Inferences of the Multi Element Stress Strength Reliability for the Burr Type X Distribution

Huibing Hao, Chunping Li

***Abstract*—This paper deals with the estimation of multi element stress strength system reliability *R* when the stress and strength follow Burr type X distributions. The maximum likelihood and Bayesian methods are used to obtain the point estimates of *R*, and the confidence intervals of *R* are obtained via the delta and MCMC methods. The performances of the different point estimation methods and their corresponding confidence intervals are compared via the extensive simulation studies. To illustrate the effectiveness of the developed model and methods, a real data set is utilized.**

***Index Terms*—Multi element stress strength model, Burr type X distribution, Maximum Likelihood method, Bayesian method, Confidence interval**

I. INTRODUCTION

THE applications of the stress strength model (SSM) are very widely, encompassing engineering applications, survival analysis, and so on. A system with a single element is referred to as a single element SSM. Within this model, a system can operate normally if its stress is lower than its inherent strength. Over the past few years, there has been a large literature to study the single element SSM under various distributional assumptions regarding stress and strength, such as Chen and Cheng[1], Rezaei et al.[2], Krishna et al.[3] and Babayi and Khorram[4], Asgharzadeh et al.[5], Akgul et al.[6], Agiwal [7], Safariyan et al.[8], and Kotz et al.[9].

If a system comprises more than one element, it is termed a multi element stress strength model (MSSM) in Ref [10]. Similar to single element SSM, many papers have been studied the MSSM under different distribution assumptions, such as Kumaraswamy distribution[11], Burr type XII distribution[12], Rayleigh distribution[13], Marshall-Olkin bivariate Weibull[14], Chen distribution[15], Topp Leone distribution[16] and generalized logistic distribution[17].

The Burr type X distribution has been widely applied to model lifetime data or survival data by Burr[18], Sartawi and Abu Salih[19], Jaheen[20] and Raqab[21]. The CDF and PDF of the Burr type X distribution can be obtained as

$$F(x)=\left(1-\exp(-x^2)\right)^{\theta},\ x>0,\theta>0, \tag{1}$$

and

$$f(x)=2\theta x\exp\left(-x^2\right)\left(1-\exp(-x^2)\right)^{\theta},\ x>0,\theta>0, \tag{2}$$

where θ is a shape parameter, and let $X\sim\text{Burr}(\theta)$.

In Refs [22-25], the reliability estimation of a single element SSM has been extensively investigated under the Burr type X distribution. In this paper, we extend the study from single element SSM to the MSSM under the same distribution.

II. RELIABILITY MODELING OF THE MSSM SYSTEM

The reliability of MSSM is introduced by Bhattacharyya and Johnson[10] as

$$R_{s,k}=P[\text{at least } s \text{ of the } (X_1, X_2, \cdots, X_k) \text{ exceed } Y]$$
$$=\sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{+\infty}\left[1-F_X(y)\right]^i\left[F_X(y)\right]^{k-i}dG_Y(y) \tag{3}$$

where Y denotes the stress variable, X_i denotes the strength variable of the i element (i=1,2,$\cdots$,k). $G(\cdot)$ and $F(\cdot)$ are the cumulative distribution functions of Y and X_i (i=1,2,$\cdots$,k), respectively.

Let X_i be (i=1,2, $\cdots$,k) random sample, and $X_i\sim\text{Burr}(\theta_1)$. Let $Y\sim\text{Burr}(\theta_2)$, then the reliability of MSSM can be calculated as

$$R_{s,k}=P[\text{at least } s \text{ of the } (X_1, X_2, \cdots, X_k) \text{ exceed } Y]$$
$$=\sum_{i=s}^{k}\binom{k}{i}\int_{0}^{+\infty}\left[1-F_X(y;\theta_1)\right]^i\left[F_X(y;\theta_1)\right]^{k-i}dG_Y(y;\theta_2)$$
$$=\sum_{i=s}^{k}\binom{k}{i}\int_{0}^{+\infty}\left[1-\left(1-\exp(-y^2)\right)^{\theta_1}\right]^i\left[\left(1-\exp(-y^2)\right)^{\theta_1}\right]^{k-i}$$
$$\times 2\theta_2 y\exp\left(-y^2\right)\left(1-\exp(-y^2)\right)^{\theta_2}dy \tag{4}$$

Let

$$u=\left(1-\exp(-x^2)\right)^{\theta_1} \tag{5}$$

Then, we have

$$R_{s,k}=\frac{\theta_2}{\theta_1}\sum_{i=s}^{k}\binom{k}{i}\int_0^1 u^{k-i+\frac{\theta_2}{\theta_1}-1}(1-u)^i du$$
$$=\frac{\theta_2}{\theta_1}\sum_{i=s}^{k}\binom{k}{i}Beta\left(k-i+\frac{\theta_2}{\theta_1},i+1\right) \tag{6}$$

where $Beta(\cdot,\cdot)$ is the standard Beta function. After the simplification, we can get

$$R_{s,k}=\frac{\theta_2}{\theta_1}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}\left(k+\frac{\theta_2}{\theta_1}-j\right)\right]^{-1} \tag{7}$$

This article is to obtain the estimates of $R_{s,k}$ via different methods when the strength and stress variables follow Burr type X distribution. This article is organized as: In Sections III and IV, the maximum likelihood estimate and asymptotic confidence interval of $R_{s,k}$ are obtained. The Bayesian estimator using the MCMC method is provided in Section V. Some simulation studies are presented in Section VI, and a

Manuscript received Nov 5, 2023; revised May 14, 2024.This work is supported by the Humanity and Social Science Foundation of Ministry of Education of China (No. 20YJAZH035), the National Bureau of Statistics of China (No.2017LY73), the Technology Creative Project of Excellent Middle & Young Team of Hubei Province (T201920), the Research Foundation for Advanced Talents of Suqian University (106-CK00042/059), the Jingdong Institute Open Fund of Suqian University (2022JDXY05), the Humanity and Social Science Foundation of Ministry of Education of China (No. 19YJAZH039).

Huibing Hao is a Professor of Management School, Suqian University, Jiangsu, 223800 China (e-mail: haohuibing@163.com).

Chunping Li is a Professor of Mathematics School, Suqian University, Jiangsu, 223800 China (corresponding author: +8613886380145; e-mail: lichunping315@163.com).

real data example is presented in Section VII. In Section VIII, some conclusions are given.

III. Maximum likelihood estimation (MLE) of $R_{s,K}$

To obtain the MLE of $R_{s,k}$, some estimations of Burr type X distribution should be obtained, and the samples can be organized as

$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nk} \end{bmatrix}, \text{ and } \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \tag{8}$$

Then, we can get

$$\begin{aligned} L(\theta_1,\theta_2 \mid X,Y) &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k} f(x_{ij})\right)g(y_i) = \prod_{i=1}^{n}\prod_{j=1}^{k} f(x_{ij})\prod_{i=1}^{n} g(y_i) \\ &= \prod_{i=1}^{n}\prod_{j=1}^{k} 2\theta_1 x_{ij}\exp\left(-x_{ij}^2\right)\left(1-\exp(-x_{ij}^2)\right)^{\theta_1} \\ &\times\prod_{i=1}^{n} 2\theta_2 y_i\exp\left(-y_i^2\right)\left(1-\exp(-y_i^2)\right)^{\theta_2} \\ &= (2\theta_1)^{nk}\left(\prod_{i=1}^{n}\prod_{j=1}^{k} x_{ij}\right)\exp\left(-\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}^2\right)\left[\prod_{i=1}^{n}\prod_{j=1}^{k}\left(1-\exp(-x_{ij}^2)\right)\right]^{\theta_1-1} \\ &\quad\times(2\theta_2)^{n}\left(\prod_{i=1}^{n} y_i\right)\exp\left(-\sum_{i=1}^{n} y_i^2\right)\left[\prod_{i=1}^{n}\left(1-\exp(-y_i^2)\right)\right]^{\theta_2-1} \end{aligned} \tag{9}$$

Take the log of both sides of Eq (9), then we can get

$$\begin{aligned} \ln L(\theta_1,\theta_2 \mid X,Y) &= nk\ln 2 + nk\ln\theta_1 + \sum_{i=1}^{n}\sum_{j=1}^{k}\ln x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}^2 \\ &\quad +(\theta_1-1)\sum_{i=1}^{n}\sum_{j=1}^{k}\ln\left(1-\exp(-x_{ij}^2)\right) \\ &= n\ln 2 + n\ln\theta_2 + \sum_{i=1}^{n}\ln y_i - \sum_{i=1}^{n} y_i^2 + (\theta_2-1)\sum_{i=1}^{n}\ln\left(1-\exp(-y_i^2)\right) \end{aligned} \tag{10}$$

By taking the derivatives of Eq (10), then we can get

$$\frac{\partial \ln L(\theta_1,\theta_2 \mid X,Y)}{\partial\theta_1} = \frac{nk}{\theta_1} + \sum_{i=1}^{n}\sum_{j=1}^{k}\ln\left(1-\exp(-x_{ij}^2)\right) = 0 \tag{11}$$

$$\frac{\partial \ln L(\theta_1,\theta_2 \mid X,Y)}{\partial\theta_2} = \frac{n}{\theta_2} + \sum_{i=1}^{n}\ln\left(1-\exp(-y_i^2)\right) = 0 \tag{12}$$

Then, the MLE of θ_1 and θ_2 can be sovled from Eqs (11) and (12) as

$$\hat{\theta}_1 = -\frac{nk}{\sum_{i=1}^{n}\sum_{j=1}^{k}\ln\left(1-\exp(-x_{ij}^2)\right)}$$

and

$$\hat{\theta}_2 = -\frac{n}{\sum_{i=1}^{n}\ln\left(1-\exp(-y_i^2)\right)} \tag{13}$$

Then, the MLE of $R_{s,k}$ can be sovled as

$$\hat{R}_{s,k} = \frac{\hat{\theta}_2}{\hat{\theta}_1}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}\left(k+\frac{\hat{\theta}_2}{\hat{\theta}_1}-j\right)\right]^{-1} \tag{14}$$

Considering that the exact confidence interval of $R_{s,k}$ is difficult to obtain, similar to Efron[26], the delta method is used.

IV. An asymptotic approach using the delta method

In order to obtain the asymptotic confidence interval of $R_{s,k}$, the delta method is used in this paper, and the MLE $\hat{R}_{s,k}$ in Eq (14) is asymptotically normal distribution with variance as

$$Var(\hat{R}_{s,k}) = \left[G'I^{-1}G\right]_{\hat{\theta},\hat{\theta}_2} \tag{15}$$

where $G' = \left(\frac{\partial R_{s,k}}{\partial\theta_1}, \frac{\partial R_{s,k}}{\partial\theta_2}\right)$, and I is Fisher's information matrix as

$$I = E\begin{pmatrix} -\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_1^2} & -\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \\ -\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_2\partial\theta_1} & -\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_2^2} \end{pmatrix} \tag{16}$$

where the partial derivatives are

$$\begin{aligned} &E\left(-\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_1^2}\right) = \frac{nk}{\theta_1^2}, \quad E\left(-\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_2^2}\right) = \frac{n}{\theta_2^2} \\ &E\left(-\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2}\right) = E\left(-\frac{\partial^2 \ln L(\theta_1,\theta_2)}{\partial\theta_2\partial\theta_1}\right) = 0 \end{aligned} \tag{17}$$

Then, the Var of $\hat{R}_{s,k}$ is computed as follows

$$\begin{aligned} Var(\hat{R}_{s,k}) &= \left[G'I^{-1}G\right]_{\hat{\theta},\hat{\theta}_2} \\ &= \left(\frac{\partial R_{s,k}}{\partial\theta_1}, \frac{\partial R_{s,k}}{\partial\theta_2}\right)\begin{pmatrix} \frac{nk}{\theta_1^2} & 0 \\ 0 & \frac{n}{\theta_2^2} \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial R_{s,k}}{\partial\theta_1} \\ \frac{\partial R_{s,k}}{\partial\theta_2} \end{pmatrix} \\ &= \frac{\theta_1^2}{nk}\left(\frac{\partial R_{s,k}}{\partial\theta_1}\right)^2 + \frac{\theta_2^2}{n}\left(\frac{\partial R_{s,k}}{\partial\theta_2}\right)^2 \end{aligned} \tag{18}$$

Considering that the derivation of $R_{s,k}$ is very difficult, in particular, for $(s,k) = (1,3)$ and $(2,4)$, the derivation of $R_{s,k}$ can be obtained as

$$\begin{aligned} &R_{1,3} = \frac{3\theta_1}{3\theta_1+\theta_2}, \\ &\frac{\partial R_{1,3}}{\partial\theta_1} = \frac{3\theta_2}{(3\theta_1+\theta_2)^2}, \\ &\frac{\partial R_{1,3}}{\partial\theta_2} = -\frac{3\theta_1}{(3\theta_1+\theta_2)^2} \\ &R_{2,4} = \frac{12\theta_1^2}{(4\theta_1+\theta_2)(3\theta_1+\theta_2)} \\ &\frac{\partial R_{2,4}}{\partial\theta_1} = -\frac{12\theta_1^2(7\theta_1+2\theta_2)}{\left[(4\theta_1+\theta_2)(3\theta_1+\theta_2)\right]^2}, \\ &\frac{\partial R_{2,4}}{\partial\theta_2} = -\frac{12\theta_1\theta_2(7\theta_1+2\theta_2)}{\left[(4\theta_1+\theta_2)(3\theta_1+\theta_2)\right]^2} \end{aligned} \tag{19}$$

Then, we can get

$$Var(\hat{R}_{1,3}) = \frac{9\hat{\theta}_1^2\hat{\theta}_2^2}{(3\hat{\theta}_1+\hat{\theta}_2)^4}\left(\frac{1}{nk}+\frac{1}{n}\right) \tag{20}$$

and

$$Var(\hat{R}_{2,4}) = \frac{144\hat{\theta}_1^4\hat{\theta}_2^2(7\hat{\theta}_1+2\hat{\theta}_2)^2}{\left[(4\hat{\theta}_1+\hat{\theta}_2)(3\hat{\theta}_1+\hat{\theta}_2)\right]^4}\left(\frac{1}{nk}+\frac{1}{n}\right) \quad (21)$$

Thus, we can get the following asymptotically distribution as

$$\frac{\hat{R}_{s,k}-R_{s,k}}{\sqrt{Var(\hat{R}_{s,k})}} \sim N(0,1) \quad (22)$$

Then, the asymptotic 100(1 − γ)% confidence interval for $R_{s,k}$ can be obtained as

$$\left(\hat{R}-Z_{1-\gamma/2}\sigma_{\hat{R}}, \hat{R}+Z_{1-\gamma/2}\sigma_{\hat{R}}\right) \quad (23)$$

where $Z_{1-\gamma/2}$ is a quantile of standard normal distribution.

V. BAYESIAN ESTIMATORS OF $R_{S,K}$

Suppose that $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_m$ are two independent random samples of sizes n and m, drawn from Burr(θ_1) and Burr(θ_2), respectively. Then, the likelihood functions of each sample can be obtained as

$$L_1(\theta) = (2\theta_1)^{nk}\left(\prod_{i=1}^{n}\prod_{j=1}^{k}x_{ij}\right)\exp\left(-\sum_{i=1}^{n}\sum_{j=1}^{k}x_{ij}^2\right) \times\left[\prod_{i=1}^{n}\prod_{j=1}^{k}\left(1-\exp(-x_{ij}^2)\right)\right]^{\theta_1-1}$$
$$\propto (2\theta_1)^{nk}\left[\prod_{i=1}^{n}\prod_{j=1}^{k}\left(1-\exp(-x_{ij}^2)\right)\right]^{\theta_1-1} \quad (24)$$

and

$$L_2(\beta) = (2\theta_2)^{n}\left(\prod_{i=1}^{n}y_i\right)\exp\left(-\sum_{i=1}^{n}y_i^2\right)\left[\prod_{i=1}^{n}\left(1-\exp(-y_i^2)\right)\right]^{\theta_2-1}$$
$$\propto (2\theta_2)^{n}\left[\prod_{i=1}^{n}\left(1-\exp(-y_i^2)\right)\right]^{\theta_2-1} \quad (25)$$

Suppose that θ_1 and θ_2 are independent gamma prior with the cumulative distribution functions as

$$\pi(\theta_1) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta_1^{a_1-1}\exp(-b_1\theta_1),\ \theta_1>0 \quad (26)$$

and

$$\pi(\theta_2) = \frac{b_2^{a_2}}{\Gamma(a_2)}\theta_2^{a_2-1}\exp(-b_2\theta_2),\ \theta_2>0 \quad (27)$$

with $\theta_1 \sim Gamma(a_1,b_1)$ and $\theta_2 \sim Gamma(a_2,b_2)$.

Therefore, the joint density of the θ_1 and θ_2, X and Y can be obtained as

$$L(\theta_1,\theta_2;X,Y) = L_1(\theta_1)\times L_2(\theta_2)\times\pi(\theta_1)\times\pi(\theta_2) \quad (28)$$

From the Bayesian theorem, the joint posterior density of θ_1 and θ_2 can be obtained as

$$L(\theta_1,\theta_2\mid X,Y) = \frac{L(\theta_1,\theta_2;X,Y)}{\int_0^{+\infty}\int_0^{+\infty}L(\theta_1,\theta_2;X,Y)\,d\theta_1 d\theta_2}$$
$$\propto \theta_1^{nk+a_1-1}\exp\left(-\left[b_1-\sum_{i=1}^{n}\sum_{j=1}^{k}\ln\left(1-\exp(-x_{ij}^2)\right)\right]\theta_1\right)$$
$$\times\theta_2^{n+a_2-1}\exp\left(-\left[b_2-\sum_{i=1}^{n}\ln\left(1-\exp(-y_i^2)\right)\right]\theta_2\right) \quad (29)$$

Then, we can get

$$L(\theta_1,\theta_2\mid X,Y) = \frac{d_1^{c_1}d_2^{c_2}}{\Gamma(c_1)\Gamma(c_2)}\theta_1^{c_1-1}\theta_2^{c_2-1}\exp\left(-d_1\theta_1-d_2\theta_2\right) \quad (30)$$

Then, we can get the posterior PDFs of θ_1 and θ_2 are as follows

$$\theta_1\mid X \sim Gamma(c_1,d_1) \quad (31)$$

and

$$\theta_2\mid Y \sim Gamma(c_2,d_2) \quad (32)$$

where

$$c_1 = nk+a_1,\ d_1 = b_1-\sum_{i=1}^{n}\sum_{j=1}^{k}\ln\left(1-\exp(-x_{ij}^2)\right),$$
$$c_2 = n+a_2,\ d_2 = b_2-\sum_{i=1}^{n}\ln\left(1-\exp(-y_i^2)\right).$$

In order to calculate the Bayesian estimator for $R_{s,k}$, we apply the approach proposed by Geman & Geman [27] to the sample by using the MCMC method. The procedure is outlined as follows:

Step 1): Generate θ_1 from $Gamma(c_1,d_1)$,

Step 2): Generate θ_2 from $Gamma(c_2,d_2)$,

Step 3): Repeat Steps 1 and 2, M times,

Step 4): Let $R_{s,k}^{(t)} = \frac{\hat{\theta}_{2t}}{\hat{\theta}_{1t}}\sum_{i=s}^{k}\frac{k!}{(k-i)!}\left[\prod_{j=0}^{i}\left(k+\frac{\hat{\theta}_{2t}}{\hat{\theta}_{1t}}-j\right)\right]^{-1}$,

$t = 1, 2, \cdots, M$.

Under the squared error loss function, the approximate posterior mean, and posterior variance of $R_{s,k}$ can be obtained as

$$\hat{E}(R_{s,k}) = \frac{1}{M}\sum_{t=1}^{M}R_{s,k}^{(t)},$$

and

$$\hat{V}(R_{s,k}) = \frac{1}{M}\sum_{i=1}^{M}\left(R_{s,k}^{(t)}-\hat{E}(R_{s,k})\right)^2 \quad (33)$$

Similar to Chen and Shao[28], we can get the credible intervals of $R_{s,k}$ as

$$\left(\hat{R}_B\left(\frac{\gamma}{2}\right), \hat{R}_B\left(1-\frac{\gamma}{2}\right)\right) \quad (34)$$

where $\hat{R}_B\left(\frac{\gamma}{2}\right)$ and $\hat{R}_B\left(1-\frac{\gamma}{2}\right)$ are $\frac{\gamma}{2}$ and $1-\frac{\gamma}{2}$ quantiles of $R_{s,k}^{(t)}$.

VI. SIMULATION STUDY

In this study, we performed a series of simulation experiments to compare the performance of the MLE and the Bayesian estimator for $R_{s,k}$. These comparisons were made across varying sample sizes of n=m=10, 15, 20, 25, 30 and under different parametric settings of (θ_1, θ_2) = (1.5,0.5), (1.5,1.0), (1.5,2.0), (1.0,0.5), (2.0,0.5) and (3.0,0.5).

For the given sample sizes and parameter settings, when (s,k) =(1,3), the true values of $R_{1,3}$ are 0.9, 0.8182, 0.6923, 0.8571, 0.9231 and 0.9474, respectively. Similarly, when (s,k) = (2,4), the true values of $R_{2,4}$ are 0.8308, 0.7013, 0.5192, 0.7619, 0.8688, 0.9095.

TABLE I
AVR, BIAS, MSE, EL AND CP UNDER TRUE VALUES CASE 1

	Methods		MLE					MCMC				
(s,k)	(n,m)	R	AVR	BIAS	MSE	EL	CP	AVR	BIAS	MSE	EL	CP
(1,3)	(10,10)	0.8182	0.8067	0.0115	0.0036	0.2186	0.9316	0.7949	0.0493	0.0040	0.2157	0.935
	(15,15)		0.8103	0.0079	0.0023	0.1774	0.9408	0.8032	0.0370	0.0023	0.1755	0.944
	(20,20)		0.8123	0.0058	0.0017	0.1526	0.9418	0.8074	0.0319	0.0017	0.1514	0.946
	(25,25)		0.8140	0.0042	0.0013	0.1360	0.9448	0.81	0.0285	0.0013	0.1351	0.958
	(30,30)		0.8142	0.0040	0.0010	0.1241	0.9483	0.8111	0.0248	0.0010	0.1235	0.953
(2,4)	(10,10)	0.7013	0.6862	0.0151	0.0076	0.3165	0.9311	0.6708	0.0688	0.0075	0.3066	0.935
	(15,15)		0.6905	0.0108	0.0048	0.2590	0.9411	0.6802	0.0551	0.0048	0.2534	0.945
	(20,20)		0.6935	0.0078	0.0035	0.2239	0.9401	0.6859	0.0468	0.0035	0.2203	0.957
	(25,25)		0.6944	0.0069	0.0028	0.2004	0.9391	0.69	0.0422	0.0027	0.1972	0.941
	(30,30)		0.6958	0.0055	0.0023	0.1858	0.9426	0.6922	0.0377	0.0023	0.1844	0.948

TABLE II
AVR, BIAS, MSE, EL AND CP UNDER TRUE VALUES CASE 2

	Methods		MLE					MCMC				
(s,k)	(n,m)	R	AVR	BIAS	MSE	EL	CP	AVR	BIAS	MSE	EL	CP
(1,3)	(10,10)	0.6923	0.6792	0.0131	0.0067	0.3023	0.9312	0.6729	0.0620	0.0064	0.2884	0.945
	(15,15)		0.6849	0.0074	0.0043	0.2486	0.9412	0.6812	0.0490	0.0039	0.2393	0.954
	(20,20)		0.685	0.0073	0.0034	0.2149	0.9421	0.683	0.0462	0.0032	0.2089	0.943
	(25,25)		0.6878	0.0045	0.0027	0.1922	0.9426	0.6838	0.0405	0.0024	0.1885	0.946
	(30,30)		0.6888	0.0034	0.0021	0.1754	0.9440	0.6872	0.0341	0.0019	0.1723	0.964
(2,4)	(10,10)	0.5192	0.5083	0.0109	0.0104	0.3883	0.9228	0.4991	0.0809	0.0102	0.3651	0.936
	(15,15)		0.5109	0.0083	0.0068	0.3210	0.9348	0.5063	0.0654	0.0066	0.3074	0.949
	(20,20)		0.5121	0.0071	0.0055	0.2794	0.9374	0.5084	0.0600	0.0053	0.2701	0.943
	(25,25)		0.5129	0.0064	0.0042	0.2508	0.9390	0.5114	0.0510	0.0040	0.2442	0.952
	(30,30)		0.5144	0.0049	0.0035	0.2295	0.9473	0.5129	0.0472	0.0033	0.2243	0.949

TABLE III
AVR, BIAS, MSE, EL AND CP UNDER TRUE VALUES CASE 3

	Methods		MLE					MCMC				
(s,k)	(n,m)	R	AVR	BIAS	MSE	EL	CP	AVR	BIAS	MSE	EL	CP
(1,3)	(10,10)	0.9231	0.9159	0.0072	0.0011	0.1085	0.9360	0.9108	0.0241	9.30E-04	0.1100	0.939
	(15,15)		0.9186	0.0045	5.92E-04	0.087	0.9437	0.9149	0.0185	5.60E-04	0.0879	0.953
	(20,20)		0.9198	0.0033	4.14E-04	0.0741	0.9473	0.9176	0.0154	3.98E-04	0.0745	0.949
	(25,25)		0.9206	0.0024	4.14E-04	0.0659	0.9483	0.9181	0.0136	2.95E-04	0.0666	0.951
	(30,30)		0.921	0.0021	3.02E-04	0.0601	0.9433	0.9183	0.0133	2.49E-04	0.0608	0.949
(2,4)	(10,10)	0.8688	0.8579	0.0108	0.0026	0.1726	0.9388	0.8497	0.0375	2.40E-03	0.1728	0.944
	(15,15)		0.8623	0.0065	0.0017	0.1380	0.9448	0.8554	0.0300	1.40E-03	0.1390	0.946
	(20,20)		0.8638	0.0051	0.0012	0.1188	0.9482	0.8573	0.0254	1.00E-03	0.1203	0.950
	(25,25)		0.8645	0.0042	9.18E-04	0.1058	0.9504	0.8599	0.0222	7.95E-04	0.1067	0.951
	(30,30)		0.8652	0.0035	7.01E-04	0.0962	0.9466	0.8623	0.0204	6.33E-04	0.0964	0.949

TABLE IV
AVR, BIAS, MSE, EL AND CP UNDER TRUE VALUES CASE 4

Methods			MLE					MCMC				
(s,k)	(n,m)	R	AVR	BIAS	MSE	EL	CP	AVR	BIAS	MSE	EL	CP
(1,3)	(10,10)	0.9474	0.9427	0.0047	4.79E-04	0.0770	0.941	0.9379	0.0177	4.75E-04	0.0794	0.947
	(15,15)		0.9440	0.0034	2.92E-04	0.0611	0.944	0.9413	0.0135	2.95E-04	0.0625	0.948
	(20,20)		0.9449	0.0025	2.03E-04	0.0525	0.948	0.9425	0.0116	2.04E-04	0.0535	0.949
	(25,25)		0.9455	0.0019	1.51E-04	0.0465	0.943	0.9437	0.0100	1.48E-04	0.0472	0.946
	(30,30)		0.9458	0.0016	1.27E-04	0.0423	0.947	0.9443	0.0090	1.26E-04	0.0428	0.947
(2,4)	(10,10)	0.9095	0.9009	0.0085	0.0013	0.1255	0.946	0.8935	0.0279	0.0014	0.1280	0.957
	(15,15)		0.9043	0.0052	7.38E-04	0.0997	0.944	0.8925	0.0223	7.49E-04	0.1018	0.948
	(20,20)		0.9060	0.0035	5.20E-04	0.0851	0.944	0.9025	0.0183	4.68E-04	0.0859	0.945
	(25,25)		0.9066	0.0029	3.96E-04	0.0758	0.952	0.9045	0.0157	3.42E-04	0.0758	0.953
	(30,30)		0.9071	0.0024	3.35E-04	0.0689	0.947	0.9051	0.0144	2.57E-04	0.0691	0.955

To compare the various estimation methods, we computed the average bias, mean squared error (MSE), expected lengths (ELs) of confidence intervals, and coverage probability (CP) of $R_{s,k}$. The results of the simulations are presented in Tables I-IV, and the parameters in Case 1 are set as θ_1=1.5, θ_2=1.0, $R_{1,3}$=0.8182, $R_{2,4}$=0.7013, the parameters in Case 2 are set as θ_1=1.5, θ_2=2.0, $R_{1,3}$=0.6923, $R_{2,4}$=0.5192, the parameters in Case 3 are set as θ_1=2.0, θ_2=0.5, $R_{1,3}$=0.9231, $R_{2,4}$=0.8688, the parameters in Case 4 are set as θ_1=3.0, θ_2=0.5, $R_{1,3}$=0.9474, $R_{2,4}$=0.9095.

From Tables I to IV, for both MLE and MCMC methods, we can observe the following patterns: (1) the average estimated value of reliability converges towards the true value as the sample size increases; (2) with increasing sample size, there is a consistent reduction in the AVR, the MSE, and the EL; (3) when θ_1 is fixed and θ_2 is increased, we can find that the MSE and EL are increasing; (4) when θ_2 is fixed and θ_1 is increased,we can find that the MSE and EL are decreasing.

In addition, by comparing the MLE and MCMC methods, it is evident that the MCMC method exhibits a greater average bias than the MLE. However, the average mean squared error (MSE) associated with the MCMC method is lower compared to the MLE. Additionally, it can be observed that the MCMC method yields slightly shorter average expected lengths (ELs) and a marginally higher average coverage probability (CP) than the MLE.

VII. REAL DATA ANALYSIS

Real data set, as reported by Bennett and Filliben [29], was considered by Tarvirdizade and Gharehchobogh [25] for the single element model. In this paper, we employ the same dataset to demonstrate the new model and method as follows:

Data Set 1: 3.051, 2.779, 2.604, 2.371, 2.214, 2.045, 1.715, 1.525, 1.296, 1.154, 1.016, 0.7948, 0.7007, 0.6292, 0.6175, 0.6449, 0.8881, 1.115, 1.397, 1.506, 1.528.

Data Set 2: 2.658, 2.434, 2.288, 2.092, 1.959, 1.814, 1.530, 1.366, 1.165, 1.041, 0.9198, 0.7241, 0.6403, 0.576, 0.5647, 0.5873, 0.8013, 1.002, 1.250, 1.347, 1.368.

By using the Kolmogorov-Smirnov test, Ref [25] showed that Burr type X distribution fit the above two data sets well. When (s,k) = (1,3) and (2,4), then, we can obtain the MLE of the parameters as $\hat{\theta}_1$ = 2.7998 and $\hat{\theta}_2$ =2.2861, and the point estimates of $R_{s,k}$ under different cases can be obtained accordingly.

Additionally, Table 6 presents the interval estimates for $R_{s,k}$ under different cases. According to Table 6, it can be observed that the lengths of the credible interval for the MLE method and MCMC methods are approximately equal to 0.16 and 0.24 for $R_{1,3}$ and $R_{2,4}$, respectively. Based on Tables 5-6, from the perspective of point estimation, we can find that the MLE method and the MCMC method yield similar results for $R_{1,3}$ and $R_{2,4}$. However, in terms of interval estimation, the ELs obtained through MCMC are slightly shorter than those obtained through the MLE method. Therefore, the MCMC method is considered the preferable approach.

V. CONCLUSIONS

This paper focuses on estimating the reliability of a multi element MSSM system, $R_{s,k}$, by using the MLE method and the MCMC method. The stress and strength distributions in the MSSM system are assumed to follow the Burr type X distributions.

TABLE V
THE POINT ESTIMATES OF R UNDER THE DIFFERENT CASES

	$R_{1,3}$	$R_{2,4}$
MLE	0.7861	0.6528
MCMC	0.7837	0.6486

TABLE IV
THE POINT ESTIMATES OF R UNDER THE DIFFERENT CASES

	MLE	MCMC
	Confidence Interval (EL)	Confidence Interval (EL)
$R_{1,3}$	(0.7030, 0.8691) (0.1661)	(0.6933, 0.8576) (0.1643)
$R_{2,4}$	(0.5331, 0.7725) (0.2394)	(0.5247, 0.7599) (0.2352)

By comparing the performances of the different methods via an extensive simulation study, we can find that the MCMC method demonstrates better performance than the MLE method. Finally, in order to illustrate the effectiveness of the developed model and methods, a real data set is utilized.

REFERENCES

[1] J. Chen, and C. Cheng, "Reliability of stress strength model for exponentiated Pareto distributions", J. Stat. Comput. Simul., vol.87, pp.791-805,2017.

[2] A. Rezaei, M. Sharafi, J. Behboodian and A. Zamani, "Inference on stress-strength parameter based on GLD distribution", Commun. Stat. Simul. Comput., vol.47, pp.1251-1263,2018.

[3] H. Krishna, M. Dube and R. Garg, "Estimation of P (Y < X) for progressively first-failure-censored generalized inverted exponential distribution", J. Stat. Comput. Simul., vol.87, pp.2274-2289,2017.

[4] S. Babayi and E. Khorram, "Inference of stress strength for the Type-II generalized logistic distribution under progressively Type-II censored samples", Commun. Stat. Simul. Comput., vol.47, pp.1975-1995,2018.

[5] A. Asgharzadeh, R. Valiollahi and M. Z. Raqab, "Estimation of Pr (Y < X) for the two-parameter generalized exponential records", Commun. Stat. Simul. Comput., vol.46, pp.379-394,2017.

[6] F. G. Akgul, S. Acıtas and B. Senoglu, "Inference on stress-strength reliability based on ranked set sampling data in case of Lindley distribution", J. Stat. Comput. Simul., vol.88, pp.3018-3032,2018.

[7] V. Agiwal, "Bayesian Estimation of stress strength reliability from Inverse Chen distribution with application on failure time data", Ann. Data. Sci., pp.1-31,2021.

[8] A. Safariyan, M. Arashi and R. A. Belaghi, "Improved point and interval estimation of the stress-strength reliability based on ranked set sampling", Statistics., vol.53, pp.101-125,2019.

[9] S. Kotz, Y. Lumelskii and M. Pensky, "The stress-strength model and its generalizations: theory and applications", Singapore: World Scientific, 2003.

[10] G. K. Bhattacharyya and R. A. Johnson, "Estimation of reliability in a multicomponent stress-strength model", J. Am. Stat. Assoc., vol.69, pp.966-970,1974.

[11] Y. S. Dey, J. Mazucheli and M. Z. Anis, "Estimation of reliability of multicomponent stress-strength for a Kumaraswamy distribution", Commun. Stat. Theor. M., vol.47, pp.1560-1572,2017.

[12] Lio, T. R. Tsai, L. Wang and I.P. Cecilio Tejada, "Inferences of the multicomponent stress-strength reliability for Burr XII distributions", Mathematics, vol.10, pp.1-28,2022.

[13] L. Wang, H. Lin, K. Ahmadi and Y. Lio, "Estimation of stress-strength reliability for multicomponent system with Rayleigh data", Energies, vol.14, pp.1-23,2021.

[14] M. Nadar and F. Kizilaslan, "Estimation of reliability in a multicomponent stress-strength model based on a Marshall-Olkin bivariate Weibull distribution", IEEE Trans. Reliab., vol.65, pp.370-380,2016.

[15] T. Kayal, Y. M. Tripathi, S. Dey and S.J. Wu, "On estimating the reliability in a multicomponent stress-strength model based on Chen distribution", Commun. Stat. Theor. M., vol.50, pp.2429-2447,2020.

[16] F. G. Akgul, "Reliability estimation in multicomponent stress–strength model for Topp-Leone distribution", J. Stat. Comput. Simul., vol.89, pp.2914-2929,2019.

[17] M. Rasekhi, M. M. Saber and H. M. Yousof, "Bayesian and classical inference of reliability in multicomponent stress-strength under the generalized logistic model", Commun. Stat. Theor. M., vol.50, pp.5114-5125,2020.

[18] I. W. Burr, "Cumulative frequency functions", Ann. Math. Stat., vol.13, pp.215-232,1942.

[19] H. A. Sartawi and M. S. Abu-Salih, "Bayesian prediction bounds for the Burr type X model", Commun. Stat. Theor. M., vol.20, pp. 2307-2330,1991.

[20] Z. F. Jaheen, "Empirical Bayes estimation of the reliability and failure rate functions of the Burr type X failure model", J. Appl. Stat. Sci., vol.3, pp.281-288,1996.

[21] M.Z. Raqab, "Order Statistics from the Burr type X Model", Comput. Math. Appl., vol.36, pp.111-120,1998.

[22] K. E. Ahmad, M. E. Fakhry and Z. F. Jaheen, "Empirical Bayes estimation of P (Y < X) and characterizations of Burr type X model", J. Stat. Plan. and Infer., vol.64, pp.297-308,1997.

[23] J. G. Surles and W. J. Padgett, "Inference for P (Y < X) in the Burr type X model", J. Appl. Stat. Sci., vol.7, pp.225-238,1998.

[24] C. Kim and Y. Chung, "Bayesian estimation of P (Y < X) from Burr-type X model containing spurious observations", Stat. Pap., vol.47, pp.643-651,2006.

[25] B. Tarvirdizade and H. K. G. H. K. "Gharehchobogh, Inference on Pr (X> Y) based on record values from the Burr Type X distribution", Hacet. J. Math. and Stat., vol.45, pp.267-278,2016.

[26] B. Efron, "The Jackknife, the Bootstrap and other resembling plans", SIAM, Philadelphia, 1982.

[27] S. Geman and D. Geman, "Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images", IEEE Trans. Pattern Anal. Mach. Intell., vol.12, pp.609-628,1984.

[28] M. H. Chen and Q. M. Shao, "Monte Carlo estimation of Bayesian credible and HPD intervals", J. Comput. Graph. Stat., vol.8, pp.69-92,1999.

[29] H. S. Bennett and J. J. Filliben, "A systematic approach for multidimensional, closed-form analytic modeling: minority electron mobilities in Ga 1-x AIxAs heterostructures", J. Res. Nat. Inst. Stan. Tech., vol.105, pp.441-452,2000.

[30] C. P. Li and H. B. Hao, "Reliability of a stress-strength model with inverse Weibull distribution", IAENG International Journal of Applied Mathematics, vol.47, no.3, pp.302-306, 2017.

[31] N.S.Y. Temraz, "Fuzzy stress-strength reliability subject to exponentiated power generalized Weibull", Engineering Letters, vol.30, no.1, pp.45-49, 2022.

Huibing Hao is a Professor of Management School, Suqian University, Jiangsu, 223800 China (e-mail: haohuibing@163.com). Huibing Hao is also a professor at Department of Mathematics and Statistics, Hubei Engineering University, Xiaogan 432100, China. He obtained his PhD degree in Mechanical Engineering from Southeast University in 2015. His research interests include reliability evaluation and maintenance decision.