Analyzing Newton's Method for Solving Algebraic Equations with Complex Variables: Theory and Computational Analysis

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Abstract—**In this work, we give a thorough examination of Newton's technique. We show that certain places outperform others in terms of where a good initial approximation may be made to assure convergence. Furthermore, to assure quicker and better convergence, certain criteria must be imposed on the function, such as dealing with additional terms from the Taylor series to achieve a technique comparable to Newton's method, but with a degree of convergence greater than two. We compare the use of Newton's technique for solving equations with a single variable to the solution of equations with many variables. While we widen our discussion to include the solution to complex-valued functions, our primary focus is on locating the roots of unity. Some new theories have been proven which is an addition to this topic, and their results are shown in the examples at the end of the paper. We investigate the incorrect choice of the starting approximation for the nth root of unity in the complex plane. When utilizing Newton's technique on a complex plane, we employ various stunning fractal graphs to explain the features and behavior of the roots of interest.**

Index Terms—**Approximate solutions, Newton's method, Complex roots, Fractals.**

I. Introduction

BECAUSE of its usefulness in many mathematical applications, solving equations, whether linear or nonlinear, is recognized as one of the fundamentals of mathematics. One of the simplest is solving nonlinear equations with a single variable. Although solving nonlinear equations might be tough, it is not difficult to acquire by grasping the geometric meaning of it. The same is true for one-variable nonlinear polynomials. However, analytical solutions are not always possible in this case, especially when the degree of the equation exceeds five. Furthermore, there are so many nonlinear equations in a multi-dimensional variable that finding solutions is challenging, if not impossible. It is worth mentioning that for situations for which we have answers, they provide complicated and lengthy computations, which are not wanted by the readers. As science progresses and the hunt for mathematical models for more complicated issues intensifies, we end up with equations that are frequently non-linear and difficult to solve. To handle such issues, we frequently employ iterative numerical algorithms that rapidly converge to an accurate solution. We describe various well-known iterative methods for solving nonlinear equations in this work, such as Fixed-Point Iteration and Newton's approach. In fact, we primarily focus on Newton's approach and have made certain improvements to several of its applications. Newton's approach is popular because it has a quicker rate of convergence than other classic iterative procedures, as long as the function is properly behaved towards the root and the initial guess is well established. Newton's approach, depending on the beginning point, can be convergent or divergent, and it is indeterminate for critical points (i.e. points that make the first derivative zero). The approach detects stable and unstable cyclic values and is sensitive to the beginning point. As a result, Newton's dynamic behavior technique is highly rich, see for example [1], and [2]. The literature contains several adjustments and improvements to Newton's technique [3]–[6]. We'd like to highlight some of the most recent discoveries in the use of Newton's approach. We show that certain places outperform others in terms of where a good initial approximation may be made to assure convergence. Furthermore, we highlight a number of earlier investigations [7]–[11], that investigated the use of Newton's technique and its rate of convergence. In this study, we look at both the pace at which the approach converges and the asymptotic error. More Taylor series terms will be considered to obtain a variation of Newton's technique with a degree of convergence greater than 2. Newton's approach is recognized to be successful in higher dimensions for solving nonlinear systems of equations [12], It has also been employed in earlier investigations [13]–[15], and numerous nice and intriguing outcomes were obtained. The similarities between single-variable and multi-variable difficulties are a major focus of this research. Newton's technique is known to have a generalization in more than one dimension, where we may solve a system consisting of two or more equations by substituting the first derivative with a Jacobian. Because complex roots are known to exist in pairs, we may consider generalizing Newton's one-dimensional approach to obtain zeros for algebraic equations in a complex system. More than a century ago, the English mathematician Arthur Cayley examined this subject for the first time [16], and this issue is currently being researched, e.g, in [17] and [18]. A detailed discussion of the relaxed Newton's approach may be found in [19], using relaxed Newton's, we can see various basins of attraction for the solution of a cubic problem. [20] investigated the ensuing dynamics when Newton's approach is applied to an exponential function. It was also demonstrated that the basins of attraction of roots have a finite area when Newton's technique is employed to compute the complex plane product of a polynomial and an exponential function. In [21], the dynamical systems of trigonometric functions

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are investigated, with an emphasis on $tan(z)$ and the fractal picture formed by iterating the Newton map, $f_t(z)$ of $tan(z)$. [3] discusses the global dynamics of Newton's approach to discovering the roots of a polynomial $p(z)$ in one variable. The behavior of Newton iteration for cubic polynomials is described by the results of [22], [23]. In [24], Newton's approach produced an intriguing and unexpected result when used to a simple polynomial $z^3 - 1 = 0$ in the complex plane. The investigation was carried out numerically and formally to determine how a basic Newton's method damping algorithm alters the gravitational basins contained in complex polynomials. Recently, a new iterative method is presented of fifth-order for solving non-linear equations [25]. For the significance of the denominator in Newton's method, Wu [26] suggested a second-order converging method for solving $f(x) = 0$ in weak conditions. In [27], authors demonstrate that the secant method with Aitken Extrapolation outperforms the Newton-Raphson method in terms of accuracy and convergence rate. A modified regularized Newton method for minimizing a convex function was introduced in [28]. A novel iterative method for solving nonlinear algebraic equations presented in [29].

In this paper, we will look at Newton's technique for functions defined in the complex plane, with a focus on finding roots for $f(z) = z^n - 1$. To discover a solution to $f(z) = 0$ for some complex number z as $z = x + iy$, we will be able to define $f(z) = 0$ as $F(x_1, x_2) = 0$, where the real and imaginary portions of $f(z) = 0$ are $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ in real, then, in two dimensions, we employ Newton's approach. We shall investigate some of the root qualities, such as symmetry. On the other hand, what we will do later is separate the real from the imaginary parts of the equation expressed in complex space, build a system of nonlinear equations, and then solve the resultant system using Newton's technique. Let us consider the complex valued functions: $f(z) = z^n - 1 = 0$, where z is the combination of two real numbers, written as $z = x + iy$, so we get, $f(x+iy) = (x+iy)^n - 1 = 0$. This leads to the development of a system of algebraic equations, in which we utilize Newton's technique in two dimensions after simplification. We demonstrate that while both strategies yield the same amount of iterations, one way has benefits over the other. The zone of attraction for Newton's technique shows amazing fractal behavior in the complex plane. Because of their attractive symmetric nature, fractal pictures created by Newton's approach will be studied. Furthermore, we discuss certain odd behavior of Newton's iterations at specific locations. Our purpose is to explain ideas through the use of images and tables created with Mathematica.

II. Improvements to Newton's Method

Newton's approach is widely known to be dependent on the initial approximation x_0. In certain circumstances, we may begin with an estimate and the procedure may not lead to the precise root. The following is an example of a graph that fails to converge:

$$y = \tan^{-1}(x).$$

This is because this function has a single root at $x = 0$, where $f'(0) = 1$, and as we travel away from both sides of the root, the slope reduces and the tangent line does not lead to zero. To demonstrate, if we start with $x_0 = 1.5$, the tangent line will pass through the $x-$axis, yielding our new approximation $x_1 = -1.69404$. We observe that the distance between the new approximation x_1 and the real zero will be bigger than the distance between the actual zero and the first approximation x_0. Also, if we start with $x_1 = -1.69404$, we get the new approximation $x_2 = 2.32113$ and note that the actual zero distance from the new approximation x_2 will be greater than the distance from the current value x_1. For $f(x) = tan^{-1}(x)$, assume that the region $R = [-x_c, x_c]$, we consider three cases,

- If we choose the initial approximation $x_0 = x_c$, then Newton's method will produce the cycle $x_1 = -x_c$, $x_2 = x_c$, $x_3 = -x_c$, ...
- If we choose the initial approximation x_0, such that $|x_0| < x_c$, then Newton's method converges to the root $x = 0$.
- If we choose the initial approximation x_0, such that $|x_0| > x_c$, then Newton's method diverges.

Now, we analyze the iteration function from equation (14) to find the point x_c when $f(x) = tan^{-1}(x)$. Suppose $x_0 = x_c$, then $x_1 = -x_c$, so,

$$-x_c = x_c - \frac{f(x_c)}{f'(x_c)},$$

hence,

$$2x_c = \frac{\tan^{-1}(x_c)}{\frac{1}{1+(x_c)2}},$$

therefor,

$$(1 + (x_c)^2)tan^{-1}(x_c) - 2x_c = 0.$$

When we use Mathematica to solve for x_c, we obtain $x_c = 1.39174$. As a result, if we start with $x_c = 1.39174$, Newton's approach will provide approximations x_c and $-x_c$ in an alternating pattern. The approach will converge for $|x_0| < 1.39174$ and diverge for $|x_0| > 1.39174$ for any starting approximation x_0. In the following theory, we will show that it can locate an area around the root, also known as a region in the basin of attraction. (1), where we can choose any initial approximation and Newton's technique will ensure convergence for the roots.

Definition 1: (**Basin of Attraction**) If r is a root of $f(x)$, the basin of attraction of r, is the set of all numbers x_0, such that iterative method's starting at x_0 converges to r.

Theorem 1: Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable in open interval $I \in \mathbb{R}$. Assume that there exists $x_* \in \mathbb{R}$, and $r, \beta > 0$, such that $[x_* - r, x_* + r] \subseteq I$, $f(x_*) = 0$, $f'(x_*)$ exists with $|\frac{1}{f'(x_*)}| \leq \beta$, and $f' \in Lip_\gamma([x_* - r, x_* + r])$. Then, there exists $\epsilon > 0$, such that for any initial approximation $x_0 \in [x_* - r, x_* + r]$ the sequence x_1, x_2, x_3,... generated by,

$$x_k = x_{k-1} - \frac{f(x_{k-1)}}{f'(x_{k-1})}, \quad k = 1, 2, 3, ...$$

is well defined, converges to x_*, and obeys,

$$|x_k - x_*| \leq \beta \cdot \gamma \cdot |x_{k-1} - x_*|^2, \quad k = 1, 2, 3, ... \qquad (1)$$

Proof: We choose $\epsilon > 0$ so that $f'(x) \neq 0$, for any $x \in (x_* - \epsilon, x_* + \epsilon)$, and then show that, since the local error in the affine model [30] used to produce each iterate of

Newton's method is at most $O(|x_k - x_*|^2)$, the convergence is quadratic. Let

$$\epsilon = min\{r, \frac{1}{2\beta\gamma}\}. \quad (2)$$

We show by induction on k that at each step (1) holds, and also that,

$$|x_k - x_*| \leq \frac{1}{2}|x_{k-1} - x_*|,$$

and so,

$$x_k \in (x_* - \epsilon, x_* + \epsilon). \quad (3)$$

Now, we first show that $f'(x_0) \neq 0$. From $|x_k - x_*| \leq \epsilon$, the Lipschitz continuity of f' at x_*, and by (2), it follows that,

$$|\frac{f'(x_0) - f'(x_*)}{f'(x_*)}| \leq |\frac{1}{f'(x_*)}||f'(x_0) - f'(x_*)|$$

$$\leq \beta \cdot \gamma \cdot |x_0 - x_*| \leq \beta \cdot \gamma \cdot \epsilon \leq \frac{1}{2}.$$

Thus, by the perturbation relation (3.1.20) [30], $f'(x_0) \neq 0$, and,

$$|\frac{1}{f'(x_0)}| \leq \frac{|\frac{1}{f'(x_*)}|}{1 - |\frac{1}{f'(x_*)}[f'(x_0) - f'(x_*]|}$$

$$\leq 2 \cdot |\frac{1}{f'(x_*)}| \leq 2 \cdot \beta. \quad (4)$$

Therefore, x_1 is well defined, and,

$$x_1 - x_* = x_0 - x_* - \frac{f(x_0)}{f'(x_0)}$$

$$= \frac{1}{f'(x_0)}[f(x_*) - f(x_0) - f'(x_0)(x_* - x_0)].$$

Notice that the term in brackets is just the difference between $f(x_*)$ and the affine model $M_c(x)$ evaluated at x_*. Therefore, using lemma (4.1.12) in [30] and (4), we get,

$$|x_1 - x_*| \leq |\frac{1}{f'(x_0)}| \cdot |[f(x_*) - f(x_0) - f'(x_0)(x_* - x_0)]|$$

$$\leq 2 \cdot \beta \cdot \frac{\gamma}{2} \cdot |x_0 - x_*|^2 = \beta \cdot \gamma \cdot |x_0 - x_*|^2.$$

This proves (1), and since $|x_0 - x_*| \leq \frac{1}{2 \cdot \beta \cdot \gamma}$, so, $|x_1 - x_*| \leq \frac{1}{2}|x_0 - x_*|$. Which shows (3), and completes the case $k = 0$. The proof of the induction step proceeds identically. ■

Now, for $y = tan^{-1}(x)$, the interval calculated by the theorem (1), is $[-0.927664, 0.927664]$. This means that, if we start by any initial approximation $x_0 \in [-0.927664, 0.927664]$, the equation (1) is satisfied, and the sequence generated by Newton's method will be converging to the root "0".

Remark 1: We previously showed that the basin of attraction for $tan^{-1}(x)$ is $(-1.39174, 1.39174)$. But if we apply the theorem (1), the number of iterations of Newton's method to reach the root is less.

III. CUBIC NEWTON'S METHOD

Newton's method has a maximum quadratic convergence rate. In this section, we'll look at how employing more terms than Taylor's series accelerates the Newton method's convergence. Suppose that $f \in C^3[a, b]$. Let $x_0 \in [a, b]$ be closed to the root r. Consider the 2^{nd} Taylor's polynomial for $f(x)$ about x_0,

$$f(x) \approx f(x_0) + (x - x_0) \cdot f'(x_0) + \frac{(x - x_0)^2}{2!} \cdot f''(x_0) + \frac{(x - x_0)^3}{3!} \cdot f'''(\xi(x)) \quad (5)$$

where $\xi(x)$ lies between x and x_0. Now, since $f(r) = 0$, equation (5) becomes after simplification,

$$f(x_0) + r \cdot f'(x_0) - x_0 \cdot f'(x_0) + (\frac{r^2}{2} - r \cdot x_0 + \frac{x_0^2}{2}) \cdot f''(x_0) \approx 0. \quad (6)$$

Solving for r, we get,

$$r \approx x_0 - \frac{f'(x_0)}{f''(x_0)} \pm \frac{\sqrt{(f'(x_0))^2 - 2 \cdot f(x_0) \cdot f''(x_0)}}{f''(x_0)} \quad (7)$$

This establishes a parallel with Newton's approach, which begins with an initial approximation x_0 and for $n \geq 1$ creates the series x_n that is equal to:

$$x_{n-1} - \frac{f'(x_{n-1})}{f''(x_{n-1})} \pm \frac{\sqrt{(f'(x_{n-1}))^2 - 2 \cdot f(x_{n-1}) \cdot f''(x_{n-1})}}{f''(x_{n-1})}, \quad (8)$$

provided that $f''(x_{n-1}) \neq 0$, and $(f'(x_{n-1}))^2 \geq 2 \cdot f(x_{n-1}) \cdot f''(x_{n-1})$. The next theory (2), will discuss how to increase the order of convergence for Newton's method.

Theorem 2: Suppose $f \in C^3[a, b]$. If $r \in [a, b]$, such that $f(r) = 0$, $f''(r) \neq 0$, $(f'(r))^2 \geq 2 \cdot f(r) \cdot f''(r)$, and $f'(r) \geq 0$. Then, for starting the initial approximation close to r, Newton's method for $n \geq 1$ generates an x_n given by,

$$x_{n-1} - \frac{f'(x_{n-1})}{f''(x_{n-1})} + \frac{\sqrt{(f'(x_{n-1}))^2 - 2 \cdot f(x_{n-1}) \cdot f''(x_{n-1})}}{f''(x_{n-1})}, \quad (9)$$

will Converge to root "r", with an order of convergence 3.

Proof: The proof is based on analyzing Newton's method as the functional iteration scheme $x_n = g(x_{n-1})$, for $n \geq 1$, with,

$$g(x) = x - \frac{f'(x)}{f''(x)} + \frac{\sqrt{(f'(x))^2 - 2 \cdot f(x) \cdot f''(x)}}{f''(x)}. \quad (10)$$

Assume $f(r) = 0$, $f'(r) \geq 0$, $f''(r) \neq 0$, and $(f'(r))^2 \geq 2f(r)f''(r)$.

Claim: $g'(r) = 0$ and $g''(r) = 0$.

Proof of claim. If we calculate $g'(r)$, $g''(r)$, and substitute $x = r$, we get,

$$g'(r) = \frac{f^{(3)}(r)f'(r)}{f''(r)^2} - \frac{f^{(3)}(r)\sqrt{f'(r)^2}}{f''(r)^2}.$$

Easy calculation we can calculate $g''(r)$, and by the given $f'(r) \geq 0$, we have $g'(r) = g''(r) = 0$, hence, the proof of claim is done. Now, if $g(x_n)$ around r, and $min(x_n, r) < \xi_n < max(x_n, r)$, then,

$$x_n - r = g(x_{n-1}) - g(r) = g'(r)(x_{n-1} - r) + g''(r)\frac{(x_{n-1} - r)^2}{2!} + g'''(\xi_{n-1})\frac{(x_{n-1} - r)^3}{3!}, \quad (11)$$

and since, $g'(r) = 0$, $g''(r) = 0$, for $x_{n-1} \neq r$, we have,

$$\frac{(x_n - r)}{(x_{n-1} - r)^3} = \frac{g'''(\xi_{n-1})}{3!}.$$

Hence, by theorem (2.3) in [31], the iterative value x_n when $n \geq 1$ is given by,

$$x_{n-1} - \frac{f'(x_{n-1})}{f''(x_{n-1})} + \frac{\sqrt{(f'(x_{n-1}))^2 - 2 \cdot f(x_{n-1}) \cdot f''(x_{n-1})}}{f''(x_{n-1})}, \quad (12)$$

which is order of convergence 3, if $g'''(r) \neq 0$. ■

A. Numerical Example

We offer an example with analytical answers to demonstrate the efficiency and rate of convergence of the cubic Newton's technique, which we compare to the classic Newton's method's rate of convergence. We apply our numerical iterative methods with tolerance 5×10^{-4} and maximum 20 iterations.

Example.1. Consider the function

$$f(x) = x^3 - 2x - 1. \quad (13)$$

The root of this function is $x = -0.618034$. We obtain the following tables by using the original Newton's technique and the cubic Newton's method on $f(x) = x^3 - 2x - 1$, with an initial approximation of $x_0 = 0.35$, to achieve the root $x = -0.618034$, and calculating the asymptotic error constant in both ways.

TABLE I
RATE OF CONVERGENCE FOR ORIGINAL NEWTON'S METHOD IN THE CALCULATION FOR ROOTS OF $f(x) = x^3 - 2x - 1$.

Original Newton's Method			
n	x_n	$f(x_n)$	asymptotic error constant "s"
0	0.35	-1.65713	
1	-0.665084	0.0359771	0.725192
2	-0.611626	-0.00554931	2.0123
3	-0.617948	-0.0000735944	1.99831
4	-0.618034	$-1.37543 * 10^{-8}$	2.00163
5	-0.618034	$-4.44089 * 10^{-16}$	
6	-0.618034	0	

TABLE II
RATE OF CONVERGENCE FOR CUBIC NEWTON'S METHOD IN THE CALCULATION FOR ROOTS OF $f(x) = x^3 - 2x - 1$.

Cubic Newton's Method			
n	x_n	$f(x_n)$	asymptotic error constant "s"
0	0.35	-1.65713	
1	-0.34996	-0.342941	2.57728
2	-0.600332	-0.0156949	2.98577
3	-0.618028	$-5.54113 * 10^{-6}$	2.95846
4	-0.618034	$-3.33067 * 10^{-16}$	
5	-0.618034	0	

The numerical findings show that the original Newton's technique quadratically converges to the zero $x = -0.618034$, with an approximate asymptotic error constant $s = 2.00163$. The cubic Newton's technique, on the other hand, converges cubically to $x = 0.618034$ with an approximate asymptotic error constant of $s = 2.95846$.

IV. Two-Dimensional Newton's Method

In sections (I), (II) and (III), we reviewed various strategies for solving nonlinear equations. Some of these approaches may also be used to solve nonlinear equation systems. In this part, we will go over some specifics of Newton's approach to solving two-variable systems of equations. We previously investigated Newton's method's iterative formula for solving nonlinear equations with a single variable $f(x) = 0$, which is provided by the following relationship:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \; n \geq 1. \quad (14)$$

The Newton iteration approach may be expressed as follows for the general case of a system of n nonlinear equations with n unknowns:

$$x^{(k)} = x^{(k-1)} - J(x^{(k-1)})^{-1} F(x^{(k-1)}), \; k \geq 1, \quad (15)$$

where

$$x = (x_1, x_2, x_3, ..., x_n)^T,$$

$$F(x) = (f_1(x), f_2(x), f_3(x), \ldots, f_n(x))^T; f_j : \mathbb{R}^n \to \mathbb{R}$$

, and $J(x)$ is **Jacobian matrix** of $F(x)$.

Remark 2: For the n-dimensional Newton's method to converge to the root, an initial guess near the root should be chosen, and the Jacobian should be singular.

Consider the following nonlinear equation system:

$$\begin{aligned} f_1(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ f_3(x_1, x_2, x_3, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, x_3, \ldots, x_n) &= 0 \end{aligned}$$

Now, when we apply the Newton iteration approach to this system in the general case, we obtain,

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ \vdots \\ x_n^{(k-1)} \end{bmatrix} - \quad (16)$$

$$\begin{bmatrix} \frac{\partial f_1(x^{(k-1)})}{\partial x_1} & \cdots & \frac{\partial f_1(x^{(k-1)})}{\partial x_n} \\ \frac{\partial f_2(x^{(k-1)})}{\partial x_1} & \cdots & \frac{\partial f_2(x^{(k-1)})}{\partial x_n} \\ \cdot & \cdot & \cdot \\ \frac{\partial f_n(x^{(k-1)})}{\partial x_1} & \cdots & \frac{\partial f_n(x^{(k-1)})}{\partial x_n} \end{bmatrix}^{-1} \begin{bmatrix} f_1(x^{(k-1)}) \\ f_2(x^{(k-1)}) \\ \cdots \\ f_n(x^{(k-1)}) \end{bmatrix}$$

If we assume $n = 2$, we have a two-dimensional system that necessitates a system of two equations in two variables, each using a function of the kind $f_j(x_1, x_2)$, $j = 1, 2$. Each of the functions represents a three-dimensional surface. And each equation $f_j(x_1, x_2) = 0$, $j = 1, 2$, represents the intersection of the x_1x_2-plane with the surface, which is a curve on the x_1x_2-plane. As a result, the intersection (s) of these curves, which indicate the system's solution, are of importance to us. To solve a 2×2 system with two variables, we begin with an initial approximation $(x_1(0), x_2(0))$ and draw a tangent plane to the surface at $(x_1(0), x_2(0))$, then take the intersection of the tangent plane with the x_1x_2-plane, which gives a line or curve. Now, two tangent planes will generate two lines or curves on the x_1x_2 plane, and their intersection will be the next Newton's method approximation. This is a geometric analysis of a two-dimensional case. For mathematical analysis, let's use the following system of equations,

$$\begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

First, we want to find the tangent planes to the two surfaces $f_1(x_1,x_2)$, $f_2(x_1,x_2)$. We get it from the first Taylor polynomials for a function of two variables around our initial approximation $(x_0^{(1)}, x_0^{(2)})$, which gives us two functions. We express the intersection of these surfaces with the x_1x_2-plane as the following system of linear equations, which represent the two lines on the x_1, x_2-plane:

$$f_1(x_1,x_2) \approx f_1(x_1^{(0)},x_2^{(0)}) + (x_1 - x_1^{(0)})\frac{\partial f_1(x_1^{(0)},x_2^{(0)})}{\partial x_1} + (x_2 - x_2^{(0)})\frac{\partial f_1(x_1^{(0)},x_2^{(0)})}{\partial x_2} = 0$$

$$f_2(x_1,x_2) \approx f_2(x_1^{(0)},x_2^{(0)}) + (x_1 - x_1^{(0)})\frac{\partial f_2(x_1^{(0)},x_2^{(0)})}{\partial x_1} + (x_2 - x_2^{(0)})\frac{\partial f_2(x_1^{(0)},x_2^{(0)})}{\partial x_2} = 0.$$

The intersection of these lines gives us the new approximation point $(x_1^{(1)}x_2^{(1)})$. Now, by solving these two equations using MATHEMATICA with respect to x_1 and x_2, we call x_1 and x_2 by $x_1^{(1)}$ and $x_2^{(1)}$, respectively, and rewrite equations as matrix form, we obtain,

$$\begin{bmatrix} x_1^{(1)} \\ \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ \\ x_2^{(0)} \end{bmatrix} - \begin{bmatrix} \dfrac{f_1 \cdot \frac{\partial f_2}{\partial x_1} - f_2 \cdot \frac{\partial f_1}{\partial x_1}}{\frac{\partial f_2}{\partial x_x}\cdot\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1}\cdot\frac{\partial f_1}{\partial x_2}} \\ \\ \dfrac{f_2 \cdot \frac{\partial f_1}{\partial x_1} - f_1 \cdot \frac{\partial f_2}{\partial x_1}}{\frac{\partial f_2}{\partial x_x}\cdot\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1}\cdot\frac{\partial f_1}{\partial x_2}} \end{bmatrix}, \tag{18}$$

where, each of the $f_1, f_2, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_1}$, and $\frac{\partial f_2}{\partial x_2}$ is evaluated at $(x_1^{(0)}, x_2^{(0)})$. On the other hand, if we take $n = 2$ in equation (16), in this case, we apply Newton's method in 2-dimensions and start with an initial guess $(x_1^{(0)}, x_2^{(0)})$, so that it is close to the root (x_1, x_2), and find the inverse and simplify, we get,

$$\begin{bmatrix} x_1^{(1)} \\ \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ \\ x_2^{(0)} \end{bmatrix} - \begin{bmatrix} \dfrac{f_1 \cdot \frac{\partial f_2}{\partial x_1} - f_2 \cdot \frac{\partial f_1}{\partial x_1}}{\frac{\partial f_2}{\partial x_x}\cdot\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1}\cdot\frac{\partial f_1}{\partial x_2}} \\ \\ \dfrac{f_2 \cdot \frac{\partial f_1}{\partial x_1} - f_1 \cdot \frac{\partial f_2}{\partial x_1}}{\frac{\partial f_2}{\partial x_x}\cdot\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1}\cdot\frac{\partial f_1}{\partial x_2}} \end{bmatrix}, \tag{19}$$

where, each of the $f_1, f_2, \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_1}$, and $\frac{\partial f_2}{\partial x_2}$ is evaluated at $(x_1^{(0)}, x_2^{(0)})$. By comparing the equation of (18) to the equation of (19), we notice that we got the same result, which means that in both (18) and (19) equations we get the same new approximation point $(x_1^{(1)}, x_2^{(1)})$. Also, we note that in terms of how the new approximation is found, it is clear that there are some parallels between the one-dimensional and two-dimensional cases.

V. Complex Newton's Method

In previous sections, we looked at Newton's method for solving equations in one variable or systems of nonlinear equations in two dimensions, and all of the roots we discussed are real. Equations can have complex roots, but we do know that complex roots occur in pairs if the coefficients of a polynomial equation are real, with each pair being a complex conjugate of the other. We shall explore and apply Newton's technique to several difficult plane functions. We'll look at the well-known $z^n - 1 = 0$ problem. Let's first analyze the following problem,

$$f(z) = z^3 - 1. \tag{20}$$

This function has three roots of unity at, $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Now, in order to solve the equation (20), we substitute $z = x + iy$, and we get,

$$(x + iy)^3 - 1 = 0.$$

After simplifying, we obtain,

$$(x^3 - 3xy^2 - 1) + i(3x^2y - y^3) = 0. \tag{21}$$

If we separate the real and imaginary parts of the equation (21) and convert to system 2×2, we get,

$$\begin{bmatrix} x^3 - 3xy^2 - 1 \\ 3x^2y - y^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{22}$$

By looking at the system (22), we note that this system can be solved by the same method as the system (17). On the other hand, if we want to solve the equation (20), we can use Newton's method in the complex plane, which is given by the following,

$$z_n = z_{n-1} - \frac{f(z_{n-1})}{f'(z_{n-1})}, \quad n \geq 1. \tag{23}$$

We aim to see if solving the equation (20) by Newton's method, and solving the system (22) by equations (18), give the same result. Let's begin, if we apply Newton's method on the function $f(z) = z^3 - 1$, when $z_0 = 3$, and $z_0 = i$, we get,

TABLE III
Newton's method in the calculation for roots of $f(z) = z^3 - 1$.

n	$z_n start"3"$	$f(z_n)$	z_n $start$ "i"	$f(z_n)$
0	3	26	i	-1-i
1	2.03704	7.4527	-0.33333+0.66667i	-0.59259-0.07407i
2	1.43836	1.9758	-0.58222+0.92444i	0.2933+0.1501i
3	1.12002	0.7050	-0.5088+0.8682i	0.019+0.020i
4	1.01240	0.0377	-0.5001+0.8660i	0
5	1.00015	0	-0.5+0.866i	0
6	1	0	-0.5+0.866i	0

On the other hand, we want to use equation (18) to solve the system (22). In this case, we need values of $x^{(0)}$ and $y^{(0)}$ until this system is resolved, but we can convert $z_0 = 3$ into $z = (3, 0)$, hence, $x^{(0)} = 3$ and $y^{(0)} = 0$. Similarly, for $z_0 = i$, we obtain $x^{(0)} = 0$, and $y^{(0)} = 1$. In the end, we get the following numerical results,

According to Tables (III) and (IV), these two strategies provide identical estimates at each repetition. Similarly, for any n, these two strategies produce the exact identical approximations to the equation $z^n - 1 = 0$. Newton's approach,

TABLE IV
NUMERICAL RESULTS FROM THE USE OF EQUATION (18) TO SOLVE THE SYSTEM (22).

n	$x^{(n)}$	$y^{(n)}$	z_n	$x^{(n)}$	$y^{(n)}$	z_n
0	3	0		0	1	
1	2.03704	0	(2.03704,0)	-0.33333	0.66667	(-0.3333, 0.6667)
2	1.43836	0	(1.43836,0)	-0.58222	0.92444	(-0.5822, 0.9244)
3	1.12002	0	(1.12002,0)	-0.5088	0.86816	(-0.5088, 0.8681)
4	1.01240	0	(1.01240,0)	-0.5001	0.86598	(-0.5001, 0.8659)
5	1.00015	0	(1.00015,0)	-0.5	0.86602	(-0.5, 0.8660)
6	1	0	(1,0)	-0.5	0.86602	(-0.5, 0.8660)

on the other hand, provides a straightforward manner of computing estimates because it works with a single variable.

Remark 3: When applying Newton's technique to the complex-valued function $f(z) = z^3 - 1$, the numerical results show that if we start with a real initial guess, the iterative converges to the real root of $f(z)$, and if we start with a complex initial guess, the iterative converges to the complex root of $f(z)$.

Since Newton's way of solving the equation (20) is equal to solving the system (22) by equation (18), we infer that the beginning values cause the Jacobian to be singular, and it is the same early approximations that cause Newton's approach to fail. So, for Newton's application to the function $f(z) = z^3 - 1$, the beginning approximations that cause Newton's method to diverge are $z_0 = 0$, and all the initial approximations that, if we start with it, cause Newton's method iterations to go to zero. The number of these starting places is now unlimited. Let's have a look at some of them. Now, if we apply Newton's method on the function $f(z) = z^3 - 1$, we get,

$$z_n = z_{n-1} - \frac{(z_{n-1})^3 - 1}{3(z_{n-1})^2}, \tag{24}$$

simplifying and renaming, we get,

$$l(z) = \frac{2(z)^3 + 1}{3(z)^2}. \tag{25}$$

Now, we put $l(z) = 0$, and solve with respect to z, we get,

$$z = \{-\sqrt[3]{\frac{1}{2}},\ \ \frac{2^{\frac{2}{3}}}{4} - i\frac{2^{\frac{2}{3}}\sqrt{3}}{4},\ \ \frac{2^{\frac{2}{3}}}{4} + i\frac{2^{\frac{2}{3}}\sqrt{3}}{4}\}. \tag{26}$$

These are the values that lead Newton's iterations to go to zero, and if we keep doing this, we get an interesting figure, (see Figure 1).

When we look at Figure 2, we can see that all of the black dots are the points if either is picked as the first approximation point for Newton's iteration approach. Newton's approach will not work. We can observe that the nearby points surrounding each of the roots of unity are unaffected. Choosing a starting estimate close to one of the roots should result in a convergent Newton's technique. In reality, the basins of attraction are the areas where Newton's technique converges for the roots of unity. Let us now display a colorful graphic representing the basins of attraction for all the roots of the equation $z^3 - 1$, where each color indicates the zone of attraction for one of the roots, (see Figure 2).

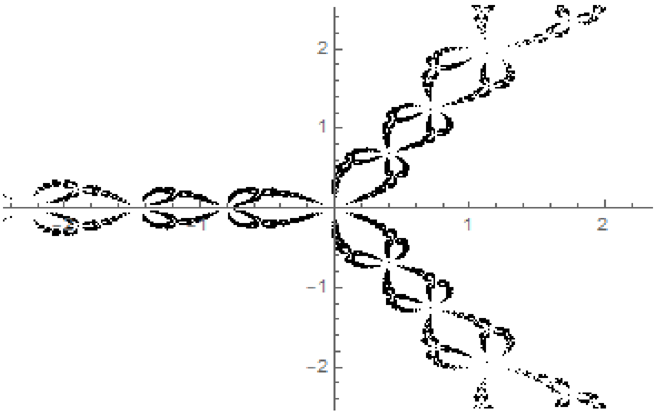

Fig. 1. The fractal image for values that lead Newton's iterations to go to zero.

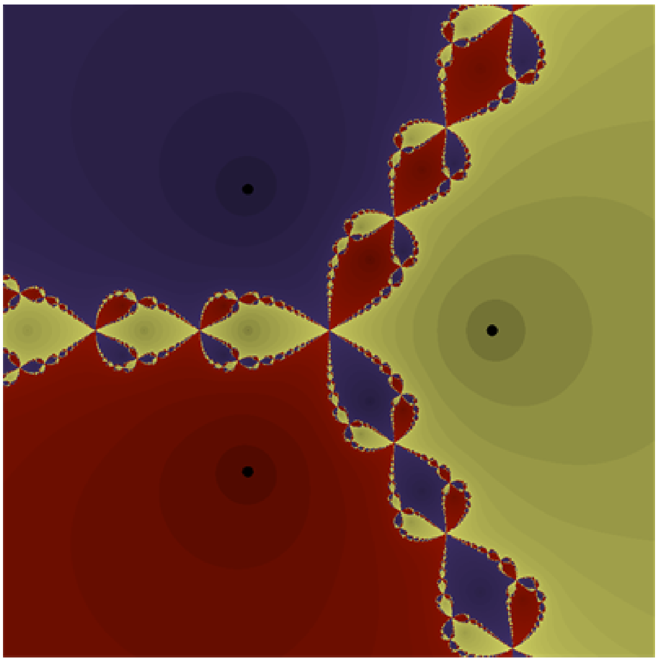

Fig. 2. Newton's basins of attraction for $f(z) = z^3 - 1$ in $\mathbb{C}$.

In this image, cumin represents the basin of attraction for the root of unity 1, burgundy represents the basin of attraction for the root of unity $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$, and dark navy represents the basin of attraction for the root of unity $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. We may identify three major zones divided by lines. Each main area is shown by a distinct hue, indicating that each root of unity has a primary basin of attraction. The fractal behavior occurs around a line that bisects the angle produced by two subsequent lines having roots, which is an intriguing discovery. This is understandable because we can expect the complement of a fractal graph to be fractal. One interesting observation is that the fractal behavior happens near or around a line (we can imagine three axes through origin around/near which all the fractal phenomena are happening and also the pre-images are distributed near and around these axes), which bisects the angle between two consecutive lines that contain roots. Let's name these fictitious axes as axes of pre-images. This phenomenon holds for all values of $n > 2$. In addition to that it seems from Fig. 2 that the axis of pre-images can be found by rotating the root-containing line by π radians. In actuality, the angle between two consecutive lines drawn from the origin to each root of $z^n - 1 = 0$ is $\frac{2\pi}{n}$. To demonstrate the concept, we'd like to provide some photographs of the roots' basin of attraction for the equation $z^n - 1 = 0$. Also, as seen in the diagrams, the angle formed by two successive lines will equal $\frac{2\pi}{n}$. Let's take a theoretical approach to this problem. To get the angle between two successive lines, we may express the equation $f(z) = z^n - 1$ in polar form.

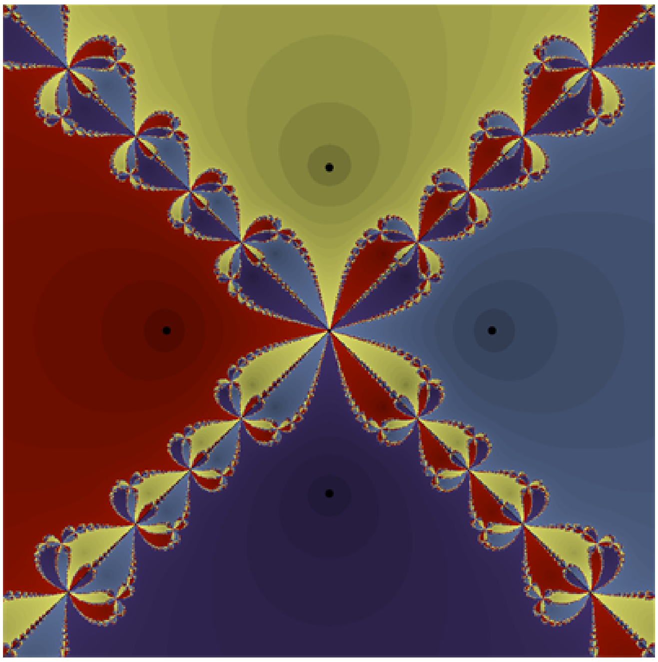

Fig. 3. Newton's basins of attraction for $f(z) = z^4 - 1$ in $\mathbb{C}$.

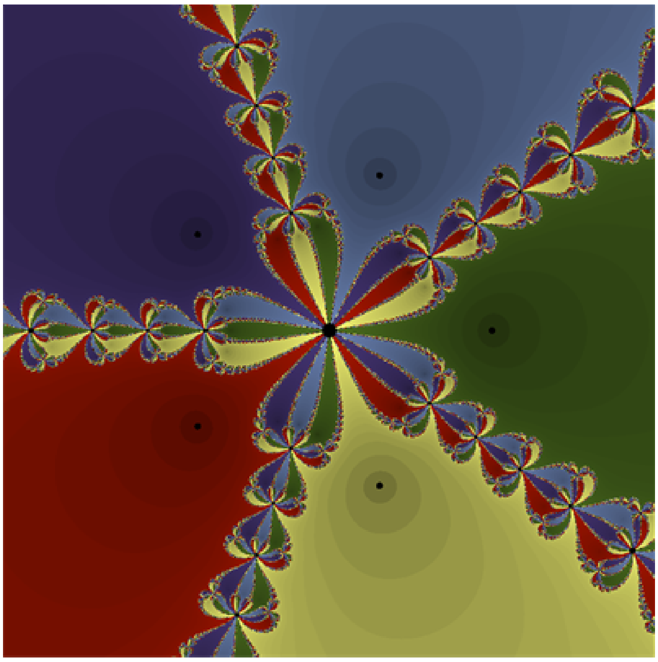

Fig. 4. Newton's basins of attraction for $f(z) = z^5 - 1$ in $\mathbb{C}$.

$$z^n = 1 = \cos(0) + i\sin(0), \tag{27}$$

hence,

$$z^n = \cos(0 + 2\pi k) + i\sin(0 + 2\pi k), \quad k = 0, 2, 4, ... \tag{28}$$

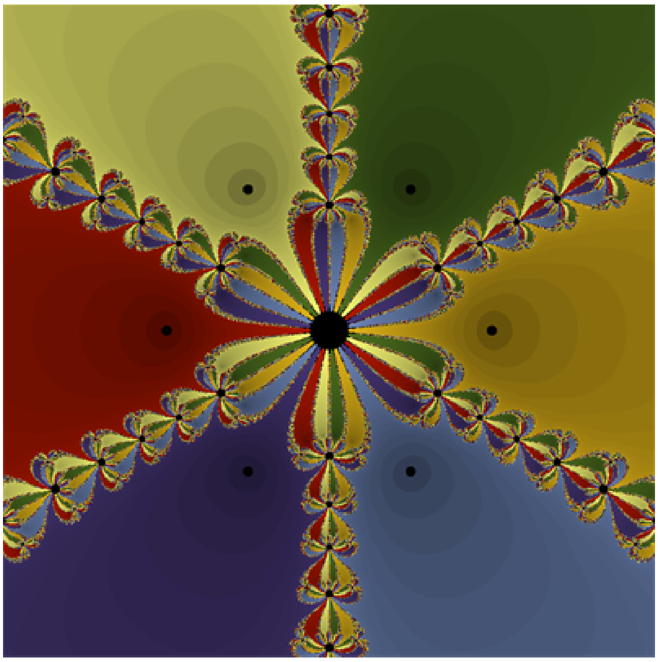

Fig. 5. Newton's basins of attraction for $f(z) = z^6 - 1$ in $\mathbb{C}$.

Taking nth root for equation (28) on both sides, we get,

$$z = (\cos(2\pi k) + i\sin(2\pi k))^{\frac{1}{n}}. \tag{29}$$

Using De Moivre's theorem [32], and Euler's formula, we obtain,

$$z = \cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n}) = e^{\frac{i2\pi k}{n}}, \quad k = 0, 2, 4, ..., 2n. \tag{30}$$

But we also know that can written z as,

$$z = re^{i\theta}. \tag{31}$$

So, by comparing equations (30) and (31), the angle each two consecutive lines is,

$$\theta = \frac{2\pi}{n}. \tag{32}$$

Remark 4: If we use Newton's approach to solve $z^n - 1 = 0$ and choose the beginning estimate on the line that divides the angle between two subsequent roots, all iterations will stay on the same line until they converge to the root, if one exists on the line.

Newton's method iteration function for $z^n - 1 = 0$ may be expressed as,

$$g(z) = z - \frac{z^n - 1}{nz^{n-1}} = \frac{(n-1)z^n + 1}{nz^{n-1}} = \frac{(n-1)z^n + 1}{nz^n} \cdot z. \tag{33}$$

Assume $z = x + iy$ is the initial guess. Now, z can be written as a Euler's form, that is,

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

So, putting $z = re^{i\theta}$ in equation (33), we get,

$$g(z) = \frac{(n-1) \cdot r^n \cdot e^{in\theta} + 1}{n \cdot r^n \cdot e^{in\theta}} \cdot z. \tag{34}$$

Any point on the lines has $r \cdot e^{\frac{ic\pi}{n}}$, for $c = 1, 2, ..., 2n$. So $\theta = \frac{c\pi}{n}$, and hence, the equation (34) becomes,

$$g(z) = \frac{(n-1) \cdot r^n \cdot e^{ic\pi} + 1}{n \cdot r^n \cdot e^{ic\pi}} \cdot z. \tag{35}$$

Now, if c is odd, then $e^{ic\pi} = -1$, and equation (35) becomes,

$$g(z) = \frac{(n-1) \cdot r^n \cdot (-1) + 1}{n \cdot r^n \cdot (-1)} \cdot z = Real\ number\ *\ z. \tag{36}$$

If c is even, then $e^{ic\pi} = 1$, and equation (35) becomes,

$$g(z) = \frac{(n-1) \cdot r^n \cdot (1) + 1}{n \cdot r^n \cdot (1)} \cdot z = Real\ number\ *\ z. \tag{37}$$

From equations (36) and (37), we can conclude that, iterations do not deviate from the line that runs through the origin. From the above, we can say that we have reached the proof of the following theorem.

Theorem 3: Iterations do not move away from the line if we select an initial approximation on any of the axes of preimages of a line that has a root of $z^n - 1 = 0$.

Before we conclude, we'll examine at an intriguing finding made when Newton's technique is applied to the nth roots of unity. We know that Newton's approach has been much improved. Among these enhancements is the following relationship:

$$x_n = x_{n-1} - \frac{kf(x_{n-1})}{f'(x_{n-1})}, \quad n \geq 1, \quad k \in \mathbb{R}. \tag{38}$$

If we set $k = 1$, we get the classic Newton's approach. We wish to know the effect of the number k on the basin of attractions for roots of unity using fractal pictures. We can see from the fractal pictures that the greater the value of k, the smaller the basins of attraction to the roots of unity are. Check out the fractal figures below for confirmation:

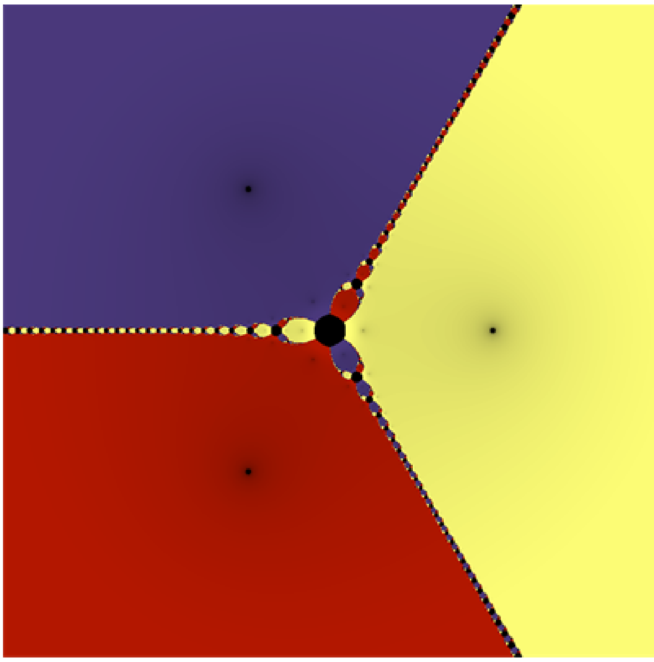

Fig. 6. Newton's basins of attraction were modified for $f(z) = z^3 - 1$ when k=0.1.

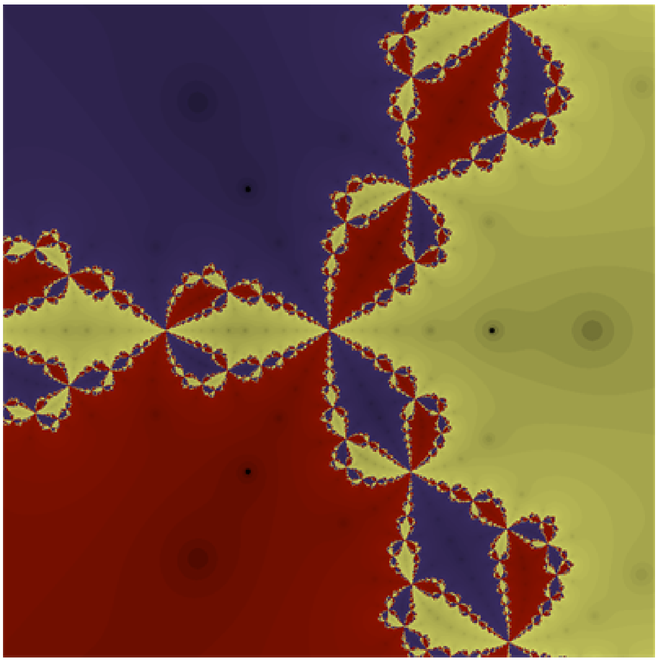

Fig. 7. Newton's basins of attraction were modified for $f(z) = z^3 - 1$ when k=1.5.

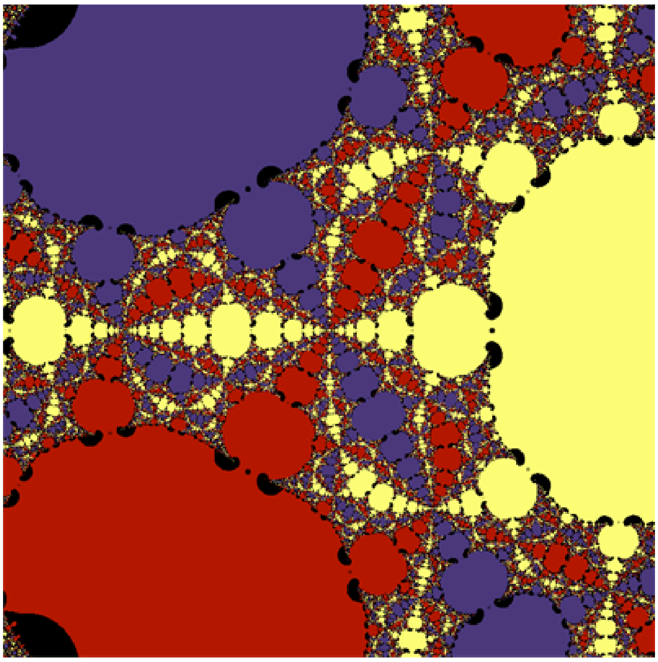

Fig. 8. Newton's basins of attraction were modified for $f(z) = z^3 - 1$ when k=2.

VI. Conclusions

The convergence of Newton's technique is generally known to be strongly reliant on the starting approximation. In the issue $y = tan^{-1}(x)$, we observed that the approach may fail to converge for some unsatisfactory beginning approximations. One of the aims of this research was to find an area where, under particular conditions, if the starting approximation is picked from it, Newton's technique converges to the root. Higher-order convergence is feasible with Newton's approach. We used the second Taylor polynomials in one dimension to demonstrate that Newton's technique will converge cubic, with constraints that may be applied to the function $f(x)$ and its derivatives. We have demonstrated theoretically and quantitatively that the approximation obtained by utilizing the first-order Taylor polynomial in two dimensions is identical to the approximation obtained by Newton's technique for two variables. Finally, we looked at Newton's technique for functions with complex values. We demonstrated that using Newton's approach, for complex z, solving $f(z) = 0$ is equivalent to solving the system $F(x, y) = 0$, in which the real and imaginary components of $f(z)$ form a system of two equations $f_1(x, y) = 0$ and $f_2(x, y) = 0$. We can also see that using Newton's method to solve $f(z) = 0$ simplifies the computations. Figures show the basin of attraction of the roots produced by Newton's technique, from which we may deduce the following:

1) The attraction basin has fractal borders and chaotic activity.
2) The modified Newton's approach produces darker basins of attraction than Newton's method.
3) Graphs of fractal patterns show that the improved approach is more successful since it converges to the root faster.
4) We analyzed Newton's method for complex valued functions. We have shown analytically that solving $f(z) = 0$ for complex z is exactly the same of solving the system $\mathbf{F}(x; y) = \mathbf{0}$.
5) We investigated where in the complex plane we could get bad initial approximations for the well-known n-th root of unity problems. We observed a fractal distribution of points in the complex plane.
6) Finally, we discovered that if we choose an initial approximation on any of the axes of preimages of the origin or a line that contains a root of the function $z^n - 1 = 0$, the iterations do not move away from the line.

Future research will compare the basins to regions of convergence resulting from Newton's method convergence theorems.

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