] studied the train timetable problem, with the optimization objective of minimizing the deviation from the original train timetable. They established an integer programming model and utilized the Lagrangian relaxation algorithm to decompose the original problem. Following a train priority strategy, a heuristic algorithm was constructed to solve the model. Zhengwen Liao et al. [8] developed an accumulation flow variable model and decomposed the combinatorial train timetabling problem into independent shortest path subproblems using Lagrangian relaxation. These subproblems can be solved by relaxing the constraints. Erfan Hassannayebi et al. [9] considered capacity and resource constraints in the study of train timetabling problems for urban rail transit and solved them using the Lagrangian relaxation algorithm. Xuesong Zhou et al. [10] studied the train scheduling problem in the existing network of high-speed passenger railways. A multi-objective programming model was established, and a heuristic algorithm was used for solving. GAO Ruhu et al. [11] constructed a 0-1 integer programming model based on space-time arcs and relaxed the coupling constraints among trains using the Lagrangian relaxation algorithm. The problem was decomposed into subproblems of finding the shortest path for individual trains in the space-time network. LIAO Zhengwen et al. [12] developed an accumulation flow variable model and used the Lagrangian relaxation algorithm to allocate the occupation time of trains to represent the tracking intervals. It considered the weight differences among trains of different grades and compared the solution efficiency using the Lagrangian relaxation algorithm with the CPLEX solver. JIANG Feng et al. [13] aimed to maximize the total profit of the entire network's train lines. It considered constraints such as train organization, station stop requirements, time window constraints, uniqueness constraints of train lines, and relevant connection constraints. The Lagrangian relaxation algorithm was used to relax the cluster constraints, and the optimization result was an improved travel speed for freight trains. LI Sihan et al. [14] solved a multi-objective optimization problem that minimized the non-fixed part of train travel time and minimized the connection time of high-speed trains. The constraints included safety interval time limitations, arrival and departure line capacity limitations, arrival and departure time domain limitations, and train unit connection limitations. GAO Ruhu et al. [15] constructed a 0-1 integer programming model based on the Time-Station-Track network, designed a Lagrangian relaxation algorithm, and proposed a heuristic algorithm based on train priority sequences to make the solution feasible. WANG Jin et al. [16] developed a model based on the discrete space-time network that aimed to maximize the number of train lines and minimize the station stop time. An improved branch-and-price algorithm was designed. LIU Yong et al. [17] studied the 3PL transportation scheduling problem in the context of supply chain management. A heuristic algorithm based on the Lagrangian relaxation method and a branch-and-bound algorithm based on the heuristic algorithm were designed to solve the problem. The study confirmed that the Lagrangian relaxation algorithm is suitable for obtaining highly suboptimal solutions in large-scale problems. CHENG Lin et al. [18] established a network flow model for time-space networks with time-varying arcs and paths. A Lagrangian relaxation heuristic algorithm was designed based on the characteristics of the model. The optimal lower bound of the original problem was obtained by constructing and solving the Lagrangian dual problem, and the optimal upper bound was obtained using the heuristic algorithm.

In summary, although the Lagrangian relaxation algorithm is widely used in research related to train timetables, it has some shortcomings. The Markov property of the key iterative algorithm, the standard subgradient algorithm, determines that the algorithm can only update based on the current subgradient information in each iteration, without utilizing past gradient information. This dependency on the current step may lead the algorithm to be highly sensitive, potentially getting trapped in local optima and failing to find the global optimum. Especially in situations with multiple local optima, the algorithm may be limited by insufficient current gradient information, preventing it from escaping local optima and restricting the optimization effectiveness. In this study, we introduce fuzzy theory into the model-solving process, replacing the standard subgradient optimization algorithm with a fuzzy subgradient optimization algorithm for problem iteration. This is done to enhance algorithm performance and convergence characteristics, aiming to achieve better model-solving results.

II. Problem Analysis and Construction of Temporal-Spatial Network for Train Timetable

This paper focuses on the optimization of train timetables for high-speed railways, involving a large number of stations and trains. It is a typical NP-hard problem that is difficult to solve efficiently within polynomial time. The problem becomes even more complex when considering the coexistence of trains with different speed levels and the train overtaking. To simplify the problem, this paper incorporates traditional constraints such as interval running time constraints and station stopping constraints into the process of constructing the temporal-spatial network for train operations. This reduces the number of constraints and simplifies the model complexity. Additionally, constraints such as the minimum departure-to-arrival safety tracking interval and overtaking constraints are represented as incompatible relationships between train occupation arcs in the temporal-spatial network. This facilitates the decomposition of the model during the Lagrangian relaxation process. However, the classical Lagrangian relaxation algorithm still need to be improved to achieve higher-quality train timetables for high-speed railways.

In order to describe the problem using discrete temporal-spatial network theory, a temporal-spatial network model is constructed. The theory of discrete temporal-spatial network utilizes a discrete time axis to replicate the nodes of the physical network in chronological order, forming a two-dimensional temporal-spatial network graph. The introduction of temporal-spatial arcs facilitates the description of the train's travel path and effectively captures the relationships between various elements in the temporal-spatial network. In this study, the train timetable is modeled as a two-dimensional network graph incorporating temporal-spatial arcs.

As a graphical representation of the specific temporal-spatial positions of trains, a directed temporal-spatial network $G=(N,A)$ can be used to describe the train timetable. In this network: The set of nodes, denoted as N, represents all the nodes that a train passes through in the temporal-spatial domain. At each station, the nodes are further divided into arrival nodes U and departure nodes W .Where, W^s represents the set of departure time nodes for train at station s, and U^s represents the set of arrival time nodes for train at station s .The set of arcs, denoted as A, represents the components of a train's timetable, determined by the arrival and departure nodes at the stations. S is the set of all stations. $L=(1,\ ...\ ,n)$ is the set of all trains. $S^i=(o_{(i)},\ ...\ ,d_{(i)})\subseteq S$ denotes the set of stations that train $i\in L$ passes through, which includes stations. A train's timetable can be represented as a path in the temporal-spatial network, subject to certain constraints, starting from the $o_{(i)}$ and passing through $o_{(i)}+1,\ ...\ ,d_{(i)}-1$ to the l_i. The set of arcs included in the path is denoted as A^i, $A^i\subseteq A$ and the set of nodes is denoted as N^i , $N^i=\{\ v\in N\setminus(\delta,\varepsilon):\sigma_i^+\neq\Phi,\ \sigma_i^-\neq\Phi\ \}$, $N^i\subseteq N$. Where, $\sigma_i^+(v)$ represents the set of entry nodes to node v on train line A^i, and $\sigma_i^-(v)$ represents the set of exit nodes from node v on train line A^i. To construct a complete train timetable, a virtual departure node σ and a virtual destination node ε are introduced in the entire network, representing the origin and destination of each train. As a result, the set of all nodes in the network (including virtual nodes) can be represented as $N=\{\delta,\varepsilon\}\cup(W^1\cup\ ...\ \cup W^{s-1})\cup(U^2\cup\ ...\ \cup U^s)$. A train's timetable, represented by the set of arcs associated with it, can be denoted as $A_i=A_i^{sta}\cup A_i^{run}\cup A_i^{stop}\cup A_i^{end}$. It specifically includes:

(1) Departure arcs from the virtual departure point σ to the departure node $v\in W^{o_{(i)}}$ of the train at its origin station. Departure arcs $A_i^{sta}=\{(\sigma_i,v)\,|\,\sigma_i=(s,t),v=(s',t')\in W^{o_{(i)}}\}$.

(2) Stop arcs at intermediate stations $s,\ s\in S^i\setminus\{o_{(i)},d_{(i)}\}$ from the arrival node $u\in U^s$ to the departure node $v\in W^s$ at the same station. These arcs represent the train's stop at an intermediate station. They connect the arrival node at the station to the departure node at the same station. The departure node $v\in W^s$ is determined based on the arrival node and the scheduled stop time of the train at that station. Stop arcs $A_i^{stop}=\{(u,v)\,|\,u=(s,t)\in U^s,v=(s,t')\in W^s\}$. Where, $ts^{\min}\le t'-t\le ts^{\max}$, $ts^{\min}$ represents the minimum stop time for train at station s, and $ts^{\max}$ represents the maximum stop time for train at station s.

(3)Running arcs within a section $(s,s+1)$ from the departure node $v\in W^s$ at a station s to the arrival node $u\in U^{s+1}$ at the next station $s+1$ .These arcs represent the train's movement within a section between two consecutive stations. They connect the departure node at the current station to the arrival node at the next station. The arrival node $u\in U^{s+1}$ is determined based on the departure node and the scheduled running time of the train within that section. Running arcs $A_i^{run}=\{(v,u)\,|\,v=(s,t)\in W^s,u=(s',t')\in U^{s'},s'=s+1\}$.

(4) Termination arcs from the arrival node $u\in U^{d_{(i)}}$ at the train's destination station to the virtual arrival point ε. This arc represents the train's arrival at the destination station and connects the arrival node at the destination station to the virtual arrival point. Termination arcs $A_i^{end}=\{(u,\varepsilon)\,|\,u=(s,t)\in U^{d_{(i)}},\varepsilon=(s,t')\}$.

The cost of each arc segment is determined by the deviation between the actual time and the scheduled time in the original train timetable. Specifically: Cost of the start arcs: The cost of the start arcs represents the deviation cost between the chosen departure time from the start station and the scheduled departure time in the original train timetable. In this case, a departure time window is defined for the train, denoted as $[d_i^{o_{(i)}}-\eta,d_i^{o_{(i)}}+\eta]$. Where, η represents the allowable fluctuation value for train departure time, and d_i^s represents the scheduled departure time of train at station s. The cost for the start arcs within this time window is zero. For the start arcs outside this time window, the cost is calculated based on a penalty factor c_{pen}^1, and the cost of initial arc (δ,v) on train line i can be expressed as

$$c(\delta,v)=\begin{cases}c_{pen}^1\times(|\,t'-d_i^{o_{(i)}}\,|-\eta)\ , & |\,t'-d_i^{o_{(i)}}\,|\ >\eta\\ 0\ , & \text{else}\end{cases}.$$

The cost of the stop arcs represents the deviation cost between the chosen dwell time at a station and the original scheduled dwell time in the train timetable. The cost is calculated based on a penalty factor c_{pen}^2, and the cost of stop arc (u,v) on the train line i can be expressed as $c(u,v)=c_{pen}^2\times|\,(t'-t)-(d_i^s-a_i^s)\,|$. Where a_i^s represents the scheduled arrival time of train at station s, and c_{pen}^1,c_{pen}^2 are the penalty coefficient for offset between initial arc and stop arc.

Using a running graph example with 5 stations and 3 train routes, we will describe the modeling process outlined above. Please refer to Fig. 1. In this example, trains K_1,K_2,K_3 represent the three train routes. Stations A, C, and E are technical stations, while the others are intermediate stations. We will represent each train route using paths in the space-time network.

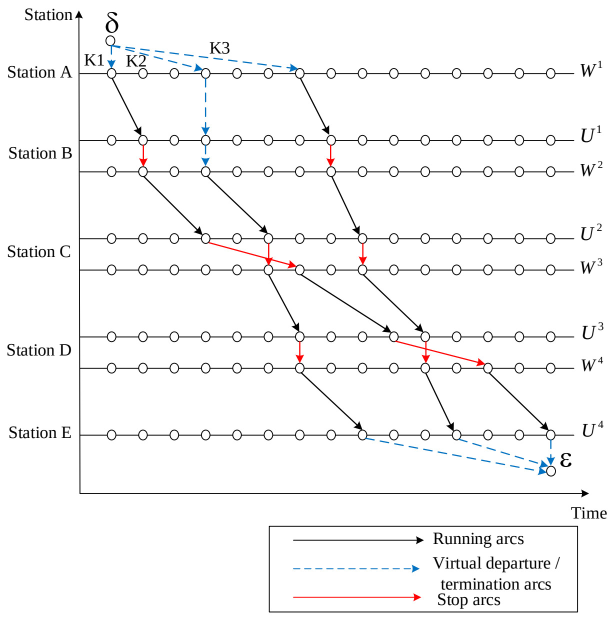


Fig. 1. Space-Time Network Representation of the Train Timetable

Due to the limited allocation of transportation resources, potential conflicts may arise among the train routes. The key to optimizing the train timetable is to resolve these potential conflicts. In the process of establishing the space-time network, each path from the virtual departure point to the virtual arrival point is unique on the feasible timetable. Solving each train route can be transformed into the problem of constructing subgraph domains for each train route. With the constraints imposed on the selection of space-time network nodes, it can be further transformed into a shortest path problem, reducing the scale and difficulty of the solution.

III. Train Timetabling Optimization Model based on Temporal-Spatial Network

A. *Model Assumptions*

(1) The station stopping plan for trains is known and cannot be changed.

(2) Trains are assumed to operate on a double-track railway, and the arrangement of approach and departure lines within stations is not considered, assuming that the capacity of the approach and departure lines is sufficient.

(3) Train operations follow automatic block signaling, with a fixed and known minimum headway time.

(4) The running times, additional dwell times at stations, number of trains, start and end points, and routes for each section are known and fixed. The timetable compilation does not consider the assignment of specific train units.

B. Construction of Train Timetabling Optimization Model

B.1. Objective Function

Taking into account the time cost of passenger travel and minimizing the deviation from the original train timetable, the objective function of the constructed temporal-spatial network is to minimize the total cost of selected arcs for all trains. Specifically, it can be described as follows:

$$Z = \min \sum_{i \in L} \sum_{a \in A^i} c_a^i x_a^i \tag{1}$$

Where $c_a^i = c(\delta, v) + c(u, v)$,

$$c(\delta, v) = \begin{cases} c_{pen}^1 \times (|t' - d_i^{o(i)}| - \eta) \;, & |t' - d_i^{o(i)}| > \eta \\ 0 \;, & \text{else} \end{cases},$$

$c(u, v) = c_{pen}^2 \times |(t' - t) - (d_i^s - a_i^s)|$,

c_a^i represents the cost of arc $a \in A^i$ on train line i,

x_a^i is the decision variable, $x_a^i = \begin{cases} 1, \text{train } i \text{ chooses arc } a \;; \\ 0, \text{else} \end{cases}$

B.2. Constraints

For two adjacent trains i and j (i、$j \in L$) at station $s \in (S^i \setminus \{d_{(i)}\}) \cap (S^j \setminus \{d_{(j)}\})$, the running arc a_1=$(s, t; s+1, t') \in A^i$, a_2=$(s, \kappa; s+1, \kappa') \in A^j$, t、t'、κ、$\kappa' \in [0, T]$ must satisfy flow balance constraints, mutually exclusive arc constraints, and variable value constraints.

(1) Flow balance constraint:

This constraint ensures that each train route can select at most one departure arc and, except for the start and end stations, the number of entering and leaving arcs for each train route at the nodes is equal.

$$\sum_{a \in \sigma_{i(\delta)}^-} x_a^i \le 1 \quad i \in L \tag{2}$$

$$\sum_{a \in \sigma_{i(v)}^+} x_a^i = \sum_{a \in \sigma_{i(v)}^-} x_a^i \quad i \in L \;, \; v \in N \setminus \{\delta, \varepsilon\} \tag{3}$$

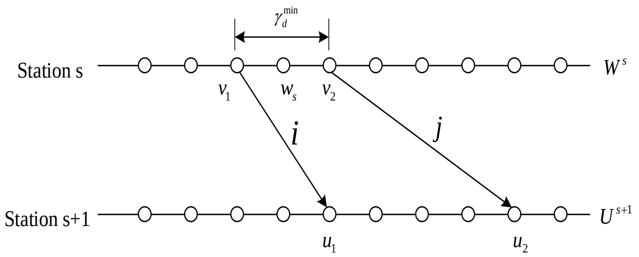
(2) Mutually exclusive arc constraint:

The constraint includes safety interval time constraint and train overtaking constraint. The security interval constraint includes the departure interval constraint and the arrival interval constraint, as shown in Fig. 2. These constraints represent the coupling relationship between multiple trains and ensure the exclusivity of temporal-spatial resource occupation by trains. In order to describe this relationship, we introduce the concept of an incompatible arc set.

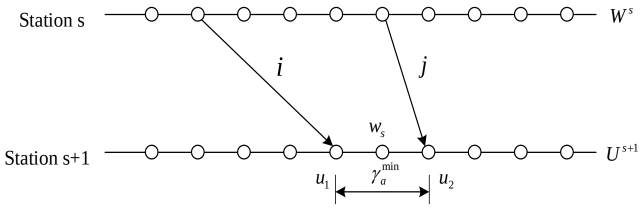
Given a specific temporal-spatial arc, the set of incompatible arcs associated with that arc is determined. Let's use arcs $a_1 = (s, t; s+1, t') \in A^{run}$ and $a_2 = (s, \kappa; s+1, \kappa') \in A^{run}$ as an example to illustrate. If both arcs a_1 *and* a_2 are occupied by trains simultaneously, the safety spacing constraint should be satisfied: $\pi_{a_1, a_2} = \max\{\gamma_d^{\min}, (t' - t) - (\kappa' - \kappa) + \gamma_a^{\min}\}$. t、t',κ、κ' are the index for time in the temporal-spatial network, t、t',κ、$\kappa' \in [0, T]$. Here, the safety spacing is explained as follows: $(t' - t)$ *and* $(\kappa' - \kappa)$ represents the interval running time for the train i and train j in the interval $(s, s+1)$. When the speeds of the trains are the same, the interval running time for both trains are the same, so the constraint simplifies to $\pi_{a_1, a_2} = \max\{\gamma_d^{\min}, \gamma_a^{\min}\}$, where γ_d^{min} is minimum departure interval time and γ_a^{min} is minimum arrival interval time. In this case, the mutually exclusive arc constraint represents the safety departure spacing constraint and the safety arrival spacing constraint. The safety spacing time is determined by taking the maximum value between the minimum arrival spacing time and the minimum departure spacing time. When train i and train j have different speed levels, the safety spacing time is determined by taking the maximum value between the minimum departure spacing time, the minimum arrival spacing time, and the difference between the interval running time of the two trains. This safety spacing constraint considers the safety departure and arrival spacing constraints as well as the train overtaking constraint.

For any given time-space arc a_1 in the graph, both the safety arrival spacing constraint and the train overtaking constraint can be transformed into a departure spacing constraint for the trains at the station.

As shown in Fig. 3(a), if $t-\pi_{a_2,a_1}<\kappa<t$, then arc a_2 is called the left incompatible arc of arc a_1. As shown in Fig. 3(b), If $t<\kappa''<t+\pi_{a_1,a_2}$, then arc a_2' is called the right incompatible arc of arc a_1. Therefore, the set of incompatible arcs $C(a_1)$ for arc a_1 can be represented as: $C(a_1)=\{a_2=(s,\kappa;s',\kappa')\in A^{run} \mid t-\pi_{a_2,a_1}<\kappa<t+\pi_{a_1,a_2}\}$.

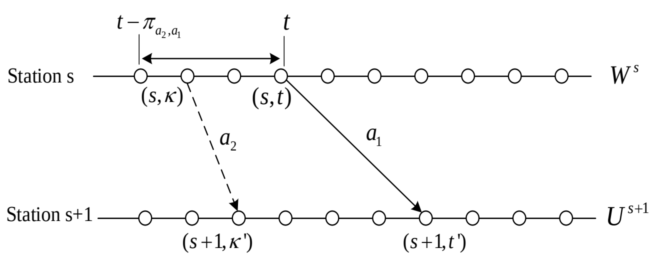


(a) Diagram of departure interval of trains

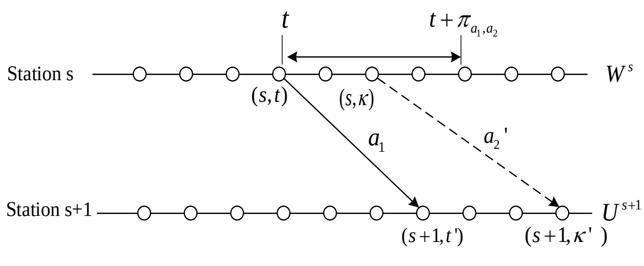


(b) Diagram of train arrival interval

Fig. 2. Diagram of departure and arrival intervals of trains



(a) The left incompatible arc of Arc a_1



(b) The right incompatible arc of Arc a_1

Fig. 3. Illustration of Incompatible Arcs on Both Sides of Arc a_1

Hence, the incompatible constraints can be described as follows:

$$\sum_{j:\,a_1\in A_j^{run}} x_{a_1}^j+\sum_{a_2\in C_{(a_1)}:\,a_1\neq a_2} x_{a_2}^i \le 1 \quad \forall i\in L,\, a_1\in A_j^{run} \tag{4}$$

(3) Variable Value Constraints.

$$x_a^i \in \{0,1\} \qquad \forall i\in L\ ,\ a\in A^i \tag{5}$$

IV. Based on the Lagrangian Relaxation Algorithm for Model Solving

A. Lagrangian Relaxation of the Original Problem

The key to Lagrangian relaxation is to determine the relaxation constraints, which are the difficult constraints that affect the speed of solving the model. In the model established in this paper, the mutual exclusion constraint (4) reflects the mutual influence between train running lines and can significantly increase the size of the model, thereby affecting the solution time.

To address this, the original model's constraint (4) is relaxed by introducing Lagrange multipliers $\lambda_i(a)\ge 0, \forall i\in L, a\in A^{run}$. By treating these difficult constraints as penalty terms and relaxing them into the objective function, a Lagrangian relaxation problem is formed.

$$Z_{LR}=\sum_{i\in L}\sum_{a\in A_i} c_a^i x_a^i + \sum_{i\in L}\sum_{a\in A^{run}} \lambda_i(a)\times\Big[\sum_{j:\,a_1\in A_j^{run}} x_{a_1}^j+\sum_{a_2\in C_{(a_1)}:\,a_1\neq a_2} x_{a_2}^i -1\Big] \tag{6}$$

By rearranging, the above equation can be transformed into:

$$Z_{LR}=-\sum_{i\in L}\sum_{a\in A_i^{run}} \lambda_i(a)+\sum_{i\in L}\sum_{a\in A^i} c_a^i * \times x_a^i \tag{7}$$

$$c_a^i *=\begin{cases} c_a^i+\sum_{j:\,a_1\in A_j^{run}} \lambda_j(a_1)+\sum_{a_2\in C_{(a_1)}:\,a_1\neq a_2} \lambda_i(a_2)\ ,\ a_1\in A_i^{run} \\ c_a^i\ , \qquad \text{else} \end{cases} \tag{8}$$

By removing the constant term, we finally obtain:

$$Z_{LR}=\sum_{a\in A^i} c_a^i * \times x_a^i \tag{9}$$

The Lagrangian Relaxation (LR) problem has the same complexity as the original problem, and if the feasible region of the original problem is non-empty, let Z^* denote the optimal objective function value of the original problem model. Then for any $\lambda\ge 0$, we have: $Z(\lambda)\le Z^*$. This is a widely recognized theorem. This definition also indicates that the LR problem provides a lower bound for the solutions of the original problem. To obtain the closest lower bound to the original problem, it is necessary to construct the Lagrangian Dual problem (LD):

$$Z_{LD}=\max\sum_{a\in A^i} c_a^i * \times x_a^i \tag{10}$$

The relaxed constraints are intended to affect only individual trains as much as possible. Therefore, the dual problem can be decomposed into several mutually independent subproblems for individual trains in mathematical programming. These subproblems can be efficiently solved using the shortest path algorithm. The solution of the Lagrangian relaxation dual problem is fixed given the Lagrangian multipliers. By applying the shortest path algorithm, the relaxed solution of the original problem, which serves as a lower bound for the original problem, can be obtained.

B. Updating Method for Lagrange Multipliers

In the iterative solution process, the standard subgradient algorithm is commonly used to update the Lagrange multipliers. However, the standard subgradient algorithm is not a monotonically decreasing algorithm and tends to exhibit oscillations during the optimization process, which severely affects its convergence efficiency. This phenomenon is believed to be caused by its Markov property, which means that the current subgradient does not have a memory effect on the historical subgradients generated in previous iterations. In this paper, fuzzy theory is introduced into the solution of subgradients, and a fuzzy subgradient algorithm is used to update the Lagrange multipliers. This algorithm utilizes membership functions to provide corresponding coefficients for all historical subgradients, thus making more effective use of all historical subgradient information.

The updating strategy for modifying the subgradient direction involves using a linear combination of historical subgradients instead of the current subgradient. Specifically, the iteration formula is modified as follows: $\lambda_i^{q+1}(a)=\max\{0,\lambda_i^q(a)-\theta^q d^q\}$, where q represents the Lagrange multipliers at the q iteration, θ^q denotes the step size at the q iteration, $\theta^q=\dfrac{1}{q+1}$, $d^q=\sum_h^q \omega_h^q g_h$. g_h represents the subgradient obtained at the h iteration, $g_h=\sum_{i:\,a_1\in A_1^{run}} x_{a_1}^i+\sum_{a_2\in C_{(a_1)}:\,a_1\neq a_2} x_{a_2}^j-1$. ω_h^q represents the weight coefficient assigned to the historical subgradient in the h iteration. To determine the weight coefficients, it is assumed that the closer the historical subgradient is to the current subgradient direction, the more information it contains and thus it should be assigned a higher weight. Based on this idea, the following definition is provided:

The weight is defined as follows:

$$\omega_h^q=\overline{\omega_h^q}\Big/\sum_{h=1}^{q}\overline{\omega_h^q} \tag{11}$$

$\overline{\omega_h^q}$ is defined by the following membership function:

$$\overline{\omega_h^q}=\begin{cases}Z_{LR}(\lambda^q,x^q)+\rho-Z_{LR}(\lambda^q,x^h)/\rho,\\ \text{if } Z_{LR}(\lambda^q,x^h)<Z_{LR}(\lambda^q,x^q)+\rho \text{ and } Z_{LR}(\lambda^q,x^h)\le Z_{LR}^*\\ 0, \qquad \text{else}\end{cases} \tag{12}$$

$$0\le\rho\le(Z_{LR}^*-Z_{LR}(\lambda^q,x^q))/a \tag{13}$$

When updating the step size, it needs to satisfy the following condition:

$$0\le\theta^q\le\frac{2(a-1)(Z_{LR}^*-Z_{LR}(\lambda^q,x^q))}{a\,\|d^q\|^2},a>1 \tag{14}$$

The reasonable weights for each historical subgradient can be determined using the aforementioned fuzzy membership function, which fully utilizes the information from historical subgradients.

The flowchart of the fuzzy subgradient algorithm can be described as follows:

Step 1: Initialization.

Set the initial iteration number $q=0$.

Initialize the Lagrange multipliers $\lambda_i^1(a)=0$.

Step 2: Iteratively update the Lagrange multipliers.

For each subgradient $\lambda_i^q(a)$:

If the stopping criterion is satisfied $\lambda_i^q(a)=0$, obtain the optimal solution and terminate the calculation.

Otherwise, calculate the weights of each historical subgradient using equations (11) to (13). Update the step size according to equation (14). Update the Lagrange multipliers $\lambda_i^{q+1}(a)=\max\{0,\lambda_i^q(a)-\theta^q d^q\}$.Repeat Step 2.

The flow chart of the fuzzy subgradient algorithm is shown in Fig.4.

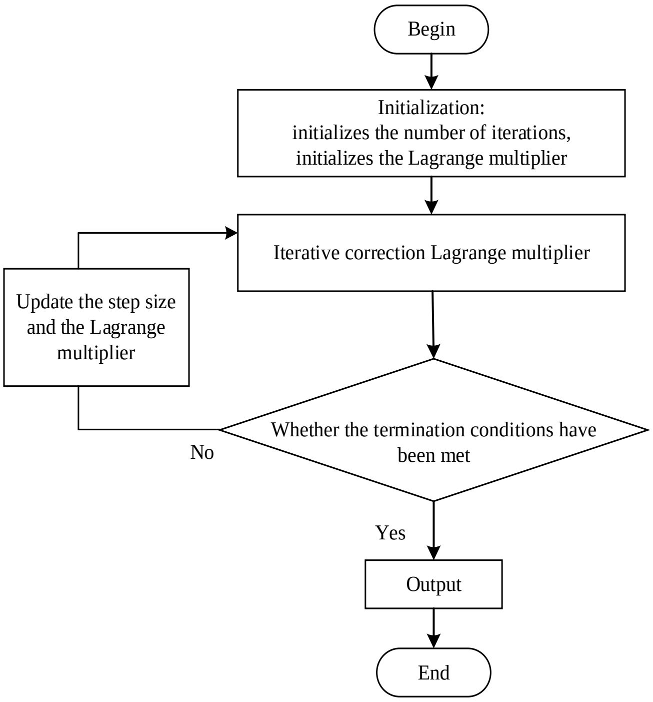


Fig. 4. Flow chart of fuzzy subgradient algorithm

The fuzzy subgradient algorithm allows the usage of different step size and adaptive adjustment during solving process in each iteration, which makes the algorithm more flexible in adapting to the characteristics of the problem. Since it only requires the calculation of subgradients of the objective function, it has relatively low computational complexity. The proof of convergence for this algorithm can be found in reference [19].

C. Heuristic Algorithm for Feasible Solutions

Due to the use of the Lagrangian relaxation method, the solution obtained is the solution to the relaxed problem. This means that the computed solution may be infeasible, requiring further processing to make it feasible.

The process of making the solution feasible also employs the shortest path method to solve the Lagrangian dual problem. The key is to consider the train organization constraints and determine the sequencing of train paths, based on the optimization order of trains according to the Lagrangian cost. As shown in equation (8), the shortest path cost for a train depends on the travel time cost of space-time arcs and the Lagrangian multipliers. In each iteration, the path selection of trains is adjusted based on the Lagrangian multipliers of the occupied arcs. The specific steps of the heuristic algorithm are as follows:

(1) Solve the Lagrangian dual problem (LD) and record the Lagrangian multipliers $\lambda_i(a)$ for all arcs in the graph.

(2) Sort the trains' path sequencing based on the Lagrangian costs of individual subproblems.

(3) Within the departure time window, considering all feasible departure nodes and $c(\delta,v)$ 、 $c(u,v)$, select the node $\lambda_i(a)=0$ for which the target is to minimize the objective. According to the depth-first search principle, traverse the train paths and perform rolling optimization to find the path with the minimum actual cost for each train.

(4) Calculate the sum of the actual costs for all train paths in the graph and record it as the upper bound solution.

D. Lagrangian Relaxation Algorithm Procedure

Step 1: Initialize the iteration number $q=0$, Lagrange multipliers $\lambda_i^q(a)=0$, step size parameter $\theta^q=0.5$, minimum upper bound after q iterations $UB=+\infty$, and maximum lower bound after q iterations $LB=-\infty$.

Step 2: Calculate and update the local/optimal lower bound. Use the shortest path algorithm to solve the Lagrangian dual subproblem and obtain the optimal solution and optimal value. If the optimal value $Z_{LR}^q > LB$, set $LB = Z_{LR}^q$.

Step 3: Calculate and update the local/optimal upper bound. Construct feasible solutions for the original problem using the method described in section D. Lagrangian Relaxation Algorithm Procedure, and calculate the corresponding objective function value Z^q. If the objective function value $Z^q < UB$, set $UB = Z^q$.

Step 4: Check if the current iteration satisfies the specified duality gap (Gap) value $Gap=\frac{UB-LB}{UB}\le \varepsilon(\varepsilon>0)$. If the condition is met, proceed to Step 6. Otherwise, continue to Step 5.

Step 5: Update the Lagrange multipliers $\lambda_i^{q+1}(a)=\max\{0,\lambda_i^q(a)-\theta^q d^q\}$ and the step size parameter $\theta^{q+1}=\theta^q \bullet e^{-0.5\varsigma^{q2}}$ using the fuzzy subgradient optimization method. Then, return to Step 2.

Step 6: Terminate the algorithm and output the final result, which corresponds to the optimal solution for the original problem, associated with the upper bound UB.

V. Verification Example

This study focuses on the optimization of train timetables for the Beijing-Shanghai high-speed railway. Based on the actual transportation production, the trains in the railway operate independently in two directions. One direction is selected for research purposes. To clearly describe the problem, we discretize one day into minutes, ranging from 1 to q , $q=1440$. The railway line consists of 23 stations. It is assumed that two different speed levels of trains operate on this line, and the travel times for different speed levels in each section are shown in Table I. Within the timetable period, a total of 82 trains are scheduled for departure, including 59 high-speed trains and 23 low-speed trains. Various parameters involved are shown in Table II. For the convenience of solving the problem, this study assumes that the train departure sequence and stopping patterns are known, and the train departure time domain is shown in Table III. Given the train departure time window, the trains can choose an appropriate departure time within the allowed fluctuation range of the departure time. The proposed algorithm in this study is implemented in C++ language on Visual Studio 2013 platform, running on a personal computer with 1 CPU Intel (R) Core (TM) i5-8250U CPU @ 1.60GHz (8CPUs), ~1.8GHz, and 4 GB of memory.

TABLE I
TRAIN TRAVEL TIMES IN SECTIONS

Operating interval	Interval running time(min)	
	high-speed train（270km/h）	low-speed train（230km/h）
Beijing South-LangFang	12	14
LangFang-Tianjin South	14	17
Tianjin South-Cangzhou West	18	21
Cangzhou West-Dezhou West	22	26
Dezhou West-Jinan West	18	22
Jinan West -Taian	12	14
Taian -Qufu East	14	17
Qufu East -Tengzhou East	11	13
Tengzhou East -Zaozhuang	7	9
Zaozhuang -Xuzhou East	13	15
Xuzhou East - Suzhou East	16	19
Suzhou East -Bengbu South	15	18
Bengbu South -Dingyuan	11	13
Dingyuan -Chuzhou	12	15
Chuzhou -Nanjing South	12	14
Nanjing South -Zhenjing North	14	17
Zhenjing North -Danyang North	5	6
Danyang North - Changzhou North	6	8
Changzhou North -Wuxi East	11	14
Wuxi East-Suzhou North	5	6
Suzhou North -Kunshan South	6	8
Kunshan South -Shanghai Hongqiao	9	10

TABLE II
RELEVANT PARAMETERS

Parameters	Quantity	Parameters	Quantity
$\gamma_{\min}^{arr}$	3min	$\gamma_{\min}^{dep}$	3min
μ	2min	μ'	3min
ts^{min}	2min	$ts^{\max}$	10min
c_{pen}^{1}	100 yuan /min	c_{pen}^{2}	100 yuan/min

TABLE III
PARTIAL TRAIN DEPARTURE TIME WINDOWS

Train Number	Train Grade	Earliest Departure Time (min)	Latest Departure Time (min)
G1	Low-Speed Train	0	20
G2	Low-Speed Train	120	140
G3	High-Speed Train	180	200
G4	High-Speed Train	300	320
G5	High-Speed Train	60	80
G6	High-Speed Train	13	33
G7	Low-Speed Train	68	88
G8	Low-Speed Train	240	260
G9	High-Speed Train	307	327
G10	High-Speed Train	128	148
G11	High-Speed Train	188	208
G12	Low-Speed Train	196	216
G13	High-Speed Train	250	270
G14	High-Speed Train	260	280
G15	High-Speed Train	314	334
G16	Low-Speed Train	320	340
G17	High-Speed Train	26	46
G18	High-Speed Train	39	59
G19	High-Speed Train	76	96
G20	High-Speed Train	84	104

Most The total time length for the problem is set to be 720 minutes. The time is discretized with a time interval of 1 minute. Each train is assigned a departure time window of 20 minutes from the origin station. The algorithm is iterated 100 times. Using the standard subgradient algorithm, the obtained optimal upper bound is 14761, the optimal lower bound is 13503.7, the computation time is 334.82 seconds, and the optimal gap value is 8.51%. With the introduction of the fuzzy subgradient algorithm in the iteration of the Lagrangian relaxation algorithm, the optimal upper bound is 14551, the optimal lower bound is 13505.8, the computation time is 260.42 seconds, and the optimal gap value is 7.18%. The iteration process of the upper and lower bounds with the fuzzy subgradient algorithm optimization in the Lagrangian relaxation algorithm is shown in Fig. 6, and the corresponding variation of the optimization gap value is shown in Fig. 7. It can be observed that the Gap values obtained using the fuzzy subgradient algorithm are superior to the optimal upper bounds. This improvement is achieved by effectively utilizing and updating historical subgradient information, which reduces the computational cost in each iteration. Standard subgradient algorithms typically require precise gradient computations of the objective function, often involving partial derivatives of each variable, resulting in higher computational complexity. In contrast, the fuzzy subgradient algorithm only requires subgradient computations of the objective function, which can simplify the computation process in certain cases. This is especially beneficial for exploring a broader solution space in large-scale problems, leading to enhanced optimization results in terms of quality and convergence speed. The fuzzy subgradient algorithm exhibits better computational performance overall. The comparison of the results between the standard subgradient algorithm and the fuzzy subgradient algorithm is shown in Table IV. Table V shows the values of upper and lower bounds and Gap values obtained by each iteration of the fuzzy sub gradient algorithm.

The optimized train schedule obtained using the Lagrangian relaxation algorithm is shown in Fig. 5. The red solid lines in Fig. 5 represent high-level trains, while the blue dotted lines represent low-level trains. Some station names are omitted in Fig. 5 to make them more clearly marked on the axis. The complete station names are shown in Table I. From the schedule, it can be seen that the departure times of some trains have been adjusted, and the trains can flexibly choose the stoppage time within the minimum and maximum stoppage time. The optimized train schedule reduces the travel time for each train, thus reducing the time cost for passengers and improving the quality of railway transportation services.

TABLE IV
COMPARISON TABLE OF SOLUTION RESULTS

Algorithm	The computation time	Lower bound	Upper bound	Gap value
The standard subgradient algorithm	334.82s	13503.7	14761	8.51%
The fuzzy subgradient algorithm	260.42s	13505.8	14551	7.18%

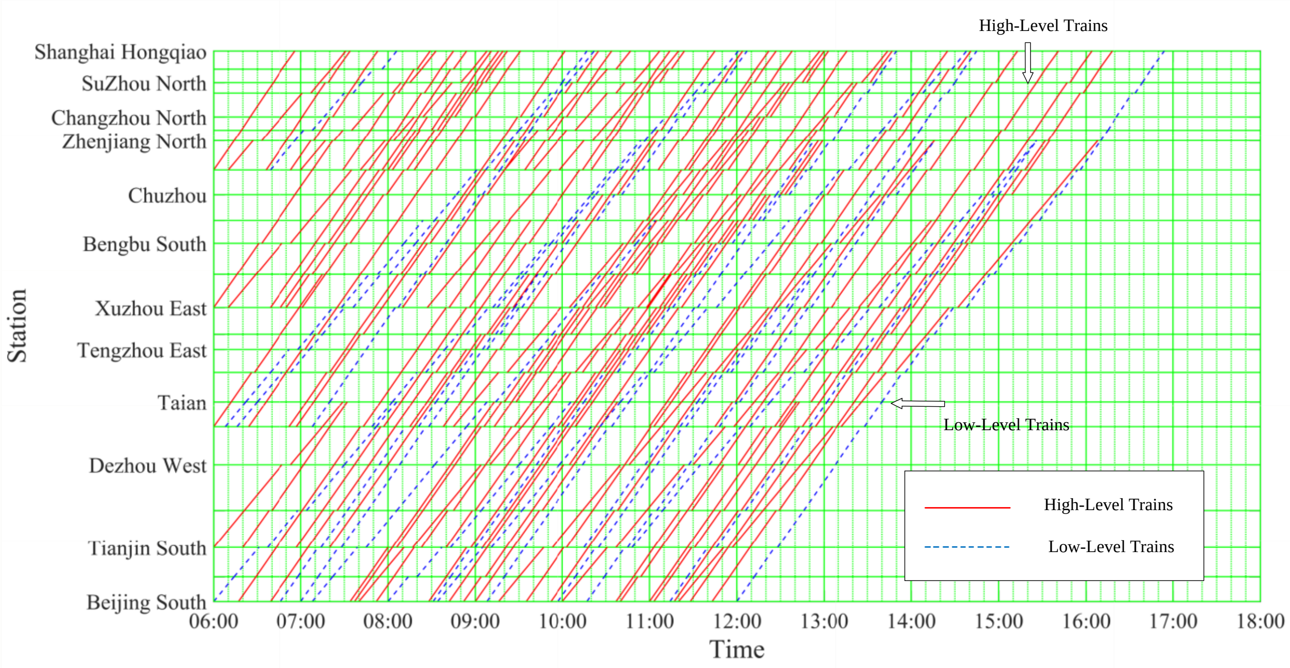


Fig. 5. Optimized Train Timetable

TABLE V
TABLE OF OPTIMIZED UPPER AND LOWER BOUNDS AND GAP VALUES

Number of iterations	Lower bound	Upper bound	Gap Value
1	0	14763	100
2	5864	14763	60.2791
3	8801	14763	40.3847
4	11164.5	14551	23.2733
5	12614.3	14551	13.3095
6	12965.8	14551	10.894
7	13231.2	14551	9.07029
8	13346.3	14551	8.27903
9	13411	14551	7.83451
10	13411	14551	7.83451
11	13464.9	14551	7.46386
12	13464.9	14551	7.46386
13	13464.9	14551	7.46386
14	13464.9	14551	7.46386
15	13464.9	14551	7.46386
16	13471	14551	7.42217
17	13505.8	14551	7.18301
⋮	⋮	⋮	⋮
100	13505.8	14551	7.18301

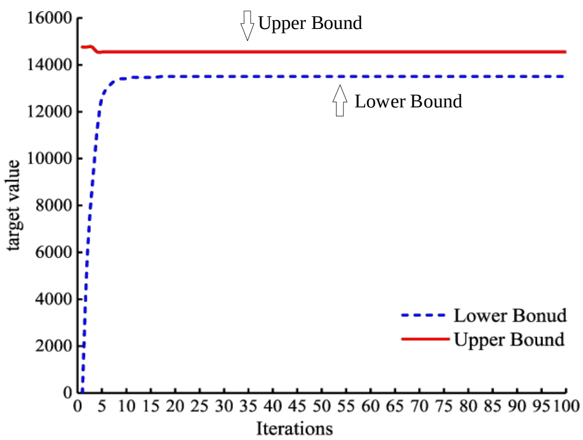


Fig. 6. Iteration Process of Upper and Lower Bounds in Lagrangian Relaxation Algorithm

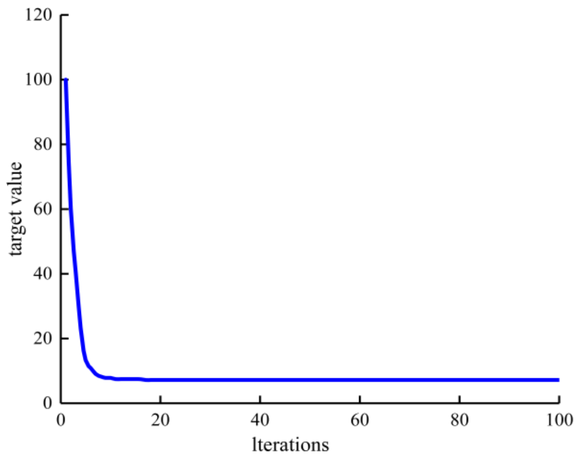


Fig. 7. Iteration Process of Gap Value in Lagrangian Relaxation Algorithm

Table V shows the values of upper and lower bounds and Gap values obtained by each iteration of the fuzzy subgradient algorithm. From table V we can see that the fuzzy subgradient optimization algorithm used in each iteration of the process of the specific values. Due to space constraints, not all of the 100 iterations are listed in full, but the iteration process is clearly shown. Among them, the lower bound value no longer changes at the 17th iteration and has reached the optimum; the upper bound value also reaches the optimum at the 4th iteration; and the Gap value no longer changes at the 17th iteration, reached the optimal Gap value. And from Fig. 6 and Fig. 7 we can see the overall trend more visually.

The results of the Beijing-Shanghai high-speed railway case study demonstrate the significant advantages of applying the fuzzy subgradient optimization algorithm in the Lagrangian relaxation algorithm. It effectively improves the quality and convergence speed of the optimization results. Using the standard subgradient algorithm, the iteratively obtained optimal upper bound is 14761, the optimal lower bound is 13503.7, the computation time is 334.82 seconds, and the optimal Gap value is 8.51%. In contrast, for the Lagrangian relaxation algorithm iteratively using the fuzzy subgradient algorithm, the optimal upper bound is 14551, the optimal lower bound is 13505.8, the computation time is 260.42 seconds, and the optimal Gap value is 7.18%. Comparing the Gap values determines the degree of optimization in the timetable, and the results of the improved fuzzy subgradient algorithm are superior to those of the standard subgradient algorithm, making it better suited for complex, nonlinear optimization problems.

VI. Conclusion

(1) This paper focuses on optimizing the high-speed railway train timetable, resulting in an optimized timetable where each train has a shorter travel time (i.e., minimized cost), thereby enhancing the quality of railway transportation services. The established 0-1 integer programming model based on space-time arcs uses incompatible constraints to characterize the safety interval constraints, providing a clearer representation of the coupling relationships between trains. Additionally, the process of decomposing the original problem using the Lagrangian relaxation algorithm becomes more convenient.

(2) In solving the model using the Lagrangian relaxation algorithm, this paper introduces fuzzy theory to improve the subgradient optimization algorithm. The fuzzy subgradient algorithm effectively utilizes historical subgradient information and overcomes the Markov property of the standard subgradient algorithm. In future research, considering the integrated optimization of train timetables and rolling stock schedules can further enhance the quality of railway transportation services.

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