On Rainbow Vertex Antimagic Coloring and Its Application on STGNN Time Series Forecasting on Subsidized Diesel Consumption

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***Abstract*—Let $G = (V, E)$ be a simple, connected and un-directed graph. We introduce a new notion of rainbow vertex antimagic coloring. This is a natural expansion of rainbow vertex coloring combined with antimagic labeling. For $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$, the weight of a vertex $v \in V(G)$ against f is $w_f(v) = \Sigma_{e \in E(v)} f(e)$, where $E(v)$ is the set of vertices incident to v. The function f is called vertex antimagic edge labeling if every vertex has distinct weight. A path is considered to be a rainbow path if for each vertex u and v, all internal vertices on the $u - v$ path have different weights. The rainbow vertex antimagic connection number of G, denoted by $rvac(G)$, is the smallest number of colors taken over all rainbow colorings induced by rainbow vertex antimagic labelings of G. In this paper we aim to discover some new lemmas or theorems regarding to $rvac(G)$. Furthermore, to see the robust application of rainbow vertex antimagic coloring, at the end of this paper we will illustrate the implementation of RVAC on spatial temporal graph neural networks (STGNN) multi-step time series forecasting on subsidized diesel consumption of some petrol stations.**

***Index Terms*—rainbow vertex antimagic coloring, STGNN, time series forecasting, subsidized diesel consumption.**

I. Introduction

In graph theory, a labeling of a graph G is essentially a function that assigns a collection of graph elements to a group of non-negative integers [1]. When this function maps the set of vertices or edges, it's termed vertex labeling or edge labeling, respectively [2].

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Graph labeling provides useful wide range of applications. Recently, there is a big effort to apply graph labeling for machine learning, namely pseudo-labeling for Graph Neural Networks (GNN), see [3]. Other GNN research results can be seen in [4], [5], [6], [7].In the realm of graph theory, numerous labeling types exist, including magic and antimagic labelings. Consider a function $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$, the weight of a vertex $v \in V(G)$ under f is $w_f(v) = \Sigma_{e \in E(v)} f(e)$, where $E(v)$ is the set of vertices incident to v. The function f is considered to have vertex antimagic labeling if it assigns a unique weight to each vertex. There have been some good results on vertex antimagic study, those can be seen in [8], [9], [10], [11].

Meanwhile, the other important study on graph is a rainbow coloring of a graph. It is a coloring of graph such that there exist a rainbow path. In rainbow vertex coloring, a path is considered to be a rainbow path if for each vertex u and v, all internal vertices on the $u - v$ path have different colors. According to Krivelevich and Yuster [12], the lower bound for rainbow vertex connection number is $rvc(G) \geq diam(G) - 1$, where $diam(G)$ is the diameter of graph G. There have been some good results on rainbow vertex coloring study, those can be found in [13], [14], [15], [16], [17], [18].

In this study, we combine the two notions, namely vertex antimagic labeling and rainbow vertex coloring. Thus we introduce a new notion, namely rainbow vertex antimagic coloring. It satisfies both properties either vertex antimagic labeling and rainbow vertex coloring (RVAC). The term "rainbow vertex antimagic connection number" of a graph G, represented as $rvac(G)$, refers to the minimal number of colors needed across all rainbow colorings that result from rainbow vertex antimagic labelings of G. This paper focuses on uncovering new lemmas or theorems related to $rvac(G)$. There have been some results on rainbow vertex antimagic coloring, it can be found in [19], [20], [21]. Furthermore, to see the robust application of rainbow vertex antimagic coloring, at the end of this paper, we will illustrate the implementation of RVAC on spatial temporal graph neural networks (STGNN) multi-step time series forecasting on subsidized diesel consumption of some petrol stations.

STGNN is a type of graph neural network specifically developed for analyzing data that has both spatial and temporal components, organized in the form of graphs [22], [23], [24]. This model is used to understand patterns and relationships among interconnected entities in both space and time. The applications of spatio-temporal graph neural networks include various fields such as weather prediction, traffic movement, social data analysis, environmental monitoring, precision agriculture, and subsidized diesel consumption distributions as well as other domains involving temporal data and graph structures. Together with RVAC, we will analyse STGNN multi-step time series forecasting on subsidized diesel consumption of some petrol stations.

II. Method

The research employed both analytical and experimental approaches. The analytical aspect involved a mathematical deductive method to detail the results, while the experimental side relied on computer programming for simulation purposes. We analysed the subsidized diesel consumption of some petrol stations across Surabaya city, East Java, Indonesia. First, we collected some data from some petrol stations regarding to five features, namely weather, sales, number of vehicles, number of buyer, and diesel supply. We developed the STGNN programming to train 60% input data initiated by doing the vertex embedding process, test the model and finally forecast the subsidized diesel consumption of some petrol stations. We used the following STGNN algorithm.

Single Layer GNN Algorithm

Step 0. Consider a graph $G(V,E)$ with an order of n and a feature matrix $H_{n\times m}$ corresponding to its n vertices and m features from some petrol stations, and give a tolerance ϵ.

Step 1. Determine the matrix adjacency A of graph G arising from spatiality of petrol stations and set a matrix $B = A+I$, where I is an identity matrix.

Step 2. Initialize weights W, bias β, learning rate α. (For simplicity, set $W_{m\times 1} = [w_1\ w_2\ \dots\ w_m]$, where $0 < w_j < 1$, bias $\beta = 0$ and $0 < \alpha < 1$)

Step 3. Multiply weight matrix with vertex features, by setting a message function $\mathbf{m_u}^l = MSG^l(h_u^{l-1})$, for linear layer $\mathbf{m_u}^l = W^l(h_u^{l-1})$.

Step 4. Aggregate the messages from vertex v's neighbors, by setting function $h_v^l = AGG^l\{m_u^{l-1}, u \in N(v)\}$, and by applying the $\mathbf{sum}(\cdot)$ function $h_v^l = SUM^l\{m_u^{l-1}, u \in N(v)\}$ in regards with matrix B.

Step 5. Determine the error, by setting $error^l = \frac{||h_{v_i} - h_{v_j}||_2}{|E|}$, where v_i, v_j are any two adjacent vertices.

Step 6. Observe whether *error* $\leq \epsilon$ or not. If **yes** then stop, if **not** then do Step 7 to update the learning weight matrix W.

Step 7. Update the learning weight matrix by adjusting $W^{l+1} = W_j^l - \alpha \times z_j \times e^l$, where z_j represents the column sum in $H_{v_i}^l$ and divide by the number of nodes.

Step 8. Save the embedding results in a vector when dealing with time series data, and repeat this procedure for subsequent time data observations.

Step 9. Load the vector data then use the time series machine learning to do training, testing and multi-step time series forecasting.

Step 10. Is RMSE $\leq \epsilon$? If YES then STOP. If No then improve W, do Step 2-9.

III. Research Findings

A. Rainbow Vertex Antimagic Coloring

In this section, first we will show rainbow vertex antimagic coloring of some graph and obtain their $rvac$. Secondly, by utilizing one of the obtained theorem, we will analyse STGNN time series forecasting for subsidized diesel consumption on some petrol stations.

Remark 1: [20] Let G be a connected graph, $rvac(G) \geq rvc(G)$.

Lemma 1: Let Bl be a bull-like graph. The rainbow vertex connection number of shackle of bull-like graph, $rvc(Shack(Bl, x_{i,j}, t)) = 6t - 1$.

Proof. $Shack(Bl, x_{i,1}, t)$ has vertex set $V(Shack(Bl, x_{i,1}, t)) = \{x_{i,j}; 1 \leq i \leq t, 1 \leq j \leq 6\} \cup \{x_{t,7}\} \cup \{z_i; 1 \leq i \leq t\}$ and edge set $E(Shack(Bl, x_{i,1}, t)) = \{x_{i,j}x_{i,j+1}; 1 \leq i \leq t, 1 \leq j \leq 5\} \cup \{x_{i,6}x_{i+1,1}; 1 \leq i \leq t-1\} \cup \{x_{t,6}x_{t,7}\} \cup \{x_{i,4}z_i, x_{i,5}z_i; 1 \leq i \leq t\}$. $Shack(Bl, x_{i,1}, t)$ has a diameter of $6t$. According to the lower bound of $rvc(G)$, we have $rvc(Shack(Bl, x_{i,1}, t)) \geq diam(Shack(Bl, x_{i,1}, t)) - 1 = 6t - 1$. Next, we will prove the upper bound of $rvc(Shack(Bl, x_{i,1}, t))$. Define a function $f : V(Shack(Bl, x_{i,1}, t)) \to \{1, 2, \cdots, |V(Shack(Bl, x_{i,1}, t))|\}$ as follows: $f(x_{1,1}) = 1; f(x_{i,1}) = 7i - 8$ for $2 \leq i \leq t; f(x_{i,j}) = 6i + j - 7$ for $1 \leq i \leq t$ and $2 \leq j \leq 6; f(x_{t,7}) = 2; f(z_i) = i$ for $1 \leq i \leq t$.

The above function is a rainbow vertex coloring of $rvc(Shack(Bl, x_{i,1}, t))$ which assure the existence of rainbow path. Now, we will show the cardinality of the obtained colors. The set of colors $\{1, 2, \cdots, 6t - 1\}$ forms an arithmetic sequence. Thus, $|f| = 6t - 1$. According to the lower bound and upper bound, we have $6t - 1 \leq rvc(Shack(Bl, x_{i,j}, t)) \leq 6t - 1$. It concludes that $rvc(Shack(Bl, x_{i,j}, t)) = 6t - 1$ with $t \geq 2$. For the illustration, the rainbow vertex coloring of the graph $Shack(Bl, x_{i,j}, t)$ can be seen in Fig. 1.

Theorem 1: For $t \geq 2$, $rvac(Shack(Bl, x_{i,j}, t)) = 6t - 1$.

Proof. By using Lemma 1 and Remark 1, we determine the lower bound $rvac(Shack(G, x_{i,1}, t)) \geq rvc(Shack(G, x_{i,1}, t)) = 6t - 1$. Now, we prove the upper bound of $rvac(Shack(G, x_{i,1}, t))$ by defining a label function $f : E(Shack(G, x_{i,1}, t)) \to$

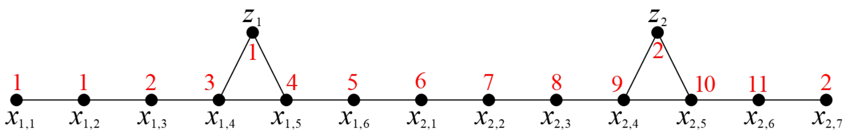

Fig. 1. Rainbow vertex coloring on $Shack(Bl, x_{i,j}, 2)$

$\{1, 2, \cdots, |E(Shack(G, x_{i,1}, t))|\}$ as follows.

$$f(x_{1,1}x_{1,2}) = 3; \quad f(x_{i,1}x_{i,2}) = 8i - 5 \text{ for } 2 \le i \le t$$
$$f(x_{1,2}x_{1,3}) = 1; \quad f(x_{i,2}x_{i,3}) = 8i - 2 \text{ for } 2 \le i \le t$$
$$f(x_{1,3}x_{1,4}) = 2; \quad f(x_{i,3}x_{i,4}) = 8i - 1 \text{ for } 2 \le i \le t$$
$$f(x_{1,4}z_1) = 5; \quad f(x_{i,4}x_{i,5}) = 8i \text{ for } 1 \le i \le t$$
$$f(x_{1,5}z_1) = 6; \quad f(x_{i,5}x_{i,6}) = 8i + 1 \text{ for } 1 \le i \le t - 1$$
$$f(x_{t,6}x_{t,7}) = 4; \quad f(x_{i,6}x_{i+1,1}) = 8i + 2 \text{ for } 1 \le i \le t - 1$$
$$f(x_{t,5}x_{t,6}) = 7; \quad f(x_{i,4}z_i) = 8i - 4$$
$$f(x_{i,5}z_i) = 8i - 3 \text{ for } 2 \le i \le t$$

Based on the label function above, we have vertex weight sets as follows:

$$w(x_{t,5}) = 16t + 4; \quad w(x_{i,1}) = 16i - 11 \text{ for } 2 \le i \le t$$
$$w(x_{1,1}) = 3; \quad w(x_{i,2}) = w(z_i) = 16i - 7 \text{ for } 2 \le i \le t$$
$$w(x_{1,2}) = 4; \quad w(x_{i,3}) = 16i - 3 \text{ for } 2 \le i \le t$$
$$w(x_{t,6}) = 11; \quad w(x_{i,4}) = 24i - 5 \text{ for } 2 \le i \le t$$
$$w(x_{1,4}) = 15; \quad w(x_{i,5}) = 24i - 2 \text{ for } 2 \le i \le t - 1$$
$$w(x_{1,5}) = 23; \quad w(x_{i,6}) = 16i + 3 \text{ for } 1 \le i \le t - 1$$
$$w(x_{1,3}) = 3; \quad w(x_{t,7}) = 4$$
$$w(z_1) = 11$$

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets as follows:
$W_{2,1} = \{w(x_{t,5})\} \cup \{w(x_{1,1})\} \cup \{w(x_{1,3})\} \cup \{w(x_{1,2})\} \cup \{w(x_{t,7})\} \cup \{w(x_{t,6})\} \cup \{w(z_1)\} \cup \{w(x_{1,4})\} \cup \{w(x_{1,5})\} = \{16t + 4, 3, 3, 4, 11, 15, 23\} \longrightarrow |W_{2,1}| = 6$.
$W_{2,2} = \{w(x_{i,1})\} = \{21, 37, \cdots, 16t - 11\} \longrightarrow U_{|W_{2,2}|} = a + (|W_{2,2}| - 1)d \longleftrightarrow 16t - 11 = 21 + (|W_{2,2}| - 1)16 \longrightarrow |W_{2,2}| = t - 1$.
$W_{2,3} = \{w(x_{i,2})\} = \{25, 41, \cdots, 16t - 7\} \longrightarrow U_{|W_{2,3}|} = a + (|W_{2,3}| - 1)d \longleftrightarrow 16t - 7 = 25 + (|W_{2,3}| - 1)16 \longrightarrow |W_{2,3}| = t - 1$.
$W_{2,4} = \{w(x_{i,3})\} = \{29, 45, \cdots, 16t - 3\} \longrightarrow U_{|W_{2,4}|} = a + (|W_{2,4}| - 1)d \longleftrightarrow 16t - 3 = 29 + (|W_{2,4}| - 1)16 \longrightarrow |W_{2,4}| = t - 1$.
$W_{2,5} = \{w(x_{i,4})\} = \{43, 67, \cdots, 24t - 5\} \longrightarrow U_{|W_{2,5}|} = a + (|W_{2,5}| - 1)d \longleftrightarrow 24t - 5 = 43 + (|W_{2,5}| - 1)24 \longrightarrow |W_{2,5}| = t - 1$.
$W_{2,6} = \{w(x_{i,5})\} = \{46, 70, \cdots, 24t - 26\} \longrightarrow U_{|W_{2,6}|} = a + (|W_{2,6}| - 1)d \longleftrightarrow 24t - 26 = 46 + (|W_{2,6}|1 - 1)24 \longrightarrow |W_{2,6}| = t - 2$.
$W_{2,7} = \{w(x_{i,6})\} = \{19, 35, \cdots, 16t - 13\} \longrightarrow U_{|W_{2,7}|} = a + (|W_{2,7}| - 1)d \longleftrightarrow 16t - 3 = 29 + (|W_{2,7}| - 1)16 \longrightarrow |W_{2,7}| = t - 1$.

Based on the above calculation, we obtain the total cardinality of $6 + (t - 1) + (t - 1) + (t - 1) + (t - 1) + (t - 2) + (t - 1) = 6t - 1$. It implies the upper bound of $rvac(Shack(G, x_{i,1}, t)) \le 6t - 1$. According to the lower bound and upper bound, we have $6t - 1 \le rvac(Shack(G, x_{i,1}, t)) \le 6t - 1$. It concludes that $rvac(Shack(G, x_{i,1}, t)) = 6t - 1$ for $t \ge 2$. For the illustration, the rainbow path of the graph $Shack(G, x_{i,1}, t)$ can be seen in Table I. For the illustration, the rainbow vertex antimagic coloring of the graph $Shack(G, x_{i,1}, t)$ can be seen in Fig. 2.

TABLE I
THE RAINBOW PATH FROM u TO v OF RAINBOW VERTEX COLORING OF $Shack(Bl, x_{i,j}, t)$

Case	u	v	Rainbow Vertex	Condition
1	$x_{i,j}$	$x_{i,k}$	$x_{i,j}, x_{i,j+1}, \cdots, x_{i,k-1}, x_{i,k}$	$j \ne k$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, x_{i,j+1}, \cdots, x_{i,5}, x_{i+1,1}, \cdots, x_{k,1}, \cdots, x_{k,l}$	$i \ne k$, $j \ne l$
3	$x_{i,j}$	z_i	$x_{i,j}, x_{i,j+1}, \cdots, x_{i,3}, z_i$	
4	$x_{i,j}$	z_k	$x_{i,j}, x_{i,j+1}, \cdots, x_{k,1}, x_{k,2}, x_{k,3}, z_k$	
5	z_i	z_j	$z_i, x_{i,4}, x_{i,5}, \cdots, x_{j,1}, x_{j,2}, x_{j,3}, z_j$	

Lemma 2: Let $B_{3,n}$ be bull graph with $n \ge 3$. Then $rvc(B_{3,n}) = 2n$.

Proof. $B_{3,n}$ has vertex set $V(B_{3,n}) = \{x_{i,j}; 1 \le i \le 2, 1 \le j \le n\} \cup \{x_1, x_2\} \cup \{y\}$ and edge set $E(B_{3,n} = \{x_{1,i}x_{1,i+1}, x_{2,i}x_{2,i+1}; 1 \le i \le n - 1\} \cup \{x_ix_{1,i}, x_iy; 1 \le i \le 2\} \cup \{x_1x_2\}$. $B_{3,n}$ has a diameter of $2n + 1$. According to the lower bound of $rvc(G)$, we have $rvc(B_{3,n}) \ge \text{diam}(B_{3,n}) - 1 = 2n + 1 - 1 = 2n$. Next, we will prove the upper bound of $rvc(B_{3,n})$. Define a function $f : V(B_{3,n}) \to \{1, 2, \cdots, |V(B_{3,n})|\}$ as follows: $f(x_{1,n}) = f(x_{1,n-1}) = f(x_{2,n}) = 1; f(x_{1,i}) = i + 2$ for $1 \le i \le n - 2; f(y) = f(x_1) = 2; f(x_{2,i}) = i + n + 1$ for $1 \le i \le n - 1, f(x_2) = n + 1$.

The above function is a rainbow vertex coloring of $rvc(B_{3,n})$ which assure the existence of rainbow path. Now, we will show the cardinality of the obtained colors. The set of colors $\{1, 2, \cdots, 2n\}$ forms an arithmetic sequence. Thus, $|f| = 2n$. According to the lower bound and upper bound, we have $2n \le rvc(B_{3,n}) \le 2n$. It concludes that $rvc(B_{3,n}) = 2n$ with $n \ge 3$. For the illustration, the rainbow vertex coloring of the graph $B_{3,n}$ can be seen in Fig. 3.

Theorem 2: For $n \ge 3$, $rvac(B_{3,n}) = 2n$.

Proof. By using Lemma 2 and Remark 1, we determine the lower bound $rvac(B_{3,n}) \ge rvc(B_{3,n}) = 2n$. Now, we prove the upper bound of $rvac(B_{3,n})$ by defining a label function $f : E(B_{3,n}) \to \{1, 2, \cdots, |E(B_{3,n})|\}$. We divide the proof into two cases.

Case 1. $n = 3$

For $n = 3$, we define a label function $f : E(B_{3,n}) \to \{1, 2, \cdots, |E(B_{3,n})|\}$ as follows: $f(x_1x_{1,1}) = 2; f(x_{1,1}x_{1,2}) = 1; f(x_{1,2}x_{1,3}) = 3; f(x_{2,2}x_{2,3}) =$

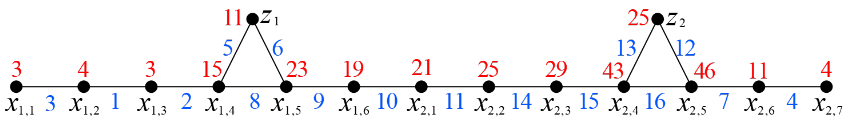

Fig. 2. Rainbow vertex antimagic coloring on $Shack(Bl, x_{i,j}, 2)$

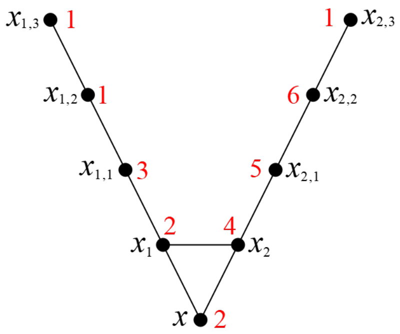

Fig. 3. Rainbow vertex coloring on $B_{3,3}$

$4; f(x_2x_{2,1}) = 8; f(x_1y) = 5; f(x_1x_2) = 9; f(x_2y) = 6; f(x_{2,1}x_{2,2}) = 7$.

Based on the label function above, we have vertex weight sets as follows: $w(y) = w(x_{2,2}) = 11; w(x_{1,1}) = w(x_{1,3}) = 3; w(x_{2,1}) = 15; w(x_{1,2}) = w(x_{2,3}) = 4; w(x_2) = 23; w(x_1) = 16$.

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and we have $rvac(B_{3,3}) \geq |\{w(V(B_{3,n}))\}| = 6$. According to the lower bound and upper bound, we have $6 \leq rvac(B_{3,3}) \leq 6$. It concludes that $rvac(B_{3,3}) = 6$ for $n = 3$.

Case 2. $n \geq 4$
Now, we prove the upper bound of $rvac(B_{3,n})$ by defining a label function $f : E(B_{3,n}) \to \{1, 2, \cdots, |E(B_{3,n})|\}$ as follows.

$$
\begin{aligned}
f(x_{1,n-1}x_{1,n}) &= 3\\
f(x_1x_2) &= 2n+3\\
f(x_2x_{2,1}) &= n+5\\
f(x_2y) &= 6\\
f(x_{1,n-3}x_{1,n-2}) &= 2\\
f(x_1y) &= 5\\
f(x_1x_2) &= 2n+3\\
f(x_{1,j}x_{1,i+1}) &= i+8 \text{ for } 1 \leq i \leq n-4\\
f(x_{2,n-1}x_{2,n}) &= 4\\
f(x_{2,i}x_{2,i+1}) &= n+i+5 \text{ for } 1 \leq i \leq n-3\\
f(x_{2,n-2}x_{2,n-1}) &= 7
\end{aligned}
$$

Based on the label function above, we have vertex weight sets as follows:

$$
\begin{aligned}
w(x_{1,n}) = w(x_{1,n-2}) &= 3\\
w(x_{2,n-2}) &= 2n+9\\
w(x_1) &= 2n+16\\
w(x_{1,n-1}) = w(x_{2,n}) &= 4\\
w(x_{2,n-1}) = w(y) &= 11\\
w(x_2) &= 3n+14\\
w(x_{1,i}) &= 2i+17 \text{ for } 2 \leq i \leq n-3\\
w(x_{2,i}) &= 2n+2i+9 \text{ for } 1 \leq i \leq n-3
\end{aligned}
$$

$$
w(x_{1,1}) = \begin{cases} 10 & \text{for } n = 4\\ i+16 & \text{for } n \geq 5 \end{cases}
$$

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets as follows:
$W_{4,1} = \{w(x_{1,n})\} \cup \{w(x_{1,n-2})\} \cup \{w(x_{1,n-1})\} \cup \{w(x_{2,n})\} \cup \{w(x_{2,n-1})\} \cup \{w(x_{2,n-2})\} \cup \{w(y)\} \cup \{w(x_1)\} \cup \{(x_2)\} = \{3, 4, 11, 2n+9, 2n+16, 3n+15\} \longrightarrow |W_{4,1}| = 6$

$W_{4,2} = \{w(x_{1,1})\} = \begin{cases} 10 & \text{for } n = 4\\ 16 & \text{for } n \geq 5 \end{cases} \longrightarrow |W_{4,2}| = 1$

$W_{4,3} = \{w(x_{1,i})\} = \{21, 23, \cdots, 2n+11\} \longrightarrow U_{|W_{4,3}|} = a + (|W_{4,3}| - 1)2 \longrightarrow 2n + 11 = 21 + (|W_{4,3}| - 1)2 \longleftrightarrow |W_{4,3}| = n - 4$

$W_{4,4} = \{w(x_{2,i})\} = \{2n+11, 2n+13, \cdots, 4n+3\} \longrightarrow U_{|W_{4,4}|} = a + (|W_{4,4}| - 1)2 \longrightarrow 4n+3 = 2n+11+(|W_{4,4}| - 1)2 \longleftrightarrow |W_{4,4}| = n - 3$

Based on the above calculation, we obtain the total cardinality of $6 + 1 + (n-4) + (n-3)$. It implies the upper bound of $rvac(B_{3,n}) \leq 2n$. According to the lower bound and upper bound, we have $2n \leq rvac(B_{3,n}) \leq 2n$. For the illustration, the rainbow path of the graph $B_{3,n}$ can be seen in Table II. For the illustration, the rainbow vertex antimagic coloring of the graph $B_{3,n}$ can be seen in Fig. 4.

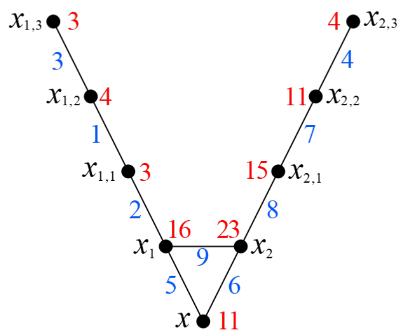

Fig. 4. Rainbow vertex antimagic coloring on $B_{3,3}$

Lemma 3: Let $N_{3,n}$ be a net graph. The rainbow vertex connection number of net graph, $rvc(N_{3,n}) = 3n$.

TABLE II
RAINBOW VERTEX COLORING FROM u TO v IN $B_{3,n}$

Case	u	v	Rainbow Vertex	Condition
1	$x_{i,j}$	$x_{i,k}$	$x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,k-2}, x_{i,k-1}$	$j \neq k$, $j \leq k$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,1}, y_i, y_k, y_{k,1}, \cdots, x_{i,k-2}, x_{i,k-1}$	$i \neq k$, $j \neq l$
3	$x_{i,j}$	y_i	$x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,1}$	-
4	$x_{i,j}$	y_k	$x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,1}, x_i$	$i \neq k$
5	$x_{i,j}$	z	$x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,1}, x_i$	-

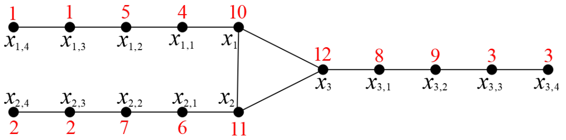

Fig. 5. Rainbow vertex coloring on $N_{3,4}$

Proof. $N_{3,n}$ has vertex set $V(N_{3,n}) = \{x_{i,j}; 1 \leq i \leq 3, 1 \leq j \leq n\} \cup \{x_1, x_2, x_3\}$ and edge set $E(N_{3,n} = \{x_1x_2, x_2x_3, x_1x_3\} \cup \{x_ix_{i,1}, 1 \leq i \leq 3\} \cup \{x_{i,j}x_{i,j+1}; 1 \leq j \leq n-1\}$. $N_{3,n}$ has a diameter of $3n+1$. The graph $N_{3,n}$ has 3 pendants where the diameter for each pendant is n. Based on this conditions, we have a rainbow vertex connection number of 3 pendants is $3(diam-1) = 3n-3$. $N_{3,n}$ also has C_3 with each pendant end connected to C_3, so 3 vertices in C_3 become internal vertices. Because of the rainbow vertex coloring requirement, the internal vertices cannot have the same color, so $rvc(N3,n) \geq 3n-3+3 = 3n$. According to the lower bound of $rvc(G)$, we have $rvc(N_{3,n}) \geq 3n$. Next, we will prove the upper bound of $rvc(N_{3,n})$. Define a function $f : V(N_{3,n}) \to \{1, 2, \cdots, |V(N_{3,n})|\}$ as follows: $f(x_{i,n}) = f(x_{i,n-1}) = i$ for $1 \leq i \leq 3; f(x_{i,j}) = i(n-2) - n + j + 5$ for $1 \leq i \leq 3, 1 \leq j \leq n-2; f(x_i) = 3n + j - 3$ for $1 \leq i \leq 3$.

The above function is a rainbow vertex coloring of $rvc(N_{3,n})$ which assure the existence of rainbow path. Now, we will show the cardinality of the obtained colors. The set of colors $\{1, 2, \cdots, 3n\}$ forms an arithmetic sequence. Thus $|f| = 3n$. According to the lower bound and upper bound, we have $3n \leq rvc(N_{3,n}) \leq 3n$. It concludes that $rvc(N_{3,n}) = 3n$. For the illustration, the rainbow vertex coloring of the graph $N_{3,n}$ can be seen in Fig. 5.

Theorem 3: For $n \geq 3$, $rvac(N_{3,n}) = 3n$.

Proof. Next, we determine the lower bound of $rvac(N_{3,n})$ using Lemma 3 and Remark 1. Based on Lemma 3 and Remark 1, we have $rvac(N_{3,n}) \geq rvc(N_{3,n}) = 3n$. Then, we prove the upper bound of $rvac(N_{3,n})$ by defining a label function $l : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$. We divide the proof into seven cases.

Case 1. $n = 3$

For $n = 3$, we prove the upper bound of $rvac(N_{3,n})$ by defining a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows: $f(x_{1,2}x_{1,3}) = 3; f(x_1x_2) = 10; f(x_2x_{2,1}) = 5; f(x_{2,2}x_{2,3}) = 4; f(x_{3,1}x_{3,2}) = 7; f(x_{1,1}x_{1,2}) = 1; f(x_1x_3) = 9; f(x_{2,1}x_{2,2}) = 6; f(x_3x_{3,1}) = 8; f(x_{3,2}x_{3,3}) = 11; f(x_1x_{1,1}) = 2; f(x_2x_3) = 12$.

Based on the label function above, we have vertex weight sets as follows: $w(x_{1,3}) = w(x_{1,1}) = 3; w(x_{2,8}) = w(x_{3,3}) = 11; w(x_2) = 27; w(x_{3,1}) = 15; w(x_3) = 29; w(x_{1,2}) = w(x_{2,3}) = 4; w(x_1) = 21; w(x_{2,2}) = 10; w(x_{3,2}) = 18$.

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and we have $rvac(N_{3,n}) \geq |W(V(N_{3,n}))| = 3n = 9$ for $n = 3$. According to the lower bound and upper bound, we have $9 \leq rvac(N_{3,n}) \leq 9$ for $n = 3$. It concludes that $rvac(N_{3,n}) = 9$ for $n = 3$.

Case 2. $n = 4$

For $n = 4$, we prove the upper bound of $rvac(N_{3,n})$ by defining a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows: $f(x_1x_{1,1}) = 5; f(x_{1,3}x_{1,4}) = 3; f(x_2x_3) = 15; f(x_{2,2}x_{2,3}) = 7; f(x_{3,1}x_{3,2}) = 10; f(x_{1,1}x_{1,2}) = 2; f(x_1x_2) = 14; f(x_2x_{2,1}) = 8; f(x_{2,3}x_{2,4}) = 4; f(x_{3,2}x_{3,3}) = 9; f(x_{1,2}x_{1,3}) = 1; f(x_1x_3) = 13; f(x_{2,1}x_{2,2}) = 6; f(x_3x_{3,1}) = 12; f(x_{3,3}x_{3,4}) = 11$.

Based on the label function above, we have vertex weight sets as follows: $w(x_{1,3}) = w(x_{2,4}) = 4; w(x_1) = 32; w(x_{2,1}) = 14; w(x_{3,1}) = 22; w(x_{2,3}) = w(x_{3,4}) = 11; w(x_{1,1}) = 7; w(x_{2,2}) = 13; w(x_{3,2}) = 19; w(x_{1,2}) = w(x_{1,4}) = 3; w(x_2) = 37; w(x_3) = 40; w(x_{3,3}) = 20$.

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and we have $rvac(N_{3,n}) \geq |\{w(V(N_{3,n}))\}| = 3n = 12$ for $n = 4$. According to the lower bound and upper bound of $rvac(N_{3,n})$, we have $12 \leq rvac(N_{3,n}) \leq 12$ for $n = 4$. It concludes that $rvac(N_{3,n}) = 12$ for $n = 4$.

Case 3. $n = 5$

For $n = 5$, we prove the upper bound of $rvac(N_{3,n})$ by defining a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows: $f(x_1x_{1,1}) = 6; f(x_{1,1}x_{1,2}) = 5; f(x_{1,2}x_{1,3}) = 2; f(x_{1,3}x_{1,4}) = 1; f(x_{1,4}x_{1,5}) = 3; f(x_2x_{2,1}) = 10; f(x_{2,1}x_{2,2}) = 9; f(x_{2,2}x_{2,3}) = 7; f(x_{2,3}x_{2,4}) = 8; f(x_{2,4}x_{2,5}) = 4; f(x_3x_{3,1}) = 15; f(x_{3,1}x_{3,2}) = 14; f(x_{3,2}x_{3,3}) = 12; f(x_{3,3}x_{3,4}) = 13; f(x_{3,4}x_{3,5}) = 11; f(x_1x_2) = 17; f(x_2x_3) = 18; f(x_1x_3) = 16$.

Based on the label function above, we ave vertex weight sets as follows: $w(x_1) = 39; w(x_{1,1}) = w(x_{3,5}) = 11; w(x_{1,2}) = 7; w(x_{1,3}) = w(x_{1,5}) = 3; w(x_2) = 45; w(x_{2,1}) = 19; w(x_{2,2}) = 16; w(x_{2,3}) = 15; w(x_3) = 49; w(x_{3,1}) = 29; w(x_{3,2}) = 26; w(x_{3,3}) = 25; w(x_{1,4}) = w(x_{2,5}) = 4; w(x_{2,4}) = 12; w(x_{3,4}) = 24$.

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and we have $rvac(N_{3,n}) \geq |\{w(V(N_{3,n}))\}| = 3n = 15$ for $n = 5$. According to the lower bound and upper bound, we have $15 \leq rvac(N_{3,n}) \leq 15$ for $n = 5$. It is concludes that $rvac(N_{3,n}) = 15$ for $n = 5$.

Case 4. $n \equiv 0(mod3), 6 \leq n \leq 9$

For $n \equiv 0(mod3), 6 \leq n \leq 9$, we define a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows: $f(x_{1,n-1}x_{x_{1,n}}) = 3; f(x_1x_{1,1}) = n+1; f(x_{1,n-2}x_{x_{1,n-1}}) = 1; f(x_{2,n-1}x_{2,n}) = 4; f(x_{3,n-1}x_{3,n}) = 11; f(x_{1,n-3}x_{1,n-2}) = 2; f(x_{1,j}x_{i,j+1}) = n-j+1$ for $1 \leq j \leq n-4; f(x_{2,j}x_{2,j+1}) = 2n-j+1$ for $1 \leq j \leq n-2; f(x_{3,j}x_{3,j+1}) = 3n-j$ for $1 \leq j \leq n-4; f(x_2x_{2,1}) = 2n+1; f(x_{3,n-2}x_{3,n-1}) = 2n+3; f(x_3x_{3,1}) = 3n; f(x_{3,n-3}x_{3,n-2}) = 2n+2; f(x_1x_3) = 3n+1; f(x_2x_3) = 3n+3; f(x_1x_2) = 3n+2$.

Based on the label function above, we have vertex weight sets as follows:

$$w(x_{2,n-1}) = \begin{cases} 12 & \text{if } n=6 \\ 16 & \text{if } n=9 \end{cases}$$
$$w(x_{2,n-2}) = \begin{cases} 17 & \text{if } n=6 \\ 25 & \text{if } n=9 \end{cases}$$
$$w(x_{2,n-3}) = \begin{cases} 19 & \text{if } n=6 \\ 27 & \text{if } n=9 \end{cases}$$

$$\begin{aligned} w(x_{1,1}) &= 2n+1; & w(x_1) &= 7n+4; \\ w(x_{1,n-3}) &= 7; & w(x_{1,n-2}) &= w(x_n) = 3; \\ w(x_{3,6}) &= 11; & w(x_{3,n-1}) &= 2n+14; \\ w(x_{3,n-2}) &= 4n+5; & w(x_{3,n-3}) &= 4n+6; \\ w(x_3) &= 9n+4; & w(x_{3,1}) &= 6n-1; \\ w(x_{2,1}) &= 4n+1; & w(x_2) &= 8n+6; \end{aligned}$$

$$\begin{aligned} w(x_{1,j}) &= 2n-j+3 \text{ for } 2 \le j \le n-4; \\ w(x_{2,j}) &= 4n-2j+3 \text{ for } 2 \le j \le n-4; \\ w(x_{3,j}) &= 6n-2j+1 \text{ for } 2 \le j \le n-4; \\ w(x_{2,n}) &= w(x_{1,n-1}) = 4. \end{aligned}$$

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and we have $rvac(N_{3,n}) \ge |W(V(N_{3,n}))| = 3n$ for $n \equiv 0(mod 3), 6 \le n \le 9$. According to the lower bound and upper bound, we have $3n \le rvac(N_{3,n}) \le 3n$ for $n \equiv 0(mod 3), 6 \le n \le 9$. It concludes that $rvac(N_{3,n}) = 3n$ for $n \equiv 0(mod 3), 6 \le n \le 9$.

Case 5. $n = 7$

For $n = 7$, we define a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows:

$$\begin{aligned} f(x_1x_{1,1}) &= 8 \\ f(x_{1,4}x_{1,5}) &= 2 \\ f(x_{1,6}x_{1,7}) &= 3 \\ f(x_2x_{2,1}) &= 15 \\ f(x_{2,6}x_{2,7}) &= 4 \\ f(x_{3,6}x_{3,7}) &= 11 \\ f(x_{1,5}x_{1,6}) &= 1 \\ f(x_3x_{3,1}) &= 21 \\ f(x_{1,j}x_{1,j+1}) &= n-j+1, 1 \le j \le 3 \\ f(x_{2,j}x_{2,j+1}) &= 2n-j+1, 1 \le j \le 3 \\ f(x_{2,j}x_{2,j+1}) &= n+j-2, 4 \le j \le 5 \end{aligned}$$

$$f(x_{3,j}) = \begin{cases} 2n-j+1 & \text{for } 1 \le j \le 3 \\ 2n+j-2 & \text{for } 4 \le j \le 5 \end{cases}$$
$$f(x_{2,j}) = \begin{cases} 2n-j+1 & \text{for } 1 \le j \le 3 \\ n+j-2 & \text{for } 4 \le j \le 5 \end{cases}$$

Based on the label function above, we have the vertex weight sets as follows:

$$\begin{aligned} & w(x_{1,n}) = w(x_{1,n-2}) = 3 \\ & w(x_{1,n-1}) = w(x_{2,n}) = 4 \\ & w(x_{1,n-3}) = 7 \\ & w(x_{1,j}) = 2n-2j+3, 2 \le j \le n-4 \\ & w(x_{1,1}) = 2n+1 \\ & w(x_1) = 7n+4 \\ & w(x_{2,j}) = 4n-2j+2, 2 \le j \le n-4 \\ & w(x_{2,n-1}) = 16 \\ & w(x_{2,n-2}) = 22 \\ & w(x_{2,n-3}) = 23 \\ & w(x_{2,1}) = 4n+1 \\ & w(x_{3,n-1}) = 30 \\ & w(x_{3,j}) = 6n-2j+1, 2 \le j \le n-4 \\ & w(x_{3,n-3}) = 38 \\ & w(x_{3,n-2}) = 37 \\ & w(x_{3,1}) = 6n-1 \\ & w(x_3) = 9n+4 \end{aligned}$$

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and we have $rvac(N_{3,n}) \ge |W(V(N_{3,n}))| = 3n$ for $n = 7$. According to the lower bound and upper bound, we have $3n \le rvac(N_{3,n}) \le 3n$ for $n = 7$. It concludes that $rvac(N_{3,n}) = 3n$ for $n = 7$.

Case 6. $n = 8$

For $n = 8$, we define a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows:

$$\begin{aligned} f(x_{1,n-1}x_{1,n}) &= 3; & f(x_{1,n-2}x_{1,n-1}) &= 1; \\ f(x_{3,n-2}x_{3,n-1}) &= 19; & f(x_{3,n-3}x_{3,n-2}) &= 18; \\ f(x_3x_{3,1}) &= 3n; & f(x_1x_2) &= 3n+2; \\ f(x_{3,n-1}x_{3,n}) &= 11; & f(x_1x_{1,1}) &= n+1; \\ f(x_{1,n-3}x_{1,n-2}) &= 2; & f(x_{3,1}x_{3,2}) &= 3n-1; \\ f(x_{2,n-1}x_{2,n}) &= 4; & f(x_1x_3) &= 3n+1; \\ f(x_2x_3) &= 3n+3; & f(x_{2,n-2}x_{2,n-1}) &= 12; \\ f(x_{2,n-3}x_{2,n-2}) &= 10; & f(x_2x_{2,1}) &= 2n+1 \end{aligned}$$

$$\begin{aligned} & f(x_{1,j}x_{1,j+1}) = n-j+1 \text{ for } 1 \le j \le n-4 \\ & f(x_{3,j}x_{3,j+1}) = 3n-j \text{ for } 1 \le j \le n-4 \\ & f(x_{2,j}x_{2,j+1}) = 2n-j+1 \text{ for } 1 \le j \le n-4 \end{aligned}$$

Based on the label function above, we have vertex weight sets as follows:

$$\begin{aligned} w(x_{1,n}) &= w(x_{1,n-2}) = 3 \\ w(x_{1,n-3}) &= 7 \\ w(x_{1,j}) &= 2n-2j+3, 2 \le j \le n-4 \\ W(x_1) &= 7n+4 \\ w(x_{2,j}) &= 4n-2j+2, 2 \le j \le n-4 \\ w(x_{2,n-1}) &= 16 \\ w(x_{2,n-2}) &= 22 \\ w(x_{2,n-3}) &= 23 \\ w(x_{2,1}) &= 4n+1 \\ w(x_{3,n-1}) &= 30 \\ w(x_{3,j}) &= 6n-2j+1, 2 \le j \le n-4 \\ w(x_{3,n-3}) &= 38 \\ w(x_{3,n-2}) &= 37 \\ w(x_{3,1}) &= 6n-1 \\ w(x_3) &= 9n+4 \\ w(x_{1,n-1}) &= w(x_{2,n}) = 4 \\ w(x_{1,1}) &= 2n+1 \end{aligned}$$

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets and it implies the upper bound of $rvac(N_{3,n}) \ge$

$|W(V(N_{3,n}))| = 3n$ for $n = 8$. According to the lower bound and upper bound, we have $3n \leq rvac(N_{3,n}) \leq 3n$ for $n = 7$. It concludes that $rvac(N_{3,n}) = 3n$ for $n = 8$.

Case 7. $n \equiv 0(mod3), n \geq 12$ and $n \equiv 1, 2(mod3), n \geq 10$

For $n \equiv 0(mod3), n \geq 12$ and $n \equiv 1, 2(mod3), n \geq 10$, we define a label function $f : E(N_{3,n}) \to \{1, 2, \cdots, |E(N_{3,n})|\}$ as follows: $f(x_{1,n-1}x_{1,n}) = 3; f(x_{1,n-2}x_{1,n-1}) = 1; f(x_{1,n-3}x_{1,n-2}) = 2; f(x_{1,j}x_{1,j+1}) = \begin{cases} n-j+2 & \text{for } 1 \leq j \leq n-10 \\ n-j+1 & \text{for } n-9 \leq j \leq n-4 \end{cases}; f(x_1x_{1,1}) = n + 2; f(x_{2,n-1}x_{2,n}) = 4; f(x_{2,n-2}x_{2,n-1}) = n+4; f(x_{2,n-3}x_{2,n-2}) = n+3; f(x_{2,j}x_{2,j+1}) = 2n-j+1$ for $1 \leq j \leq n-4; f(x_2x_{2,1}) = 2n+1; f(x_{3,n-1}x_{3,n}) = 11; f(x_{3,n-2}x_{3,n-1}) = 2n+3; f(x_{3,n-3}x_{3,n-2}) = 2n+2; f(x_{3,j}x_{3,j+1}) = 3n-j$ for $1 \leq j \leq n-4; f(x_3x_{3,1}) = 3n; f(x_1x_2) = 3n+2; f(x_1x_3) = 3n+1; f(x_2x_3) = 3n+3$. Based on the label function above, we have vertex weight sets as follows:

$$w(x_{1,j}) = \begin{cases} 3, & \text{if } j = n, j = n-2 \\ 4, & \text{if } j = n-1 \\ 7, & \text{if } j = n-3 \\ 11, & \text{if } j = n-4 \\ 2n-2j+3, & \text{if } n-8 \leq j \leq n-5 \\ 22, & \text{if } j = n-9 \\ 2n-2j+5, & \text{if } 2 \leq j \leq n-10 \\ 2n+3, & \text{if } j = 1 \end{cases}$$

$$w(x_{2,j}) = \begin{cases} 4n+1, & \text{if } j = 1 \\ 4n-2j+3, & \text{if } 2 \leq j \leq n-4 \\ 2n+8, & \text{if } j = n-3 \\ 2n+7, & \text{if } j = n-2 \\ n+8, & \text{if } j = n-1 \\ 4, & \text{if } j = n \end{cases}$$

$$w(x_{3,j}) = \begin{cases} 6n-1, & \text{if } j = 1 \\ 6n-2j+1, & \text{if } 2 \leq j \leq n-4 \\ 4n+6, & \text{if } j = n-3 \\ 4n+5, & \text{if } j = n-2 \\ 2n+14, & \text{if } j = n-1 \\ 11, & \text{if } j = n \end{cases}$$

$$w(x_j) = \begin{cases} 7n+5, & \text{if } j = 1 \\ 8n+6, & \text{if } j = 2 \\ 9n+4, & \text{if } j = 3 \end{cases}$$

The above sets will induce the rainbow vertex coloring of graph. We can calculate the cardinality of vertex weight sets as follows:

$W_{6,1} = \{w(x_{1,n}), w(x_{1,n-2}), w(x_{1,n-9}), w(x_{1,n-3}), w(x_{1,n-4}), w(x_{1,1}), w(x_{2,1}), w(x_{2,n-3}), w(x_{2,n-2}), w(x_{2,n-1}), w(x_{2,n}), w(x_{3,1}), w(x_{3,n-3}), w(x_{3,n-2}), w(x_{3,n-1}), w(x_{3,n}), w(x_1), w(x_2), w(x_3)\} = \{3, 4, 7, 22, 11, 2n+3, 4n+1, 2n+8, 2n+7, n+8, 6n-1, 4n+6, 4n+5, 2n+14, 7n+5, 8n+6, 9n+4\}$, so $|W_{6,1}| = 17$.

$W_{6,2} = \{w(x_{1,j}) | n-8 \leq j \leq n-5\}, W_{6,3} = \{w(x_{1,j}) | 2 \leq j \leq n-10\}, W_{6,4} = \{w_{2,j} | 2 \leq j \leq n-4\}, W_{6,5} = \{w(3,j) | 2 \leq j \leq n-4\} \longrightarrow W_{6,2} = \{13, 15, 17, 19\} \longrightarrow |W_{6,2}| = 4$

$W_{6,3} = \{25, 27, \cdots, 2n+1\} \longrightarrow U_{|W_{6,3}|} = a + (|W_{6,3}| - 1)2 \longrightarrow 2n+1 = 25 + (|W_{6,3}| - 1)2 \longleftrightarrow |W_{6,3}| = n - 11$.

$W_{6,4} = \{2n+11, 2n+13, \cdots, 4n-1\} \longrightarrow U_{|W_{6,4}|} = a + (|W_{6,4}| - 1)2 \longleftrightarrow 4n-1 = 2n+11 + (|W_{6,4}| - 1)2 \longleftrightarrow |W_{6,4}| = n - 5$.

$W_{6,5} = \{4n+9, 4n+11, \cdots, 6n-3\} \longrightarrow U_{|W_{6,5}|} = a + (|W_{6,5}| - 1)2 \longleftrightarrow 6n-3 = 4n+9 + (|W_{6,5}| - 1)2 \longleftrightarrow |W_{6,5}| = n - 5$.

Based on the above calculation, we obtain the total cardinality of $|W_6| = |W_{6,1}| + |W_{6,2}| + |W_{6,3}| + |W_{6,4}| + |W_{6,5}| = 17 + 4 + (n-11) + (n-5) + (n-5) = 3n$. It implies the upper bound of $rvac(N_{3,n}) \leq 3n$. According to the lower and upper bound, we have $3n \leq rvac(N_{3,n}) \leq 3n$ for $n \equiv 0(mod3), n \geq 12$ and $n \equiv 1, 2(mod3), n \geq 10$. It concludes that $rvac(N_{3,n}) = 3n$ for $n \equiv 0(mod3), n \geq 12$ and $n \equiv 1, 2(mod3), n \geq 10$.

According to the lower bound and upper bound of Case 1 until Case 7, we have $3n \leq rvac(N_{3,n}) \leq 3n$. It concludes that $rvac(N_{3,n}) = 3n$ for $n \geq 3$. For the illustration, the rainbow path of the graph $N_{3,n}$ can be seen in Table III. For the illustration, the rainbow vertex antimagic coloring of the graph $N_{3,n}$ can be seen in Fig. 6.

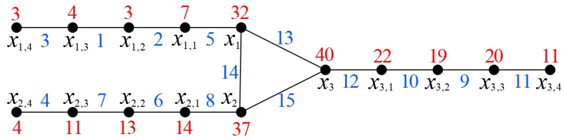

Fig. 6. Rainbow vertex antimagic coloring on $N_{3,4}$

TABLE III
RAINBOW VERTEX COLORING FROM u TO v IN $N_{3,n}$

Case	u	v	Rainbow Vertex	Condition
1	$x_{i,j}$	$x_{i,k}$	$x_{i,j}, x_{i,j+1}, \cdots,$ $x_{i,k-1}, x_{i,k}$	$j \neq k$, $j < k$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,j}, x_{i,j-1}, \cdots,$	$i \neq k$
2	$x_{i,j}$	$x_{k,l}$	$x_{i,1}, x_i, x_k, x_{k,1}, \cdots,$	$i \neq k$
2	$x_{i,j}$	$x_{k,l}$	$x_{k,l-1}, x_{k,l}$	$i \neq k$
3	$x_{i,j}$	x_i	$x_{i,j}, x_{i,j-1}, \cdots, x_{i,1}, x_i$	
4	$x_{i,j}$	x_k	$x_{i,j}, x_{i,j-1}, \cdots, x_{i,1}, x_i, x_k$	$i \neq k$
5	x_i	x_j	x_i, x_j	$i \neq j$

B. The Application of Rainbow Vertex Antimagic Coloring

The next research result is implementing the rainbow vertex antimagic coloring scheme on the subsidized petrol distribution. We utilizes the graph representation of eight petrol stations in Surabaya, Indonesia. The selected graph representation is in the form of $Shack(G, x_{i,j}, t)$ graph which has been shown in Theorem 1. The petrol station locations and the graph representation can be depicted in Figure 7.

Based on Theorem 1, the graph representation of the eight petrol stations is the base graph of $Shack(Bl, x_{i,j}, t)$. Theorem 1 shows that $Shack(Bl, x_{i,j}, t) = 6t - 1$. When we have a graph $Shack(Bl, x_{i,j}, 1)$, thus we have $rvac(Shack(Bl, x_{i,j}, 1)) = 5$, see Figure 8(a). We denote Petrol Station 54.602.69 as x_1, Petrol Station 54.602.30 as x_2, Petrol Station 51.601.77 as x_3, Petrol Station 54.601.100 as x_4, Petrol Station 54.602.48 as x_5, Petrol Station 54.602.64 as x_6, Petrol Station 54.602.52 as x_7, and Petrol Station 54.602.68 as z. By Theorem 1, we

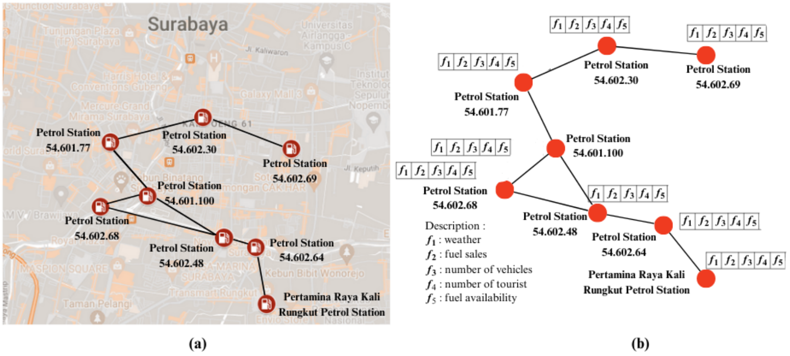

Fig. 7. (a) The map of eight petrol stations, (b) Its graph representation

have RVAC on $Shack(Bl, x_{i,j}, 1)$, see Figure 8(a). Since $rvac(Shack(Bl, x_{i,j}, 1)) = 5$, it means that we need to have computer five admins to monitor subsidized petrol distribution, see Table V. This table tell us that Admin 3 monitors the subsidized diesel distribution on Petrol Station x_2, Admin 15 monitors the subsidized diesel distribution on Petrol Station x_1, x_3, z, Admin 21 monitors the subsidized diesel distribution on Petrol Station x_4, x_7, Admin 11 monitors the subsidized diesel distribution on Petrol Station x_5, Admin 4 monitors the subsidized diesel distribution on Petrol Station x_6. The biggest workload is Admin 15, since it monitors three petrol stations.

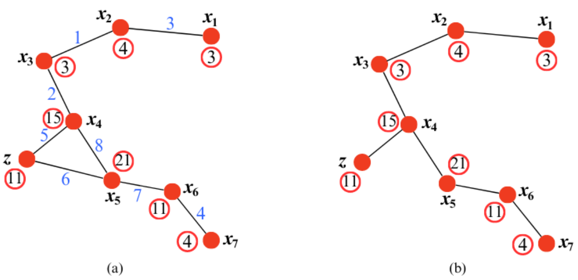

Fig. 8. (a) The Illustration of RVAC on $Shack(Bl, x_{i,j}, 1)$, (b) The spanning tree of RVAC and its rainbow vertex antimagic coloring

TABLE IV
THE COMPUTER FIVE ADMINS REPRESENTED BY ALL VERTEX WEIGH OF RVAC

Admin	3	15	21	11	4
Vertex Weight	4	3, 3, 11	15, 4	21	11
Vertex	x_2	x_1, x_3, z	x_4, x_7	x_5	x_6

Furthermore, the rainbow vertex antimagic coloring concept can be utilized for Petrol Delivery Services (PDS) with 135 as call center. Delivery services can be accessed 24 hours with the time delivery from 08.00 - 20.00. Using the rainbow vertex antimagic coloring concept, the PDS center can determine which petrol stations are close to the costumers, see Table V. This table tell us if there is consumer who needs PDS service on road 6, PDS center can easily obtain that the closest petrol stations $\{11, 21\} = \{z, x_6\}$. If there is consumer who needs PDS service on road 5, PDS center can easily obtain that the closest petrol stations $\{11, 15\} = \{z, x_4\}$.

TABLE V
THE COMPUTER FIVE ADMINS REPRESENTED BY ALL VERTEX WEIGH OF RVAC

3	4	11	15	21
$\{3\}$	$\{4\}$	$\{5, 6\}$	$\{2, 5, 8\}$	$\{6, 7, 8\}$
$\{1, 2\}$	$\{1, 3\}$	$\{4, 7\}$		

C. STGNN Multi-Step Time Series Forecasting

From now on, we will discuss STGN multi-step time series forecasting on subsidized diesel consumption of some petrol stations. several key metrics are employed to assess the accuracy and effectiveness of the model's predictions. They are as follows.

1) Evaluation Metrics: Root Mean Squared Error (RMSE) is a commonly utilized metric that calculates the average size of the errors between the predicted values and the actual ones, offering an indicator of the model's accuracy. Mean Absolute Error (MAE) complements RMSE by evaluating the average absolute differences between predictions and true values, offering insights into the model's overall accuracy. Accuracy is a crucial metric in classification, measuring the ratio of accurate predictions to the total made by the model, especially vital for classification tasks. Additionally, the Coefficient of Determination (R^2) evaluates how much variance in the dependent variable can be predicted from the independent variables, with values near 1 suggesting a stronger model fit. Together, these metrics offer a detailed insight into a model's effectiveness, addressing various facets of its predictive accuracy and reliability.

$$
\begin{aligned}
RMSE &= \sqrt{\frac{1}{n}\sum_{i=1}^{n} ||y_i - \hat{y}_i||^2} \\
MAE &= \frac{1}{n}\sum_{i=1}^{n} |y_i - \hat{y}_i| \\
Accuracy &= \frac{TP + TN}{TP + TN + FP + FN} \\
R^2 &= 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}
\end{aligned}
$$

Where y_i and $\hat{y}_i$ denote the actual and forecasted values at time t, TP is True Positives, TN is True Negatives, FP is False Positives, and FN is False Negatives.

2) Hyperparameters: The first hyperparameters related to PyTorch and Torch Geometric versions. It then proceeds to install necessary libraries, including 'torch-geometric', 'torch-geometric-temporal', and 'networkx', ensuring the required dependencies are in place. Five main variables are extracted: weather, sales, number of vehicles, number of tourists, and availability. These variables constitute the time series data used for analysis and forecasting.

Next, we involve building a temporal graph convolution model (RecurrentGCN) using Torch Geometric. This model is designed to capture temporal dependencies in the graph structure. Furthermore, the model is trained for a certain number of epochs, and the mean square error is used as the loss function.

The Temporal Graph Convolutional Network (TGCN) model, a key component of the script, operates with specific hyperparameters. The 'node_features' parameter dictates the number of features associated with each node in the graph, offering a comprehensive representation of the characteristics inherent to each entity within the temporal structure. Additionally, the periods parameter determines the number of time steps considered during the application of temporal convolutional layers, influencing the model's capacity to capture temporal dependencies within the graph.

In the context of training, several hyperparameters shape the learning process. The learning rate (lr) is a crucial factor, and in this case, it is set to 0.01 within the optimizer, affecting the magnitude of adjustments made to the model's parameters during training. The epochs parameter denotes the number of training cycles the model undergoes, with the script employing 200 epochs for robust learning. The step_ahead parameter specifies the number of steps the model foresees into the future, facilitating forecasting capabilities. Additionally, the lags parameter indicates the number of lagged time steps utilized as features, contributing to the model's understanding of temporal patterns. Lastly, the train_ratio parameter determines the proportion of the dataset allocated for training, with a value of 0.6 ensuring an appropriate balance between training and testing data. Collectively, these hyperparameters orchestrate the behavior and performance of the script's temporal graph convolutional model.

3) Some Models to Compare: The first model that we use for the comparison of a number of models is the historical average (HA). HA refers to a forecasting method that relies on the historical average of a time series as a prediction for future values. It is a simple and intuitive approach where the forecast for a specific time point is based on the average of past observations up to that point.

The ARIMA model, short for AutoRegressive Integrated Moving Average, is a prevalent model for forecasting in time series analysis, blending autoregressive and moving average elements. This model requires differencing the data to achieve stationarity, followed by examining how an observation correlates with its previous values. ARIMA excels in identifying trends and seasonal patterns in time series datasets.

The third model in our lineup is SVR, or Support Vector Regression, a machine learning approach designed for regression challenges. It utilizes support vector machines to identify an optimal hyperplane that closely aligns with the data, aiming to reduce the error margin. SVR shines in scenarios involving non-linear relationships and is adept at handling time series forecasting tasks.

The fourth model that we use is the GCN. GCN, or Graph Convolutional Network, is a specialized neural network variant tailored for analyzing data structured in graph form. It leverages the graph structure to perform convolutional operations on the nodes of the graph. GCNs are commonly used in tasks involving graph data, such as social network analysis or spatial-temporal data analysis.

The fifth model we use is the GRU (Gated Recurrent Unit). GRU, standing for Gated Recurrent Unit, is a recurrent neural network (RNN) architecture crafted to recognize and retain long-term dependencies within sequential data. It includes gating mechanisms to control the flow of information, addressing some of the challenges faced by traditional RNNs in learning and remembering patterns over longer sequences.

The last model is the STGNN (Spatial Temporal Graph Neural Network). STGNN is a specialized type of graph neural network designed for spatio-temporal data. It combines the concepts of GCN with the ability to model temporal dependencies in sequential data. STGNNs are commonly applied in traffic forecasting tasks, where spatial and temporal patterns are crucial for accurate predictions.

Prior to run the programming, we start with the following observations.

Observation 1: Consider a graph G with an order of n. The vertex set is defined as $V(G) = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$, and the edge set is $E(G) = \{v_i v_j | v_i, v_j \in V(G)\}$, respectively. The features of the vertices are as follows $h_{v_i} = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,m} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,m} \end{bmatrix}$. The vertex embedding can be determined using the messages passing from vertex v's neighbors $h_v^{l+1} = AGG\{m_u^{l+1}, u \in N(v)\}$ under the aggregation $\mathbf{sum}(\cdot)$, where $l = 0, 1, 2, 3, \ldots, k$. Thus $h_v^{l+1} = SUM\{m_u^{l+1}, u \in N(v)\}$ in regards to the matrix $B = A + I$ where A, I are adjacency and identity matrix, respectively.

Proof. With graph G, we can derive the adjacency matrix A. To account for self-adjacency for each vertex in G, we add the identity matrix I to A, resulting in matrix B as follows.

$$B = A + I = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix}$$

According to the single layer GNNN algorithm, we need to initialize the learning weight matrix as follow.

$$W = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,m} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,m} \end{bmatrix}$$

This weight is utilized to determine the value of m_{v_i} and to update the weight for the upcoming iteration. The vertex embedding process in GNN is segmented into two phases: message passing and aggregation. During the initial phase, we perform message passing with $\mathbf{m_u} = MSG(h_u)$. For linear layer we have $\mathbf{m_u}^{l+1} = W^l \cdot h_u^l$, where $l = 0, 1, 2, \ldots k$. We can iteratively start the calculation as follows:

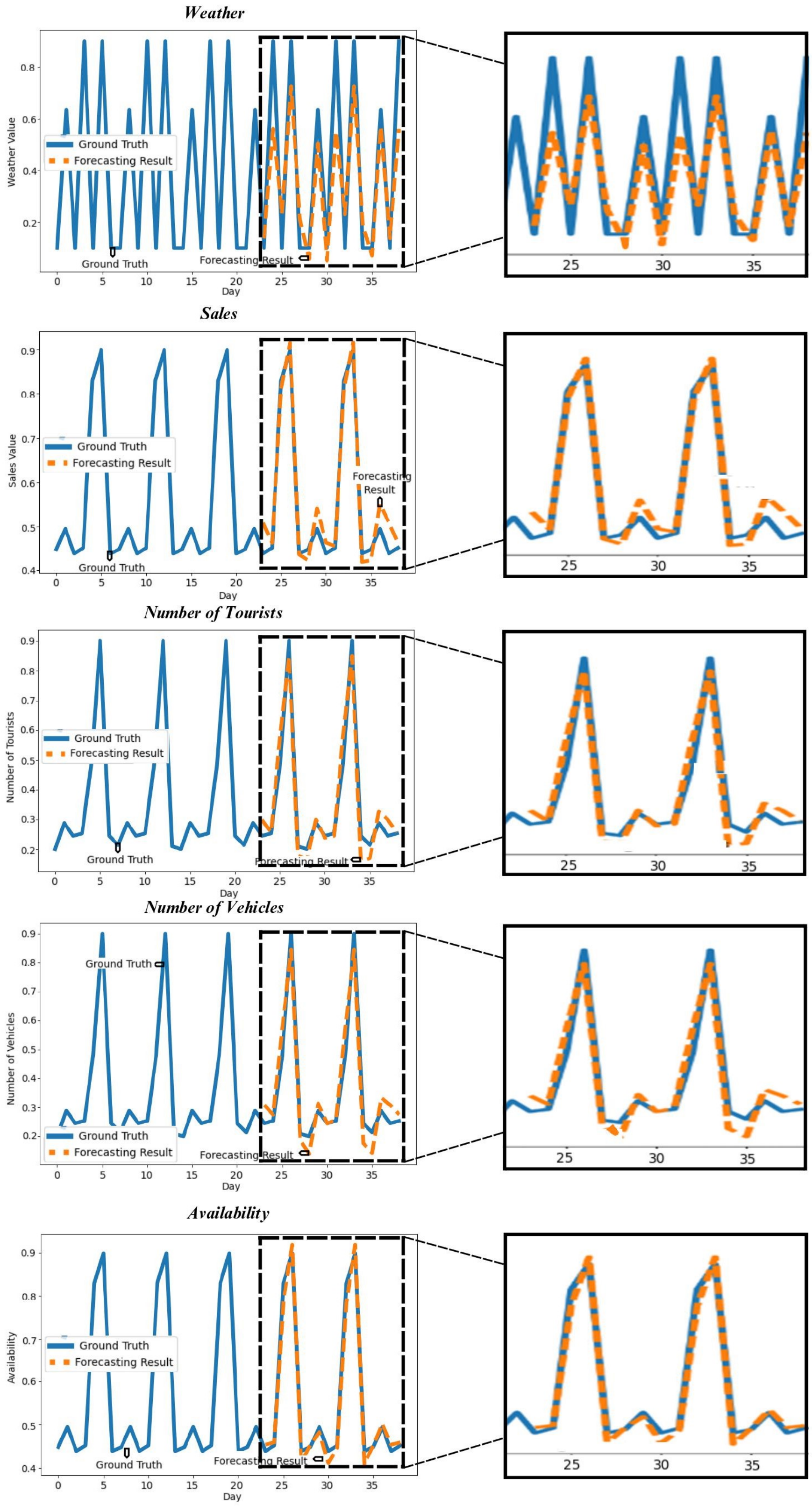

Fig. 9. The STGNN time series testing results on Petrol Station 54.602.69

TABLE VI
THE PREDICTION RESULTS OF THE STGNN MODEL AND OTHER BASELINE METHODS ON DATASET.

T	Matric	Dataset of Eight Petrol Stations					
		HA	ARIMA	SVR	GCN	GRU	STGNN
5 Days	RMSE	7.9067	8.1040	7.5346	9.7695	4.0471	3.9041
	MAE	5.4847	6.1081	4.9187	7.2654	2.6792	2.6050
	Accuracy	0.6595	0.3067	0.6860	0.6411	0.7057	0.8195
	R^2	0.7813	0.0721	0.8091	0.6026	0.8426	0.8721
10 Days	RMSE	7.9076	8.2101	7.4725	9.3429	4.0757	3.7695
	MAE	5.4858	6.2143	4.9719	7.3101	2.7007	2.5441
	Accuracy	0.6706	0.4271	0.6976	0.6305	0.7147	0.7565
	R^2	0.7813	0.0713	0.8120	0.6075	0.8356	0.8402
15 Days	RMSE	7.9076	8.2111	7.4742	9.4011	4.0901	3.9539
	MAE	5.4847	6.2132	5.0310	7.3693	2.7096	2.6544
	Accuracy	0.6695	0.4259	0.6964	0.6362	0.7021	0.7431
	R^2	0.7813	0.0815	0.8120	0.6016	0.8339	0.8498
20 Days	RMSE	7.9076	8.2031	7.4861	9.4393	4.2120	4.0020
	MAE	5.4847	6.2097	5.0593	7.4109	2.7310	2.5867
	Accuracy	0.6695	0.4261	0.6960	0.6244	0.7004	0.7218
	R^2	0.7813	0.0803	0.8133	0.5877	0.8421	0.8792

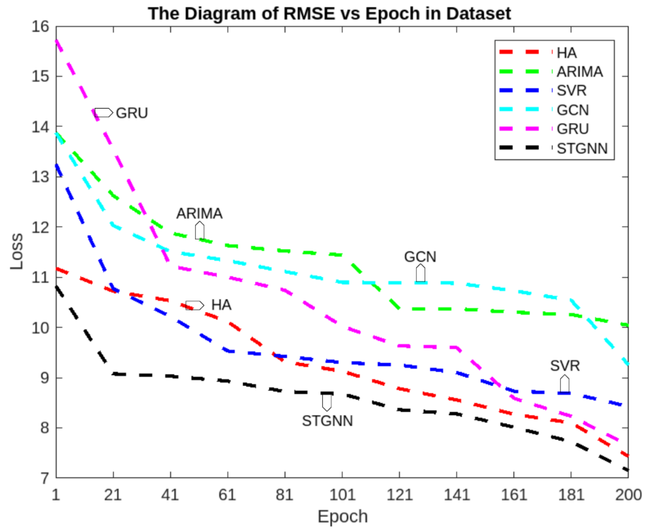

Fig. 10. Comparison of predicted performance

$$m_{v_i}^1 = H_{v_i}^0 \cdot W^0 = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,m} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,m} \end{bmatrix} \times \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,m} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m,1} & w_{m,2} & \cdots & w_{m,m} \end{bmatrix}$$

After completing the initial step, we move on to the second phase: aggregation concerning the neighbors of v. Through the aggregation operation $\mathbf{sum}(\cdot)$, we calculate $h_v^{l+1} = AGGm_u^{l+1}, u \in N(v)$, resulting in $h_v^{l+1} = |SUMm_u^{l+1}, u \in N(v)|$. With reference to the matrix $B = A + I$, the embedding vector $h_{v_i}^1$ is expressed as follows.

$$h_{v_i}^{l+1} = \begin{bmatrix} m_{v_{1,1}}^{l+1} & m_{v_{1,2}}^{l+1} & \cdots & m_{v_{1,m}}^{l+1} \\ m_{v_{2,1}}^{l+1} & m_{v_{2,2}}^{l+1} & \cdots & m_{v_{2,m}}^{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{v_{n,1}}^{l+1} & m_{v_{n,2}}^{l+1} & \cdots & m_{v_{n,m}}^{l+1} \end{bmatrix}$$

Subsequently, it's essential to calculate the error value, which reflects the proximity of two adjacent vertices in the embedding space. A lower error value signifies a shorter distance between the vertices. The error is defined as follows: $error^l = \frac{||h_{v_i} - h_{v_j}||_{inf}}{|E(G)|}$ where $i, j \in \{1, 2, ..., n\}$. We need to check whether $error \leq \epsilon$. If no, we need to update new W^l using the obtained $h_{v_i}^l$ in the previous iteration. We update the learning weight matrix by using $W^{l+1} = W^l - \alpha \times error^l$ until $error \leq \epsilon$. □

Applying the **Single layer GNN Algorithm** above, we can develop and run the programming to analyse the subsidized diesel consumption of some petrol stations across Surabaya city, East Java, Indonesia. First, we collected some data from the petrol stations regarding to five features, namely weather, sales, number of vehicles, number of buyer, and diesel supply within 42 days observations. We developed the STGNN programming to train 60% input data, test and finally forecast the subsidized diesel consumption of eight petrol stations for several times ahead.

In regards with the algorithm above, first we need to deal with graph embedding along with eight petrol stations. We assume that each node will affect the other nodes whenever they are adjacent. Thus, we need the message passing process and consider the adjacency matrix of eight petrol stations. Later we develop STGNN multi-step time series forecasting to train 60% data and obtain the smallest Root Mean Square

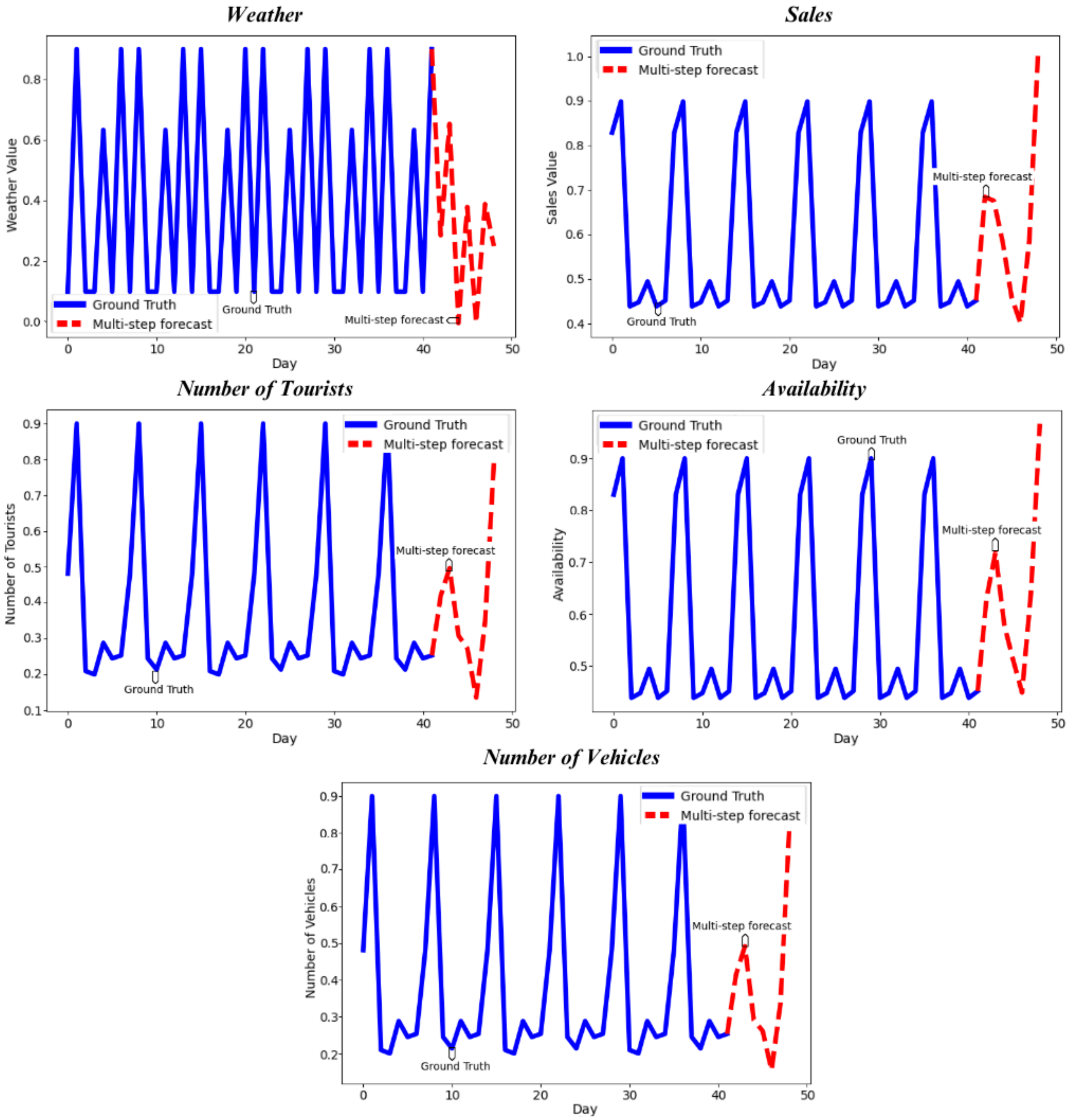

Fig. 11. The results of multi-step time series forecasting of petrol station 54.602.69.

Error (RMSE) or Mean Square Error (MSE) of the testing data. Figure 9 shows the results of testing process on Petrol Station x_1.

To convince the robustness of STGNN model, We compared six models, namely Historical Average (HA), Auto Regressive Integrated Moving Average (ARIMA), Support Vector Regression (SVR), Graph Convolutions Networks (GCN), Gated Recurrent Unit (GRU), Spatio Temporal Graph Neural Networks (STGNN). The comparison results between the six models can be seen in Figure 10 and Table VI. Figure 10 show that STGNN need 200 epochs to tend to the smallest error and the convergence history of loss decreasing steadily. The graphic does not show a big oscillation compared with other models. The STGNN model also shows that time series forecasting for 5, 10, 15, 20 days ahead, either RMSE, MAE, Accuracy and R^2 show the best values compare with other models. Thus, we conclude that we can use this models for doing forecasting and monitoring of eight petrol stations. By using RVAC concept, the number of admins to do monitoring process are only five admins, namely admin 3, 4, 11, 15, 21. Figure 11 shows the results of multi-step time series forecasting of petrol station x_1.

IV. Concluding Remarks

We have studied the rainbow vertex antimagic coloring of shackle of bull-like graph, bull graph, and net graph. Whe have obtained the best exact values of those $rvac(G)$. However, finding the rainbow vertex antimagic chromatic number is not easy task, even it is consider to be a NP-hard problem if the order of graph is unbounded. Thus, we propose the following open problems. (i) Find the exact values of $rvac$ on any other graph operations; (ii) Characterize the existence of rainbow vertex antimagic coloring of any graph having a specific properties, (iii) Apply the obtained theorem into STGNN multi-step time series forecasting for specific real-life input data.

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