Variance Gamma Model in Determining the Default Probability of Coupon Bond Issuing Company

Abdul Hoyyi, Abdurakhman, and Dedi Rosadi*

Abstract— **Bond is one of the most attractive financial instruments for both investors in the capital market and companies seeking funds for benefit. Investing in bonds often yields fixed income through coupons and also exposes investors to investment risk, such as credit risk. This credit risk encompasses the potential loss arising from a failure to meet credit payment obligations upon maturity, leading to a declaration of default. To proactively address this issue, there is a need to calculate the default probability of a company in order to gain insights into the entire default potential. Previous studies had predominantly employed bond models that assumed ln returns on assets follow a normal distribution. However, real-world ln returns on traded assets exhibited characteristics including excess kurtosis and heavy tails, which diverged from the assumptions of a normal distribution. The models developed on this assumption did not accurately reflect the nature of the data. In order to bridge this gap, this study aimed to introduce a novel approach for gauging default probability through the use of the Variance Gamma model. The chi-square test is used to determine goodness of fit. The Variance Gamma parameter estimation used Maximum Likelihood Estimation (MLE), with the initial value being the outcome of parameter estimation through the moment method. This approach assessed default probability in the context of both one-period and two-period coupon value payments.**

Index Terms— **Variance Gamma, bond, coupon, default probability**

I. Introduction

THE capital market in Indonesia is undergoing rapid growth, encompassing a diverse array of assets and volumes. As a result, there is a need to employ precise models in order to accurately value financial instruments including bonds, options, and securities. An example of such a prevalent model is the Geometric Brownian Motion (GBM), frequently employed to depict the dynamics of asset price shifts. The simplicity of the model facilitates its application in modeling the prices of diverse corporate assets and measuring tool for risk management. The foundational assumption of GBM lies in the normal distribution of natural logarithm returns from assets.

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Studies conducted using data from traded asset prices in Indonesia showed the presence of excess kurtosis and tails in the distribution of natural logarithm returns. Consequently, the performance of the GBM falls short of accurately depicting asset price dynamics [1]. By adding parameters to control kurtosis and volatility in the distribution of natural logarithm asset returns, the Variance Gamma (VG) model presents an advantage [2]. Furthermore, a three-parameter VG process has been developed by [3], encompassing an additional parameter to control skewness. This sentiment was shared by [4], stating that the VG managed volatility and also influenced the slope and kurtosis of the distribution of natural logarithm asset returns. The results were compared against GBM, culminating in the conclusion that VG more effectively characterizes the dynamics of asset prices in the market. Several papers in finance have supported the use of process variance-gamma. For example, [5] and [6] confirm that an excellent model for managing financial data is the variance-gamma distribution. There are various procedures available for computing the Variance-Gamma model, such as those described by Avramidis et al. [7] and [8] . The closed form for this model is presented in [3] and was later developed by [9] and [10] .

Investing in bond entails considerations of risk, particularly credit risk, which arises when a company becomes incapable of meeting its debt obligations upon maturity, effectively defaulting. To measure credit risk, bond price valuation scenarios can be generated to assess the ability of a company to fulfill its debt commitments. This process empowers investors to select or evaluate secure bond for investment. There are two models of credit risk, namely structural and reduced model. The structural model relies on internal company information, such as assets and liabilities, while the reduced leverages market-derived data, specifically company ratings [11] .

The concept of bond valuation was first introduced by Merton in 1974. He proposed a method to measure the risk associated with corporate bonds by analyzing changes in the value of corporate assets and debts. The Merton model is used to estimate the probability of a company going bankrupt, which is similar to the Black-Scholes option pricing method [12] . A company is considered default when, at maturity, the value of its assets is lower than the value of its debt, assuming the company has only issued one zero-coupon bond. The landscape of bond valuation models is expanding, but there remains a limited discourse on coupon bond valuation. It is crucial to note that several contemporary bond is issued with coupon. Studies considered the valuation of coupon bond with normally distributed assets, including the valuation of corporate bond featuring coupon using the GBM model [13] Additional study explored bond valuation, specifically coupling the Black-Scholes model with equations related to higher statistical moments—skewness and kurtosis. This comprehensive valuation entails estimating equity and default probabilities for bond-issuing companies, based on the Normal distribution and employing the standard Gram-Charlier expansion model with the Hermite polynomial approach [14].

Further studies on the valuation of coupon bond within non-normally distributed datasets have been pursued. The method used includes the Fast Fourier Transform (FFT) in conjunction with the characteristic function of the Normal Inverse Gaussian distribution [15], the Variance Gamma distribution [16], and the GBM jump-diffusion model [17]. It should be noted that these investigations generally revolved around one-period coupon. This study distinguishes itself by introducing a novel approach, namely ascertaining default probabilities integral to coupon bond valuation through the Variance Gamma. The development of default probability assessments encompasses both one-period and two-period coupon bond, factoring in default at maturity. The resultant modeling aims to contribute to the advancement of financial statistical theory, particularly within the domain of bond investment.

A. Variance Gamma Distribution

The density function of the multivariate Variance Gamma distribution with dimension d with shape parameter (v) is as follows:

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{2\left(\frac{1}{v}\right)^{\frac{1}{v}}\left(\frac{2}{v}+\boldsymbol{\theta}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}\right)^{\frac{d}{2}-\frac{1}{v}}}{(2\pi)^{\frac{d}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}\Gamma\left(\frac{1}{v}\right)} \times \frac{\mathcal{K}_{\frac{1}{v}-\frac{d}{2}}\left(\sqrt{\mathrm{Q}(x)\left(\frac{2}{v}+\boldsymbol{\theta}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}\right)}\right)\exp((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta})}{\left(\sqrt{\mathrm{Q}(x)\left(\frac{2}{v}+\boldsymbol{\theta}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}\right)}\right)^{\frac{d}{2}-\lambda}} \quad (1)$$

Where:

$\boldsymbol{X}' = (X_1, X_2, \dots, X_d)$, is a vector of random variables

v = a parameter shape

$\mathrm{Q}(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$, representing the Mahalanobis distance

$\boldsymbol{\mu}' = (\mu_1, \mu_2, \dots, \mu_n)$, indicating the location parameter vector

$\boldsymbol{\theta}' = (\theta_1, \theta_2, \dots, \theta_n)$, signifying the *skewness* parameter vector

$$\boldsymbol{\Sigma} = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_d) \\ Cov(X_2,X_1) & Var(X_2) & \cdots & Cov(X_2,X_d) \\ \vdots & \vdots & & \vdots \\ Cov(X_d,X_1) & Cov(X_d,X_2) & \dots & Var(X_d) \end{bmatrix}$$

is the covariance matrix [18].

For $d = 1$, Equation (1) is the univariate Variance Gamma distribution as stated by [3], namely:

$$f_X(x\,;\,\mu,\sigma,\theta,v) = C(\mu,\sigma,\theta,v) \times |x-\mu| \exp\left(\frac{\theta(x-\mu)}{\sigma^2}\right)\mathcal{K}_{\frac{1}{v}-\frac{1}{2}}\left(\frac{|x-\mu|\sqrt{\frac{2\sigma^2}{v}+\theta^2}}{\sigma^2}\right) \quad (2)$$

Where:

$\mathcal{K}_v\,(\,)$: a modified Bessel function of a second order type v

$$C(\mu,\sigma,\theta,v) = \frac{2}{\sigma\sqrt{2\pi}v^{\frac{1}{v}}\Gamma\left(\frac{1}{v}\right)}\left(\frac{1}{\sqrt{\frac{2\sigma^2}{v}+\theta^2}}\right)^{\frac{1}{v}-\frac{1}{2}}$$

μ : a location parameter

σ : a dispersion parameter

v : a *shape* parameter

θ : a *skewness* parameter

The modified Bessel function of the second kind $\kappa_n(x^*)$ is defined as follows:

$$\kappa_n(x^*) = \frac{\pi(I_{-n}(x^*) - I_n(x^*))}{2\sin(\pi n)}$$

$I_n(x^*)$ (The modified Bessel function of the first kind x^*) is a special solution of a second-order differential equation, as follows:

$$x^{*2}y'' + x^*y - (x^{*2} + n^2)y = 0$$

and can be expressed by the following infinite series:

$$I_n(x^*) = \sum_{n^*=0}^{\infty} \frac{1}{n^*!\,\Gamma(n+n^*+1)}\left(\frac{x^*}{2}\right)^{n+n^*}$$

Where n is the real number (order) [19].

B. Parameter Estimation

The moment method is one way for estimating Variance Gamma parameters. [3], which is straightforward to execute and yields a closed-form solution. For example, at a given time interval t, the random variable X_t follows a Variance Gamma distribution with mean θg and variance $\sigma\sqrt{g}$, expressed as:

$$X_t = \theta g + \sigma\sqrt{g}z \quad (3)$$

Where $z \sim$ Normal (0,1) and $g \sim$ Gamma$(t.\,vt)$.

Finding the first four moments (mo) of X_t is the first step in estimating the VG parameter as follows :

1. $$mo_1 = E(X_t) = E(\theta g + \sigma\sqrt{g}z) = \theta t \quad (4)$$
2. $mo_2 = E\left[\left(X_t - E(X_t)\right)^2\right]$

   By assuming $x = X_t - E(X_t)$

   $$x = (g-t)\theta + \sigma\sqrt{g}$$

   $$mo_2 = E(x^2) = (\theta^2 v + \sigma^2)t \quad (5)$$
3. $mo_3 = E(x^3)$

   $$E(x^3) = \left[(g-t)\theta + \sigma\sqrt{g}z\right]^2.\left[(g-t)\theta + \sigma\sqrt{g}z\right], = (2\theta^3v^2 + 3\sigma^2\theta v)t \quad (6)$$
4. $mo_4 = E(x^4)$

   $$E(x^4) = E\left[(g-t)\theta + \sigma\sqrt{g}z\right]^2.\left[(g-t)\theta + \sigma\sqrt{g}z\right]^2 = (3\sigma^4 v + 12\sigma^2\theta^2v^2 + 6\theta^4v^3)t + (3\sigma^4 + 6\sigma^2\theta^2 v + 3\theta^2v^2)t^2 \quad (7)$$

According to [20], the solution of those equations may in practice require iterative procedure, but the following simple arguments can be used to begin such a procedure, and will sometimes suffice for estimation. From equation (5)-(7) for very small values of θ the values of θ^2, θ^3, θ^4 are close to zero.

a. Estimated Parameter σ :

$$\begin{aligned} \mathrm{Var}(X_t) &= E(x^2) \\ &= \theta^2 v + \sigma^2 \\ &= (0)v + \sigma^2 \\ &= \sigma^2 \end{aligned}$$

So that $\hat{\sigma} = \sqrt{\mathrm{Var}(X_t)}$ (8)

b. Estimated Parameter θ :

$$\begin{aligned} \text{Skewness }(X_t) &= \frac{mo_3}{\sigma^3} \\ &= \frac{E(x^3)}{\sigma^3} \\ &= \frac{2\theta^3 v^2 + 3\acute{o}^2 \theta v}{(\theta^2 v + \sigma^2)^{3/2}} \\ &= \frac{2(0)v^2 + 3\sigma^2 \theta v}{((0)v + \sigma^2)^{3/2}} \\ &= \frac{3\theta v}{\sigma} \end{aligned}$$

So that

$$\hat{\theta} = \frac{\sigma\, Skewness(X_t)}{3v} \quad (9)$$

c. Estimated Parameter v :

$$\begin{aligned} \text{Kurtosis }(X_t) &= \frac{mo_4}{\sigma^4} \\ &= 3(v+1) \end{aligned}$$

$$\hat{v} = \frac{\text{Kurtosis }(X_t)}{3} - 1 \quad (10)$$

In this study, the maximum likelihood method was employed for VG parameter estimation, a technique discussed in a paper authored by [21]. Partitioning the data into k intervals, the goodness of fit test uses the Chi-square test. The Chi-square formula as follow :

$$\chi^2 = \sum_{i=1}^{k} \frac{(\mathcal{O}_i - \mathcal{N} p_i)^2}{\mathcal{N} p_i} \quad (11)$$

Where $\mathcal{N}$ is the sample size; $\mathcal{O}_i$ is the observed value in the ith sub-interval; and p_i is the probability that the observed value will occur by random in the ith sub-interval. This Chi-square statistic is then compared to the $\chi^2_{\alpha;\,k-1-m}$ value, where α signifies the significance level, and m represents the VG model's number of parameters [22].

C. Geometric Brownian Motion (GBM) Model

The GBM model encompasses two parameters, the first one, μ, corresponds to the expected value of ln returns on assets, while the second parameter, σ, signifies asset price volatility.

GBM model is mathematically formulated as follows:

$$dA_t = \mu A_t dt + \sigma A_t dW_t \quad (12)$$

This equation is a Stochastic Differential Equation. In the context of Equation (12), A represents the asset price, t signifies time [23], and W stands for standard Brownian motion with a Normal distribution characterized by a mean of 0 and a variance equal to $t_j - t_{j-1}$, signifying the expected value of asset returns. The parameter σ represents asset price volatility. The resolution of Equation (12) to attain the GBM asset price model is achieved through the Ito theorem. The theorem is expressed in the equation below:

$$dA_t = \mu A_t dt + \sigma A_t dW_t$$

According to the Ito theorem, the function $\mathcal{H} = \mathcal{H}(A,t)$ is as follows:

$$d\mathcal{H} = \left(\frac{\partial \mathcal{H}}{\partial A_t}\mu A_t + \frac{\partial \mathcal{H}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}}{\partial A_t^2}\sigma^2 A_t^2\right)dt + \frac{\partial \mathcal{H}}{\partial A_t}\sigma A_t dW_t$$

For example $\mathcal{H} = \ln A_t$ function, with the condition $\frac{\partial \mathcal{H}}{\partial A_t} = \frac{1}{A_t}$, $\frac{\partial^2 \mathcal{H}}{\partial A_t^2} = -\frac{1}{A_t^2}$, $\frac{\partial \mathcal{H}}{\partial t} = 0$

From the above expression, the following equation is formulated:

$$d\mathcal{H} = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dW_t$$

By integrating both sides of the equation from 0 to *t*,

$$\begin{aligned} \int_0^t d\mathcal{H} &= \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dW_t \\ \ln A_t \big|_0^t &= \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^t \sigma\, dW_t \\ \ln A_t - \ln A_0 &= \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^t \sigma\, dW_t \\ \ln A_t - \ln A_0 &= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \\ A_t &= A_0.\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \end{aligned} \quad [24]\ (13)$$

When the process follows the VG process equation (13), then it can be written as follows:

$$A_t = A_0.\exp[(\omega + r - q)t + X_t]$$

Where q represents the dividend. If there is no dividend $(q = 0)$ the model becomes:

$$A_t = A_0.\exp[(\omega + r)t + X_t] \quad (14)$$

Where $\omega = \frac{1}{v}\ln\left(1 - \theta v - \frac{1}{2}\sigma^2 v\right)$ and r is a risk-free interest rate [25]. Equation (14) is the Variance Gamma asset model.

II. One-Period Coupon Default Probability

According to [14], a one-period coupon bond granted investors coupon payments once during the stated duration, issued upon maturity. Alongside the obligation to pay a coupon K, the bond issuer would also be obligated to pay the bondholders the face value. For example, when a company had a total asset value of A_t, a face value obligation of K, a risk-free interest rate represented as r, a bond coupon size denoted with c, and a bond maturity T_1, two possible scenarios emerged upon maturity. Firstly, when the asset value was greater than or equal to the face value plus the coupon $(K + c = K_1)$, specifically when $A_{T_1} < K_1$, the bond issuer would remit K_1 to investors. The capital or equity of the issuer was $A_{T_1} - K_1$, and when the asset value was less than the face value plus the coupon $A_{T_1} < K_1$, the values would be 0, indicating a default situation. As detailed in [13], Fig. 1 showed an overview of the circumstances for a single coupon payment period.

Considering default at maturity, it was assumed that a company defaulted exclusively at the maturity of the bond (T_1). Consequently, the time of default (η) became a discrete random variable defined as follows:

$$\eta = \begin{cases} \infty & \text{If } A_{T_1} \geq K_1 \\ T_1 & \text{If } A_{T_1} < K_1 \end{cases}$$

The probability of default at maturity was expressed as followed:

$$\begin{aligned} P(\eta = T_1) &= P(A_{T_1} < K_1) \\ &= P(A_0 \exp\{(r+\omega)T_1 + X\} < K_1) \\ &= P\left(X < \left(Ln\left(\frac{K_1}{A_0}\right) - (r+\omega)T_1\right)\right) \\ &= \int_{-\infty}^{z} f_X(x)dx \end{aligned} \tag{15}$$

Where,

$$z = \ln\left(\frac{K_1}{A_0}\right) - (r+\omega)T_1$$

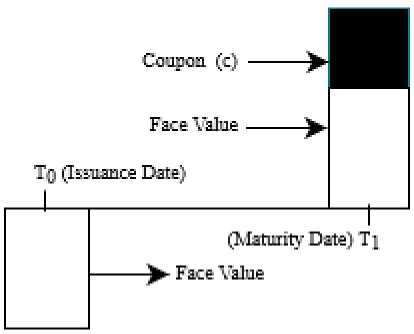

Fig. 1. Bond with a one-period coupon.

III. Two-Period Coupon Default Probability

Based on [13], an overview of the situation for the two coupon payment periods was shown in Fig. 2. A bond with a two-period coupon could be exemplified as follows. A bond issuer (or obligor) company issued a bond with a coupon featuring a face value K and maturing at T_2. The coupon value was paid twice during the period of the bond, at coupon payment T_1 and at maturity T_2, each with a fixed interest rate, c_1 and c_2, respectively. As a result, the obligor was obligated to make payments at $T_1(c_1)$ and at $T_2(K_1 = K + c_2)$. Default considerations indicated that at T_1, two scenarios arose: either the obligor paid the debt, remitting c_1, or the company defaulted. Similarly, at T_2, the company either fulfilled its obligations by paying the face value of the bond (K) plus the coupon value (c_2), or defaulted [14]

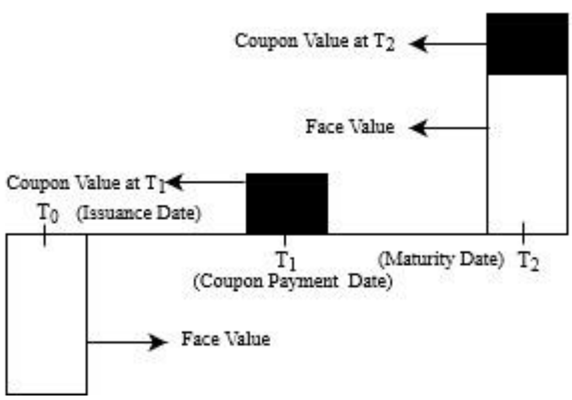

Fig. 2. Bond with a two-period coupon.

TABLE I
Capital Structure Of The Company Issuing Two-Period Coupon Bond At Maturity

Situation	Asset	Liability	Equity
Non Default	$A_{T_2} \geq K_1$	K_1	$A_{T_2} - K_1$
Default	$A_{T_2} < K_1$	A_{T_2}	0

The equity of the company at maturity T_2, can be written as:

$$\varphi_{T_2} = \begin{cases} A_{T_2} - K_1 & \text{If } A_{T_2} \geq K_1 \\ 0 & \text{If } A_{T_2} < K_1 \end{cases}$$

$$\begin{aligned} \varphi_{T_2} &= (A_{T_2} - K_1)^+ \\ &= \text{Max}\,(A_{T_2} - K_1, 0) \end{aligned}$$

According to [13], the present value of the equity at the $t = T_1$ is as follows:

$$\begin{aligned} \varphi_{T_1}^{T_2} &= \exp(-r(T_2 - T_1))\, E[(A_{T_2} - K_1)^+] \\ &= \exp(-r(T_2 - T_1))\left[\text{Max}\left(A_{T_2} - K_1, 0\right)\right] \end{aligned} \tag{16}$$

The solution to equation (16) is the European VG call option (Co) price according to [3], which is expressed as:

$$\begin{aligned} &Co\left(A_{T_1}; K_1, (T_2 - T_1)\right) \\ &= A_{T_1}\Psi\left(d\sqrt{\frac{1-\mathcal{C}_1}{v}}, (\zeta + s)\sqrt{\frac{v}{1-\mathcal{C}_1}}, \gamma\right) \\ &\quad -K_1 \exp(-r(T_2 - T_1)) \\ &\quad \times \Psi\left(d\sqrt{\frac{1-\mathcal{C}_2}{v}}, \zeta s\sqrt{\frac{v}{1-\mathcal{C}_2}}, \gamma\right) \end{aligned} \tag{17}$$

Where $\Psi(\mathcal{A}, \mathcal{B}, \gamma)$ is a solution expressed below:

$$\int_0^\infty N\left(\frac{\mathcal{A}}{\sqrt{u}} + \mathcal{B}\sqrt{u}\right)\frac{\exp(-u)u^{\gamma-1}}{\Gamma(\gamma)}du$$

From the above formula, the following equations are formulated:

$$\begin{aligned} \Psi(\mathcal{A}, \mathcal{B}, \gamma) &= \frac{\mathcal{C}^{\gamma+\frac{1}{2}} \exp[sign(\mathcal{A})\mathcal{C}]\,(1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \\ &\times \mathcal{K}_{\gamma+\frac{1}{2}}(\mathcal{C})\Phi\left[\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -sign(\mathcal{A})\mathcal{C}(1+u)\right] \\ &-sign(\mathcal{A})\frac{\mathcal{C}^{\gamma+\frac{1}{2}} \exp[sign(\mathcal{A})\mathcal{C}]\,(1+u)^{1+\gamma}}{\sqrt{2\pi}\Gamma(\gamma)(1+\gamma)} \times \mathcal{K}_{\gamma-\frac{1}{2}}(\mathcal{C}) \\ &\quad \Phi\left[1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -sign(\mathcal{A})\mathcal{C}(1+u)\right] \\ &\quad +sign(\mathcal{A})\frac{\mathcal{C}^{\gamma+\frac{1}{2}} \exp[sign(\mathcal{A})\mathcal{C}](1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \times \\ &\quad \mathcal{K}_{\gamma-\frac{1}{2}}(\mathcal{C})\Phi\left[\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -sign(\mathcal{A})\mathcal{C}(1+u)\right] \end{aligned}$$

Where :

$$\gamma = \frac{(T_2 - T_1)}{v}, \mathcal{C}_1 = \frac{v(\alpha+s)^2}{2}, \; \mathcal{C}_2 = \frac{v\alpha^2}{2}$$

$$d = \frac{1}{s}\left[\ln\left(\frac{A_{T_1}}{K_1}\right) + r(T_2 - T_1) + \frac{(T_2 - T_1)}{v}\ln\left(\frac{1-\mathcal{C}_1}{1-\mathcal{C}_2}\right)\right]$$

$$s = \frac{\sigma}{\sqrt{1+\left(\frac{\theta}{\sigma}\right)^2\frac{v}{2}}}$$

$$\zeta = -\frac{\theta}{\sigma^2}\, s\, , \mathcal{C} = |\mathcal{A}|\sqrt{2+\mathcal{B}^2}\, , u = \frac{\mathcal{B}}{\sqrt{2+\mathcal{B}^2}}$$

Φ represents a degenerate hypergeometric function of two variables with the integral form of Humbert, expressed as follows:

$$\Phi(\alpha_1,\beta_1,\gamma_1;x_1,y_1)=\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1-\alpha_1)}\times$$
$$\int_0^1[u_1{}^{\alpha_1-1}(1-u_1)^{\gamma_1-\alpha_1-1}\,(1-u_1x_1)^{-\beta_1}\exp(u_1y_1)]du_1.$$

The equity of the obligor at coupon payment time will not be equal to zero if $A_{T_1}\geq A_1^*$, with A_1^* representing an estimate of A_{T_1} satisfying

$$F(A_1^*)=\varphi_{T_1}^{T_2}-c_1=0.$$
$$\exp(-r(T_2-T_1))[\text{Max}\,(A_1^*-K_1,0)]-c_1=0.$$

A_1^* is estimated with a numerical approach. In this study, the numerical approach is used the bisection method. The algorithm can be described like this [26],

1. Set the initial values $a_0=A_l$ and $b_0=A_r$ where $F(A_l)$ and $F(A_r)$ have opposite signs.
2. Given two rational numbers a_{i-1} and b_{i-1} with the property that $F(a_{i-1})$ and $F(b_{i-1})$ have the opposite signs, set $A_1^*=\frac{a_{i-1}+b_{i-1}}{2}$,
   a) If $F(A_1^*)=0$, stop.
   b) If $F(A_1^*)\times\ F(b_{i-1})<0$, set $a_i=A_1^*$ and $b_i=b_i$.
   c) If $F(A_1^*)\times\ F(a_{i-1})<0$, set $a_i=a_i$ and $b_i=A_1^*$.
3. Increase *i* by 1 and go back to step 2 as desired.

The capital structure of the bond issuer with a two-period coupon at maturity, based on default at maturity, is described as follows:

TABLE II
CAPITAL STRUCTURE OF TWO-PERIOD COUPON BOND ISSUING COMPANIES DURING PAYMENT

Situation	Asset	Liability	Equity
Non Default	$\varphi_{T_2}^{T_1}\geq c_1$	c_1	$\varphi_{T_2}^{T_1}-c_1$
Default	$\varphi_{T_2}^{T_1}<c_1$	c_1	0

During the coupon payment time (T_1), the obligor was considered to be in default when $A_{T_1}<A_1^*$. Consequently, the probability of default at the time could be expressed as follows:

$$P(\eta=T_1)=P(A_{T_1}<A_1^*)$$
$$=P(A_0\exp\{(r+\omega)T_1X\}<A_1^*)$$
$$=P\left(X<\left(\ln\left(\frac{A_1^*}{A_0}\right)-\ (r+\omega)T_1\right)\right)$$
$$=\int_{-\infty}^{\ln\left(\frac{A_1^*}{A_0}\right)-\ (r+\omega)T_1}f(x)dx \qquad (18)$$

From Equation (14), the following expressions can be deduced:

$A_{T_1}=A_0\exp((r+\omega)T_1+X_1)$, and
$A_{T_2}=A_0\exp((r+\omega)T_2+X_2)$

$$P\big(A_{T_2}<K_1\big|\,A_{T_1}\geq A_1^*\big)=1-$$
$$\frac{P\big((r+\omega)T_2+X_2\big)\geq\ln\dfrac{K_1}{A_0}\cap((r+\omega)T_1+X_1)\geq\ln\dfrac{A_1^*}{A_0})}{P(A_{T_1}\geq A_1^*)}$$

$$=1-$$
$$\frac{P(X_2<\ln\dfrac{A_0}{K_1}+\ (r+\omega)T_2\cap X_1<\ln\dfrac{A_0}{A_1^*}+\ (r+\omega)T_1)}{1-P(M_{T_1}<M_1^*)}$$

$$=1-\frac{\int_{-\infty}^{z_1}\int_{-\infty}^{z_2}f(x_1,x_2)dx_1dx_2}{1-P(A_{T_1}<A_1^*)}$$
$$=1-\frac{\int_{-\infty}^{z_2}\int_{-\infty}^{z_1}f_{X_1,X_2}(x_1,x_2)dx_1dx_2}{1-\int_{-\infty}^{z}f(x)dx}$$

where

$z_1=\ln\frac{A_0}{A_1^*}+\ (r+\omega)T_1$, $z_2=\ln\frac{A_0}{K_1}+\ (r+\omega)T_2$,

$z=\ln\left(\dfrac{A_1^*}{A_0}\right)-\ (r+\omega)T_1$

$f_{X_1,X_2}(x_1,x_2)$ is obtained from Equation (1) for $d=2$. This equation is a bivariate Variance Gamma distribution, namely:

$$f_{X_1,X_2}(x_1,x_2)$$
$$=\frac{2\left(\frac{1}{v}\right)^{\frac{1}{v}}\left(\frac{2}{v}+\boldsymbol{\theta}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}\right)^{1-\frac{1}{v}}}{2\pi|\boldsymbol{\Sigma}|^{\frac{1}{2}}\Gamma(\frac{1}{v})}$$
$$\times\frac{\mathcal{K}_{\frac{1}{v}-1}\left(\sqrt{\mathrm{Q}(\boldsymbol{x})\left(\frac{2}{v}+\boldsymbol{\theta}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}\right)}\right)\exp((\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta})}{\left(\sqrt{\mathrm{Q}(\boldsymbol{x})\left(\frac{2}{v}+\boldsymbol{\theta}'\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}\right)}\right)^{1-\lambda}}$$

Where:

$\boldsymbol{x}'=(x_1,x_2)$
$\boldsymbol{\mu}=(\mu_1,\mu_2)$
$\boldsymbol{\theta}=(\theta_1,\theta_2)$
$Q(\boldsymbol{x})=\ (\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$

$$\boldsymbol{\Sigma}=\begin{pmatrix}\sigma_1^2v_1 & \sigma_1\sigma_2\rho\sqrt{v_1v_2}\\ \sigma_1\sigma_2\rho\sqrt{v_1v_2} & \sigma_2^2v_2\end{pmatrix}$$

$\boldsymbol{\sigma}=(\sigma_1,\sigma_2)$

The shape of the covariance matrix is ($\boldsymbol{\Sigma}$), as stated in [27].

IV. DATA AND METHOD

This study employed bond data from an Indonesian banking company, specifically Bank CIMB Niaga Sub-Ordination Bond III 2018 Series A. The bond was issued on November 16, 2018, and reached maturity on November 15, 2023. The relevant data was sourced from [28].

Company asset data were extracted from the financial statements of PT Bank CIMB Niaga Tbk, spanning from November 2018 to April 2023. These asset details could be accessed through [29], and the stages undertaken in this study encompassed:

1) Exploration of data using quantiles plots (Q-Q plots), testing the Variance Gamma distribution fit using the Chi-Square test, and estimating parameters using maximum likelihood.
2) Determining the Variance Gamma asset model, and the estimated value of the asset at the time of payment of the first coupon value.

3) Calculating the default probability of the bond issuer company for both one-period and two-period coupon using the Variance Gamma distribution.

Data were processed using software R. Some of the R packages used included moments [30], Variance Gamma [31], BAS [32], Bessel [33], and ghyp [34].

V. RESULT AND DISCUSSION

Previous studies developed numerous prediction models for assets or stocks. Predictive accuracy was measured using the mean absolute percentage error (MAPE). For the asset model, prediction accuracy was observed with GBM (MAPE = 6.11%) and GBM with jump diffusion (MAPE = 3.87%). This comparison showed the superiority of the model with jump diffusion [17]. A similar investigation using the same data employed the VG model (MAPE = 2.59%), highlighting its superiority. It should be noted that modeling for predicting daily stock prices had also been conducted by [35], consisting of GBM (MAPE = 9.71%), GBM model with jump diffusion (MAPE = 11.77%), and VG (MAPE = 5.75%). These results collectively underscored the accuracy of VG, causing it to emerge as a more appropriate choice for bond valuation. The first step carried out in this study was to explore the data, namely making quantiles plots (Q-Q plots) and formal tests to see distribution compatibility.

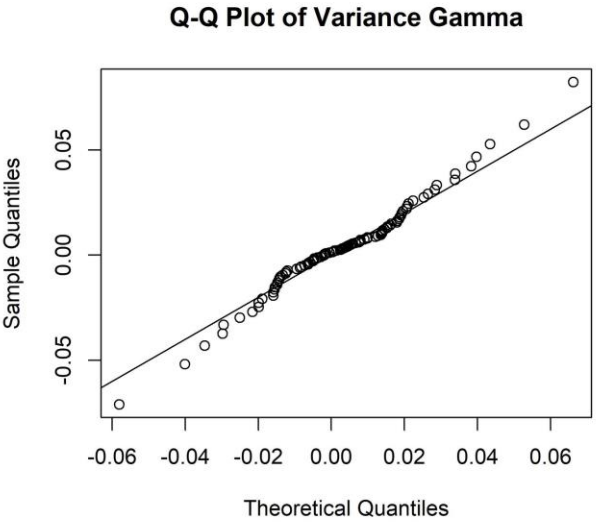

Fig. 3. Quantile plots of the variance gamma distribution.

Fig. 3 showed the quantile plots for the Variance Gamma distribution, which formed a linear pattern, signifying alignment between the data and distribution. The VG distribution model was subjected to a fit test, employing the following formal hypothesis tests:

H_0: ln returns on assets of PT Bank CIMB Niaga Tbk adhered to VG distribution

H_1: ln returns on assets of PT Bank CIMB Niaga Tbk do not conform to VG distribution

The Chi-Square test was employed for this analysis, with a significance level α of 5%. The obtained p-value of 0.0840 led to the conclusion that the distribution of ln returns on the assets of PT Bank CIMB Niaga Tbk followed the VG distribution. Parameter estimation for the Variance Gamma model was carried out using the Maximum Likelihood approach, yielding the following results:

TABLE III
ESTIMATION OF VARIANCE GAMMA MODEL PARAMETERS

Company Name	$\hat{\mu}$	$\hat{\sigma}$	$\hat{v}$	$\hat{\theta}$
PT Bank CIMB Niaga Tbk.	0.00317	0.02214	1.51210	0.00071

Table III showed the results of parameter estimation for ln returns of assets. The value $\hat{v}$ = 1.512104 was employed to determine kurtosis, and from Equation (10), the value kurtosis = 57.43574. This result indicated a presence of excess kurtosis, but a non-zero value of $\hat{\theta}$ implied distribution asymmetry. In addition to VG parameters, essential bond data variables encompassed face value (K), bond term (T), coupon percentage (k), and risk-free interest rate (r).

TABLE IV
BOND DATA VARIABLES

Data	Information
Bond Name	Bank CIMB Niaga Subordinated Bond III Year 2018 Series A
Bond Code	IDA0000946A6
Issuer	PT Bank CIMB Niaga Tbk
Face Value (K)	IDR 75,000,000,000
Issuance Date	November 16, 2018
Maturity Date	November 15, 2023
Bond Term (T)	5 Years
Company Rating	idAAA
Coupon Percentage (k)	9.85% per year

The risk-free interest rate (r) used referred to the interest rate of Bank Indonesia, namely 5.75% [25].

A. Determining The One-Period Coupon Default Probability

In this study, it was assumed that the company solely provided one coupon value (c) during the bond duration, with payment occurring upon maturity (T_1) . The determination of coupon value (c) and K_1 (face value plus coupon value) was detailed as follows:

$$c = \left(\frac{Kk}{4}\right)24 = 6Kk,$$
$$K_1 = K + c$$
$$= K + \left(\frac{Kk}{4}\right)24$$
$$= K(1 + 6k)$$
$$= 75000000000(1 + 6(0.09850))$$
$$K_1 = \ Rp\ 119.325.000.000{,}00$$

The values of the remaining variables required to calculate the default probability were summarized as follows:

TABLE V
THE VALUE OF VARIABLES TO DETERMINE THE PROBABILITY OF A ONE-PERIOD COUPON DEFAULT

Variable Name	Variable Value	Information
c	IDR 44,325,000,000	Coupon Value
K_1	IDR 119,325,000,000	Debt Value and Coupon Value
T_1	60	Maturity Time
A_0	IDR 256,211,135,000,000	Company assets at T_0 (bond issuance time, November 2018)
ω	-0.0009583582	Constant

Based on Equation (15), the default probability value was formulated as follows:

$$P(\eta = T_1)$$
$$= P\left(X_1 < \left(Ln\left(\frac{K_1}{A_0}\right) - (r+\omega)T_1\right)\right)$$
$$= P\left(X_1 < \left(Ln\left(\frac{119325000000}{256211135000000}\right) - (0.0575 + (-0.0009583582)60\right)\right)$$
$$= P(X_1 < -8.76182)$$

The probabilities were computed using the R package, and the function used was pvg., which produced the following:

$$P(X_1 < -8.76182) = 1.511977 \times 10^{-203}$$

The probability value was very small (close to zero), indicating that this company was not default at maturity (time T_1).

B. Determining The Two-Period Coupon Default Probability

The two-period coupon default probability pertained to dual coupon payments across the bond duration, specifically at $T_1$and T_2. At T_1, only the coupon value was disbursed, while at T_2, payment comprised the coupon value and bond payable. In this study, T_1 was set at November 2021 (with the possibility of exploring other months), and T_2 signified the maturity date, namely November 2023. Estimation of A_{T_1} (A_1^*) employed a numerical approach through the bisection method. The result of the iterative is A_1^* = IDR 25,113,294,359. In addition to the VG distribution parameters shown in Table III, the remaining variables pivotal for determining two-period coupon default probability were summarized in Table VI. The default probability for the bond issuer at the time of coupon value payment (time T_1) based on default maturity was expressed below:

$$P(\eta = T_1) = P(A_{T_1} < A_1^*)$$
$$= P\left(X_1 < \left(Ln\left(\frac{A_1^*}{A_0}\right) - (r+\omega)T_1\right)\right)$$
$$= P\left(X_1 < \left(Ln\left(\frac{25113294359}{256211135000000}\right) - (0.0575 + (-0.0009583582)\right)\right)$$
$$= P(X_1 < -11.42841)$$
$$= 1.947979 \times 10^{-265}$$

TABLE VI
VALUE OF VARIABLES FOR CALCULATING THE PROBABILITY OF A TWO PERIOD COUPON DEFAULT

Variable Name	Variable Value	Information
c_1	IDR 22,162,500,000	Coupon value at time T_1
c_2	IDR 22,162,500,000	Coupon value at time T_2
K_1	IDR 97,162,500,000	Value of bond payable and coupon value
T_1	36	Time for payment of the coupon value (November 2021)
T_2	60	Maturity Date (November 2023)
A_0	IDR 256,211,135,000,000	Company Asset at T_0 (bond issuance time, November 2018)
A_1	IDR 301,435,478,000,000	Asset Value at Time T_1
A_1^*	IDR 25,113,294,359	Estimated value of assets
ω	-0.0009583582	Constant

The default probability for the bond issuer at the time of coupon value payment (time T_1) based on default maturity was expressed below:

$$P(\eta = T_1) = P(A_{T_1} < A_1^*)$$
$$= P\left(X_1 < \left(Ln\left(\frac{A_1^*}{A_0}\right) - (r+\omega)T_1\right)\right)$$
$$= P\left(X_1 < \left(Ln\left(\frac{25113294359}{256211135000000}\right) - (0.0575 + (-0.0009583582)\right)\right)$$
$$= P(X_1 < -11.42841)$$
$$= 1.947979 \times 10^{-265}$$

The exceedingly small probability value (close to zero) indicated the non-default status of the company during coupon value payment c_1 (time T_1). Therefore, the analysis needs to be continued for time T_2 (during coupon payment c_2 and face value K). The probability of the issuing company facing defaults at maturity (Time T_2)) was expressed using the following formula:

$$P(\eta = T_2) = P(A_{T_2} < K_1 | A_{T_1} > A_1^*)$$
$$= 1 - \frac{\int_{-\infty}^{z_2}\int_{-\infty}^{z_1} f_{X_1,X_2}(x_1,x_2)dx_1dx_2}{1-\int_{-\infty}^{z} f(x)dx}$$

To calculate $\int_{-\infty}^{z_2}\int_{-\infty}^{z_1} f(x_1,x_2)dx_1dx_2$, the package R, namely pghyp function, was used, and the following equation was obtained:

$$P(\eta = T_2) = 1 - \frac{1}{1-1{,}947979\times 10^{-265}}$$
$$\approx 0$$

Where z_1 and z_2 indicate integral limit.

$$z_1 = \ln\frac{A_0}{A_1^*} + (r+\omega)T_1 = 8.035915 \text{ and}$$
$$z_2 = \ln\frac{A_0}{K_1} + (r+\omega)T_2 = 4.647428$$

The very small probability value indicates the ability of the company to pay the face value (K) and the bond coupon (c_2) at T_2.

VI. CONCLUSION

In conclusion, the use of the VG model in applying the coupon bond theory showed an extremely low probability value concerning maturity-based default. This result indicated a high level of safety for investors who considered bond from PT Bank CIMB Niaga Tbk. The minimal probability of default, closely approaching zero, stemmed from the substantial asset value that was capable of covering the bond obligations. Furthermore, PT Bank CIMB Niaga Tbk, being a robust banking entity, demonstrated impressive performance and showed a capacity to meet long-term financial commitments. This was evidenced by the exemplary corporate rating of idAAA. As a suggestion for future study, the equity and liabilities of companies using the VG model approach could be explored. Additionally, there was potential for the enhancement of coupon value payment spanning two periods.

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