Bipolar Complex Fuzzy Interior Ideals in Semigroups

P. Khamrot, A. Iampan, T. Gaketem

***Abstract*—In 2022, Rehman et al. presented bipolar complex fuzzy sets and proved the properties of bipolar complex fuzzy ideals in semigroups. In this research, we give the concept of bipolar complex fuzzy interior ideals in semigroups. We prove the basic properties of bipolar complex fuzzy interior ideals and study the relationship between bipolar complex fuzzy ideals and bipolar complex fuzzy interior ideals in semigroups. Finally, we characterize a semisimple semigroup in terms of bipolar complex fuzzy interior ideals.**

***Index Terms*—BCF sets, BCF ideals, BCF interior ideals, semisimple**

I. INTRODUCTION

THE THEORY of bipolar complex fuzzy sets is an extension of bipolar fuzzy set. The complex fuzzy set is a tool for dealing with uncertainty and complex information. It is studied in the structure of real number positive, negative and imaginary numbers positive, negative with generalizations of bipolar fuzzy set.

The concept of fuzzy sets by Zadeh in 1975, [1]. After that it has applications in several areas like medical science, image processing, decision-making method, etc. After, Kuroki [2] studied fuzzy subsemigroups and types fuzzy ideals in semigroups. Jun and Song [3] present fuzzy interior ideals in semigroups. In 1994 Zhang [4] developed the notion of fuzzy set go to bipolar fuzzy sets whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$, and used them for modeling and decision analysis. In 2000, Lee [5] used the term bipolar valued fuzzy sets and applied it to algebraic structures. The theory of complex fuzzy sets interesting by Ramot et al. [6]. Tamir et al. [7] studied the complex fuzzy set in structure cartesian by transforming the range from the unit circle to the complex plane. Al-Husban [8] discussed complex fuzzy groups. Hu et al. [9] developed the complex fuzzy set in othogonality and application to signal detection. The complex intuitionistic fuzzy soft sets introduced by Kumar and Bajaj [10]. Moreover, research in types bipolar fuzzy ideals, such as Kang [11], studied bioplar fuzzy subsemigroups in semigroups. Chinnadurau and Arulmozhi [12] discussed the bipolar fuzzy ideal in orederd Γ-semigroups, Khamrot and Siripitukdet [13] explained generalized bipolar fuzzy subsemigroups in semigroups. Gaketem and Khamrot [14] studied bipolar weakly interior ideals in semigorups. Mahmood [15] introduced bipolar soft set. Gaketem et al. [16] expand cubic biploar fuzzy subsemigroups and ideals in semigroups. Recently, Rehman et al. [17] presented bipolar complex fuzzy sets and bipolar complx fuzzy ideals in semigroups.

In this study, we give the concept of bipolar complex fuzzy interior ideals in semigroups and investigate the properties of bipolar complex fuzzy interior ideals in semigroups. The remainder of this paper is organized in the following. In Section 3, we study the relationship biploar complex fuzzy ideals and biploar complex fuzzy interior ideals in semigroups. In Section 4, we characterize a semisimple semigroup in terms of bipolar complex fuzzy interior ideals. The conclusion are presented in Section 5.

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II. PRELIMINARIES

In this topic, we literature review some basic definitions and theorems of semigroups, fuzzy sets, bipolar fuzzy sets, and bipolar complex fuzzy sets, which will be helpful in the next topic. This paper will denote a semigroup by $\mathfrak{F}$.

By a subsemigroup of $\mathfrak{F}$ we mean a non-empty subset $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{K}^2 \subseteq \mathfrak{K}$.
A non-empty subset $\mathfrak{K}$ of $\mathfrak{F}$ is called a *left* (right) ideal of $\mathfrak{F}$ if $\mathfrak{F}\mathfrak{K} \subseteq \mathfrak{K}$ ($\mathfrak{K}\mathfrak{F} \subseteq \mathfrak{K}$). By an *ideal* $\mathfrak{K}$ of $\mathfrak{F}$ we mean a left ideal and a right ideal of $\mathfrak{F}$. A subsemigroup $\mathfrak{K}$ of $\mathfrak{F}$ is called an *interior ideal* of $\mathfrak{F}$ if $\mathfrak{F}\mathfrak{K}\mathfrak{F} \subseteq \mathfrak{K}$. A *regular* of $\mathfrak{F}$ if for each $h \in \mathfrak{F}$, there exists $r \in \mathfrak{F}$ such that $h = hrh$. A *left* (right) regular of $\mathfrak{F}$ if for each $h \in \mathfrak{F}$, there exists $r \in \mathfrak{F}$ such that $h = rh^2$ ($h = h^2r$). An *intra-regular* of $\mathfrak{F}$ if for each $h \in \mathfrak{F}$, there exist $r, t \in \mathfrak{F}$ such that $h = rh^2t$.
A *semisimple* $\mathfrak{F}$ if for every $h \in \mathfrak{F}$ there exist $r, t, e, d \in \mathfrak{F}$ such that $h = rhtehd$.

For any $h_i \in [0,1]$, $i \in \mathcal{F}$, define

$$\underset{i\in\mathcal{F}}{\vee} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \underset{i\in\mathcal{F}}{\wedge} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We see that for any $h_1, h_2 \in [0,1]$, we have

$$h_1 \vee h_2 = \max\{h_1, h_2\} \quad \text{and} \quad h_1 \wedge h_2 = \min\{h_1, h_2\}.$$

A fuzzy set ω of a non-empty set $\mathfrak{F}$ is a function $\omega : \mathfrak{F} \to [0,1]$.

Definition 2.1. *A* bipolar fuzzy set *(shortly, BF set) ω on $\mathfrak{F}$ is an object having the form*

$$\omega := \{(\mathfrak{F}, \omega^P(h), \omega^N(h)) \mid h \in \mathfrak{F}\},$$

where $\omega^P : \mathfrak{F} \to [0,1]$ and $\omega^N : \mathfrak{F} \to [-1,0]$.

Remark 2.2. *For the sake of simplicity we shall use the symbol $\omega = (\mathfrak{F}; \omega^P, \omega^N)$ for the BF set $\omega = \{(\mathfrak{F}, \omega^P(h), \omega^N(h)) \mid h \in \mathfrak{F}\}$.*

Definition 2.3. *[18] A BF set $\omega = (\mathfrak{F}; \omega^P, \omega^N)$ on $\mathfrak{F}$ is called a*

(1) BF subsemigroup *on $\mathfrak{F}$ if $\omega^P(h_1h_2) \geq \omega^P(h_1) \wedge \omega^P(h_2)$ and $\omega^N(h_1h_2) \leq \omega^N(h_1) \vee \omega^n(h_2)$ for all $h_1, h_2 \in \mathfrak{F}$.*
(2) BF left (right) ideal *on $\mathfrak{F}$ if $\omega^P(h_1h_2) \geq \omega^P(h_2)$ ($\omega^P(h_1h_2) \geq \omega^P(h_1)$) and $\omega^N(h_1h_2) \leq \omega^N(h_2)$ ($\omega^N(h_1h_2) \leq \omega^N(h_1)$) for all $h_1, h_2 \in \mathfrak{F}$. A BF set $\omega = (\mathfrak{F}; \omega^P, \omega^N)$ on $\mathfrak{F}$ is called a* BF ideal *on $\mathfrak{F}$ if it is both a BF left ideal and a BF right ideal ob $\mathfrak{F}$.*
(3) BF interior ideal *on $\mathfrak{F}$ if $\omega = (\mathfrak{F}; \omega^P, \omega^N)$ is a BF subsemigroup on $\mathfrak{F}$, $\omega^P(h_1h_2h_3) \geq \omega^P(h_2)$ and $\omega^N(h_1h_2h_3) \leq \omega^N(h_2)$ for all $h_1, h_2, h_3 \in \mathfrak{F}$.*

Definition 2.4. *[17] A* bipolar complex fuzzy set *(shortly, BCF set) ω^{RI} on $\mathfrak{F}$ is an object having the form $\omega^{RI} := \{(\mathfrak{F}, \omega^P(h) = \omega^{RP}(h) + \iota\omega^{IP}(h), \omega^N(h) = \omega^{RN}(h) + \iota\omega^{IN}(h)) \mid h \in \mathfrak{F}\}$, is called the positive supportive grade and negative supportive grade respectively, where $\omega^{RP}, \omega^{IP} : \mathfrak{F} \to [0,1]$, and $\omega^{RN}, \omega^{IN} : \mathfrak{F} \to [-1,0]$.*

Remark 2.5. *For the sake of simplicity we shall use the symbol $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ for the BCF set $\omega^{RI} = \{(\mathfrak{F}, \omega^{RP}(h) + \iota\omega^{IP}(h), \omega^{RN}(h) + \iota\omega^{IN}(h)) \mid h \in \mathfrak{F}\}$.*

Definition 2.6. *[17] A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ on $\mathfrak{F}$ is called a* BCF subsemigroup *on $\mathfrak{F}$ if for all $h_1, h_2 \in \mathfrak{F}$,*

(1) $\omega^P(h_1h_2) \geq \omega^P(h_1) \wedge \omega^P(h_2) \Rightarrow \omega^{RP}(h_1h_2) \geq \omega^{RP}(h_1) \wedge \omega^{RP}(h_2)$ *and* $\omega^{IP}(h_1h_2) \geq \omega^{IP}(h_1) \wedge \omega^{IP}(h_2)$
(2) $\omega^N(h_1h_2) \leq \omega^N(h_1) \vee \omega^N(h_2) \Rightarrow \omega^{RN}(h_1h_2) \leq \omega^{RN}(h_1) \vee \omega^{RN}(h_2)$ *and* $\omega^{IN}(h_1h_2) \leq \omega^{IN}(h_1) \vee \omega^{IN}(h_2)$.

Definition 2.7. *[17] A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ on $\mathfrak{F}$ is called a* BCF left (right) ideal *on $\mathfrak{F}$ if for all $h_1, h_2 \in \mathfrak{F}$,*

(1) $\omega^P(h_1h_2) \geq \omega^P(h_2)(\omega^P(h_1h_2) \geq \omega^P(h_1)) \Rightarrow \omega^{RP}(h_1h_2) \geq \omega^{RP}(h_2)$
$(\omega^{RP}(h_1h_2) \geq \omega^{RP}(h_1))$
and $\omega^{IP}(h_1h_2) \geq \omega^{IP}(h_2)(\omega^{IP}(h_1h_2) \geq \omega^{IP}(h_1))$
(2) $\omega^N(h_1h_2) \leq \omega^N(h_2)(\omega^N(h_1h_2) \geq \omega^N(h_1)) \Rightarrow \omega^{RN}(h_1h_2) \leq \omega^{RN}(h_2)(\omega^{RN}(h_1h_2) \geq \omega^{RN}(h_1))$ *and* $\omega^{IN}(h_1h_2) \leq \omega^{IN}(h_2)(\omega^{IN}(h_1h_2) \geq \omega^{IN}(h_1))$.

A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ on $\mathfrak{F}$ is called a *BCF ideal* on $\mathfrak{F}$ if it is both a BCF left ideal and a BCF right ideal on $\mathfrak{F}$.

Next, we review the definition of the characteristic bipolar complex fuzzy function.

Let $\mathfrak{K}$ be a non-empty subset of $\mathfrak{F}$. The characteristic bipolar complex fuzzy set (shortly, CBCF set) $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{F}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ is defined as follows:

$$\chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}(h) = \begin{cases} 1 + \iota 1 & \text{if} \quad h \in \mathfrak{K} \\ 0 + \iota 0 & \text{if} \quad h \notin \mathfrak{K}, \end{cases}$$

$$\chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}}(h) = \begin{cases} -1 - \iota 1 & \text{if} \quad h \in \mathfrak{K} \\ 0 + \iota 0 & \text{if} \quad h \notin \mathfrak{K}. \end{cases}$$

for all $h \in \mathfrak{F}$ and $\chi^{RI}_{\mathfrak{K}}$ is a characteristic bipolar complex fuzzy set.

In the following theorem, we give a relationship between a subsemigroup (left ideal, right ideal, ideal) and the BCF function which is proved easily.

Theorem 2.8. *[17] Let $\mathfrak{K}$ be a non-empty subset on $\mathfrak{F}$. Then $\mathfrak{K}$ is a subsemigroup (left ideal, right ideal, ideal) of $\mathfrak{F}$ if and only if $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{F}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ is a BCF subsemigroup (left ideal, right ideal, ideal) on $\mathfrak{F}$.*

Definition 2.9. *[17] A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ on $\mathfrak{F}$ with $\pi, \eta \in [0,1]$ and $\varrho, \sigma \in [-1,0]$. Define the set*

(1) $\mathcal{P}(\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)) = \{h \in \mathfrak{F} \mid \omega^{RP}(h) \geq \pi, \omega^{IP}(h) \geq \eta\}$ *is called* ***positive*** (π, η)***-cut*** *of a CBF set of $\mathfrak{F}$.*
(2) $\mathcal{N}(\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)) = \{h \in \mathfrak{F} \mid \omega^{RN}(h) \leq \varrho, \omega^{IN}(h) \leq \sigma\}$ *is called* ***negative*** (ϱ, σ)***-cut*** *of a CBF set of $\mathfrak{F}$.*
(3) $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\varpi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma))) = \mathcal{P}(\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)) \cap \mathcal{N}(\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma))$ *is called* $((\varpi, \eta), (\varrho, \sigma))$***-cut*** *of a CBF set on $\mathfrak{F}$.*

In the following theorems, we give a relationship between a subsemigroup (left ideal, right ideal, ideal) and the $((\pi, \eta), (\varrho, \sigma))$-cut of a BCF set which proved easily.

Theorem 2.10. *[17] A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup (left ideal, right ideal, ideal) of a semigroup $\mathfrak{F}$ if and only if the non-empty subset $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$ is a subsemigroup (left ideal, right ideal, ideal) on $\mathfrak{F}$ for all $\pi, \eta \in [0,1]$ and $\varrho, \sigma \in [-1,0]$.*

Next, we study intersection and product of BCF sets as define.
Let $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ and $\psi^{RI} = (\mathfrak{F}; \psi^P, \psi^N) = (\mathfrak{F}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ are BCF sets of $\mathfrak{F}$. Define

(1) $(\omega^{RI} \cap \psi^{RI})(h) = \omega^{RP}(h) \wedge \psi^{RP}(h)$, $\omega^{IP}(h) \wedge \psi^{IP}(h)$ and $\omega^{RN}(h) \vee \psi^{RN}(h)$, $\omega^{IN}(h) \vee \psi^{IN}(h)$ for all $h \in \mathfrak{F}$.
(2) $\omega^{RI}(h) \precsim \psi^{RI}(h) = \omega^{RP}(h) \leq \psi^{RP}(h)$, $\omega^{IP}(h) \leq \psi^{IP}(h)$ and $\omega^{RN}(h) \geq \psi^{RN}(h)$,
$\omega^{IN}(h) \geq \psi^{IN}(h)$ for all $h \in \mathfrak{F}$.
(3) $\omega^{RI} \circledcirc \psi^{RI} = (\mathfrak{F}; \omega^P \circ \psi^P, \omega^N \circ \psi^N) = (\mathfrak{F}; \omega^{RP} \circ \psi^{RP} + \iota\omega^{IP} \circ \psi^{IP}, \omega^{RN} \circ \psi^{RN} + \iota\omega^{IN} \circ \psi^{IN})$ where;
$$(\omega^{RP} \circ \psi^{RP})(h) = \begin{cases} \bigvee_{(k,o) \in A_h} \{\omega^{RP}(k) \wedge \psi^{RP}(o)\} & \text{if } A_h \neq \emptyset \\ 0 & \text{if } A_h = \emptyset, \end{cases}$$
$$(\omega^{IP} \circ \psi^{IP})(h) = \begin{cases} \bigvee_{(k,o) \in A_h} \{\omega^{IP}(k) \wedge \psi^{IP}(o)\} & \text{if } A_h \neq \emptyset \\ 0 & \text{if } A_h = \emptyset, \end{cases}$$
$$(\omega^{RN} \circ \psi^{RN})(h) = \begin{cases} \bigwedge_{(k,o) \in A_h} \{\omega^{RN}(k) \vee \psi^{RN}(o)\} & \text{if } A_h \neq \emptyset \\ 0 & \text{if } A_h = \emptyset, \end{cases}$$
$$(\omega^{IN} \circ \psi^{IN})(h) = \begin{cases} \bigwedge_{(k,o) \in A_h} \{\omega^{IN}(k) \vee \psi^{IN}(o)\} & \text{if } A_h \neq \emptyset \\ 0 & \text{if } A_h = \emptyset. \end{cases}$$

Obviously, the operation $\circledcirc$ is associative [17]. For $h \in \mathfrak{F}$, define $A_h := \{(k, o) \in \mathfrak{F} \times \mathfrak{F} \mid h = ko\}$.

Next, we study equivalent conditions are important properties for BCF subsemigroups of semigroups which are shown in the following theorems.

Theorem 2.11. *[17] A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$ if and only if $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$.*

III. Bipolar complex fuzzy interior ideals

In this part, we give the concepts of bipolar complex fuzzy interior ideals in semigroups and we study important properties of bipolar complex fuzzy interior ideals in semigroups.

Definition 3.1. *A BCF subsemigroup $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ on $\mathfrak{F}$ is called a* BCF interior ideal *on $\mathfrak{F}$ if for all $h_1, h_2, h_3 \in \mathfrak{F}$,*

(1) $\omega^P(h_1h_2h_3) \geq \omega^P(h_2) \Rightarrow \omega^{RP}(h_1h_2h_3) \geq \omega^{RP}(h_2)$ *and* $\omega^{IP}(h_1h_2h_3) \geq \omega^{IP}(h_2)$

(2) $\omega^N(h_1h_2h_3) \leq \omega^N(h_2) \Rightarrow \omega^{RN}(h_1h_2h_3) \leq \omega^{RN}(h_2)$ *and* $\omega^{IN}(h_1h_2h_3) \leq \omega^{IN}(h_2)$.

The following example is a BCF interior ideal of a semigroup.

Example 3.2. *Consider a semigroup $(\mathfrak{F}, \cdot)$ defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	c	b	a

A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ in $\mathfrak{F}$ as follows:
$\omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN} = \{(a, (0.7 + \iota0.8, -0.6 - \iota0.7)), (b, (0.4 + \iota0.6, -0.5 - \iota0.6)), (c, (0.6 + \iota0.7, -0.5 - \iota0.6)), ((d, 0.3 + \iota0.5, -0.3 - \iota0.5))\}$. *By routine calculation, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$.*

Theorem 3.3. *Every BCF ideal of a semigroup $\mathfrak{F}$ is a BCF interior ideal of $\mathfrak{F}$.*

Proof: Let $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ be a BCF ideal of $\mathfrak{F}$ and let $h_1, h_2 \in \mathfrak{F}$. Then $\omega = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF left ideal and BCF right ideal of $\mathfrak{F}$. Thus,

(1) $\omega^P(h_1h_2) \geq \omega^P(h_2) \Rightarrow \omega^{RP}(h_1h_2) \geq \omega^{RP}(h_2)$ and $\omega^{IP}(h_1h_2) \geq \omega^{IP}(h_2)$,

(2) $\omega^N(h_1h_2) \leq \omega^N(h_2) \Rightarrow \omega^{RN}(h_1h_2) \leq \omega^{RN}(h_2)$ and $\omega^{IN}(h_1h_2) \leq \omega^{IN}(h_2)$.

Hence,

(1) $\omega^P(h_1h_2) \geq \omega^P(h_1) \wedge \omega^P(h_2) \Rightarrow \omega^{RP}(h_1h_2) \geq \omega^{RP}(h_1) \wedge \omega^{RP}(h_2)$ and $\omega^{IP}(h_1h_2) \geq \omega^{IP}(h_1) \wedge \omega^{IP}(h_2)$,

(2) $\omega^N(h_1h_2) \leq \omega^N(h_1) \vee \omega^N(h_2) \Rightarrow \omega^{RN}(h_1h_2) \leq \omega^{RN}(h_1) \vee \omega^{RN}(h_2)$ and $\omega^{IN}(h_1h_2) \leq \omega^{IN}(h_1) \vee \omega^{IN}(h_2)$.

This show that $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$.

Let $h_1, h_2, h_3 \in \mathfrak{F}$. Then,

(1) $\omega^P(h_1h_2h_3) = \omega^P(h_1(h_2h_3)) \geq \omega^P(h_2h_3) \geq \omega^P(h_2) \Rightarrow \omega^{RP}(h_1h_2h_3) = \omega^{RP}(h_1(h_2h_3)) \geq \omega^{RP}(h_2h_3) \geq \omega^{RP}(h_2)$ and $\omega^{IP}(h_1h_2h_3) = \omega^{IP}(h_1(h_2h_3)) \geq \omega^{IP}(h_2h_3) \geq \omega^{IP}(h_2)$,

(2) $\omega^N(h_1h_2h_3) = \omega^N(h_1(h_2h_3)) \leq \omega^N(h_2h_3) \leq \omega^N(h_2) \Rightarrow \omega^{RN}(h_1h_2h_3) = \omega^{RN}(h_1(h_2h_3)) \leq \omega^{RN}(h_2h_3) \leq \omega^{RN}(h_2)$ and $\omega^{IN}(h_1h_2h_3) = \omega^{IN}(h_1(h_2h_3)) \leq \omega^{IN}(h_2h_3) \leq \omega^{IN}(h_2)$.

Therefore, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. ■

Remark 3.4. *In example 3.2 we can show that the converse of the above theorem is not true in general.*
Consider $\omega^{RP} + \iota\omega^{IP}(bd) = \omega^{RP} + \iota\omega^{IP}(d) = 0.3 + \iota0.5$ and $\omega^{RN} + \iota\omega^{IN}(bd) = \omega^{RN} + \iota\omega^{IN}(d) = -0.3 - \iota0.5$. Then,

(1) $\omega^P(bd) = \omega^P(d) = 0.3 \ngeq 0.4 = \omega^P(b) \Rightarrow \omega^{RP}(bd) = \omega^{RP}(d) = 0.3 \ngeq 0.4 = \omega^{RP}(b)$ *and* $\omega^{IP}(bd) = \omega^{IP}(d) = 0.5 \ngeq 0.6 = \omega^{IP}(b)$.

(2) $\omega^N(bd) = \omega^N(d) = -0.3 \nleq -0.5 = \omega^N(b) \Rightarrow \omega^{RN}(bd) = \omega^{RN}(d) = -0.3 \nleq -0.5 = \omega^{RN}(b)$ *and* $\omega^{IN}(bd) = \omega^{IN}(d) = -0.5 \nleq -0.6 = \omega^{IN}(b)$.

Thus, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is not a BCF ideal of $\mathfrak{F}$.

The following theorem shows that the BCF interior ideals and BCF ideals coincide for some types of semigroups.

Theorem 3.5. *In regular, left (right) regular, intra-regular, and semisimple semigroup, the BCF interior ideals and BCF ideals coincide.*

Proof: Let $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ be a BCF interior ideal of a regular semigroup and let $h_1, h_2 \in \mathfrak{F}$. Since $\mathfrak{F}$ is regular, we have there exists $k \in \mathfrak{F}$ such that $h_1 = h_1kh_1$. Thus,

(1) $\omega^P(h_1h_2) = \omega^P((h_1kh_1)h_2) = \omega^P((h_1k)h_1h_2) \geq \omega^P(h_1) \Rightarrow \omega^{RP}(h_1h_2) = \omega^{RP}((h_1kh_1)h_2) = \omega^{RP}((h_1k)h_1h_2) \geq \omega^{RP}(h_1)$ and $\omega^{IP}(h_1h_2) = \omega^{IP}((h_1kh_1)h_2) = \omega^{IP}((h_1k)h_1h_2) \geq \omega^{IP}(h_1)$

(2) $\omega^N(h_1h_2) = \omega^N((h_1kh_1)h_2) = \omega^N((h_1k)h_1h_2) \leq \omega^N(h_1) \Rightarrow \omega^{RN}(h_1h_2) = \omega^{RN}((h_1kh_1)h_2) = \omega^{RN}((h_1k)h_1h_2) \leq \omega^{RN}(h_1)$ and $\omega^{IN}(h_1h_2) = \omega^{IN}((h_1kh_1)h_2) = \omega^{IN}((h_1k)h_1h_2) \leq \omega^{IN}(h_1)$.

Hence, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF right ideal of $\mathfrak{F}$. Similarly, we can prove that $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF left ideal of $\mathfrak{F}$. Thus, $\omega^{RI} = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF ideal of $\mathfrak{F}$.

Similarly, we can prove the other cases also. ■

The following theorems are basic properties.

Theorem 3.6. *Let $\mathfrak{K}$ be a non-empty subset on $\mathfrak{F}$. Then $\mathfrak{K}$ is an interior ideal of $\mathfrak{F}$ if and only if $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{F}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ is a BCF interior ideal of $\mathfrak{F}$.*

Proof: Suppose that $\mathfrak{K}$ is an interior ideal on $\mathfrak{F}$. Then $\mathfrak{K}$ is a subsemigroup on $\mathfrak{F}$. Thus by Theorem 2.8, $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{F}; \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ is a BCF subsemigrup of $\mathfrak{F}$. Let $h_1, h_2, h_3 \in \mathfrak{F}$.

If $h_2 \in \mathfrak{K}$, then $h_1h_2h_3 \in \mathfrak{K}$. Thus, $1 + \iota1 = \chi^{RP}_{\mathfrak{K}}(h_2) = \chi^{IP}_{\mathfrak{K}}(h_2) = \chi^{RP}_{\mathfrak{K}}(h_1h_2h_3) = \chi^{IP}_{\mathfrak{K}}(h_1h_2h_3)$ and $-1 - \iota1 =$

$\chi_{\mathfrak{K}}^{RN}(h_2) = \chi_{\mathfrak{K}}^{IN}(h_2) = \chi_{\mathfrak{K}}^{RN}(h_1h_2h_3) = \chi_{\mathfrak{K}}^{IN}(h_1h_2h_3)$. Hence,

(1) $\chi_{\mathfrak{K}}^{P}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{P}(h_2) \Rightarrow \chi_{\mathfrak{K}}^{RP}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{RP}(h_2)$ and $\chi_{\mathfrak{K}}^{IP}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{IP}(h_2)$,
(2) $\chi_{\mathfrak{K}}^{N}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{N}(h_2) \Rightarrow \chi_{\mathfrak{K}}^{RN}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{RN}(h_2)$ and $\chi_{\mathfrak{K}}^{IN}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{IN}(h_2)$.

If $h_2 \notin \mathfrak{K}$, then $0 + \iota 0 = \chi_{\mathfrak{K}}^{P}(h_2)$ and $0 + \iota 0 = \chi_{\mathfrak{K}}^{N}(h_2)$. Thus,

(1) $\chi_{\mathfrak{K}}^{P}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{P}(h_2) \Rightarrow \chi_{\mathfrak{K}}^{RP}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{RP}(h_2)$ and $\chi_{\mathfrak{K}}^{IP}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{IP}(h_2)$,
(2) $\chi_{\mathfrak{K}}^{N}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{N}(h_2) \Rightarrow \chi_{\mathfrak{K}}^{RN}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{RN}(h_2)$ and $\chi_{\mathfrak{K}}^{IN}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{IN}(h_2)$.

Hence, $\chi_{\mathfrak{K}}^{RI} = (\mathfrak{F}; \chi_{\mathfrak{K}}^{P}, \chi_{\mathfrak{K}}^{N}) = (\mathfrak{F}; \chi_{\mathfrak{K}}^{RP} + \iota\chi_{\mathfrak{K}}^{IP}, \chi_{\mathfrak{K}}^{RN} + \iota\chi_{\mathfrak{K}}^{IN})$ is a BCF interior ideal of $\mathfrak{F}$.

Conversely, suppose that $\chi_{\mathfrak{K}}^{RI} = (\mathfrak{F}; \chi_{\mathfrak{K}}^{P}, \chi_{\mathfrak{K}}^{N}) = (\mathfrak{F}; \chi_{\mathfrak{K}}^{RP} + \iota\chi_{\mathfrak{K}}^{IP}, \chi_{\mathfrak{K}}^{RN} + \iota\chi_{\mathfrak{K}}^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. Then $\chi_{\mathfrak{K}}^{RI} = (\mathfrak{F}; \chi^{P}, \chi^{N}) = (\mathfrak{F}; \chi_{\mathfrak{K}}^{RP} + \iota\chi_{\mathfrak{K}}^{IP}, \chi_{\mathfrak{K}}^{RN} + \iota\chi_{\mathfrak{K}}^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$. Thus by Theorem 2.8, $\mathfrak{K}$ is a subsemigroup of $\mathfrak{F}$.
Let $h_1, h_2, h_3 \in \mathfrak{F}$ and $h_2, \in \mathfrak{K}$. Then $1 + \iota 1 = \chi_{\mathfrak{K}}^{P}(h_2) = \chi_{\mathfrak{K}}^{RP}(h_2) = \chi_{\mathfrak{K}}^{IP}(h_2)$ and $-1 - \iota 1 = \chi_{\mathfrak{K}}^{N}(h_2) = \chi_{\mathfrak{K}}^{RN}(h_2) = \chi_{\mathfrak{K}}^{IN}(h_2)$. By assumtion,

(1) $\chi_{\mathfrak{K}}^{P}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{P}(h_2) \Rightarrow \chi_{\mathfrak{K}}^{RP}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{RP}(h_2)$ and $\chi_{\mathfrak{K}}^{IP}(h_1h_2h_3) \geq \chi_{\mathfrak{K}}^{IP}(h_2)$
(2) $\chi_{\mathfrak{K}}^{N}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{N}(h_2) \Rightarrow \chi_{\mathfrak{K}}^{RN}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{RN}(h_2)$ and $\chi_{\mathfrak{K}}^{IN}(h_1h_2h_3) \leq \chi_{\mathfrak{K}}^{IN}(h_2)$.

Thus, $h_1h_2h_3 \in \mathfrak{K}$. Therefore, $\mathfrak{K}$ is an interior ideal on $\mathfrak{F}$. ∎

Theorem 3.7. *A BCF set $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of a semigroup $\mathfrak{F}$ if and only if the non-empty subset $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$ is an interior ideal of $\mathfrak{F}$ for all $\pi, \eta \in [0, 1]$ and $\varrho, \sigma \in [-1, 0]$.*

Proof: Let $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. Then $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$. Thus by Theorem 2.10, $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$ is a subsemigroups of $\mathfrak{F}$.
Let $h_1, h_2, h_3 \in \mathfrak{F}, \pi, \eta \in [0, 1]$ and $\varrho, \sigma \in [-1, 0]$.

If h_2 is an element of $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$, then $\omega^{RP}(h_2) \geq \pi, \omega^{IP}(h_2) \geq \eta$ and $\omega^{RN}(h_2) \leq \varrho, \omega^{IN}(h_2) \leq \sigma$. By assumption,

(1) $\omega^{P}(h_1h_2h_3) \geq \omega^{P}(h_2) \Rightarrow \omega^{RP}(h_1h_2h_3) \geq \omega^{RP}(h_2)$ and $\omega^{IP}(h_1h_2h_3) \geq \omega^{IP}(h_2)$
(2) $\omega^{N}(h_1h_2h_3) \leq \omega^{N}(h_2) \Rightarrow \omega^{RN}(h_1h_2h_3) \leq \omega^{RN}(h_2)$ and $\omega^{IN}(h_1h_2h_3) \leq \omega^{IN}(h_2)$.

Thus, $h_1h_2h_3$ is an element of $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$.
Hence, $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$ is an interior ideals of $\mathfrak{F}$.

Conversely, suppose that $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$ is an interior ideals of $\mathfrak{F}$. By assumption, $\mathcal{PN}((\omega^{RP} + \iota\omega^{IP}, (\pi, \eta)), (\omega^{RN} + \iota\omega^{IN}, (\varrho, \sigma)))$ is a subsemigroups of $\mathfrak{F}$. Thus by Theorem 2.10, $\omega^{RI} = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$. Let $h_1, h_2, h_3 \in \mathfrak{F}, \pi, \eta \in [0, 1]$ and $\varrho, \sigma \in [-1, 0]$. By assumetion, $h_1h_2h_3$ is an element of $\omega^{RI} = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$. Thus, $\omega^{RP}(h_1h_2h_3) \geq \pi = \omega^{RP}(h_2)$, $\omega^{IP}(h_1h_2h_3) \geq \eta = \omega^{RP}(h_2)$, $\omega^{RN}(h_1h_2h_3) \leq \varrho = \omega^{RN}(h_2)$ and $\omega^{IN}(h_1h_2h_3) \leq \sigma = \omega^{RN}(h_2)$. Hence, $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. ∎

Some equivalent conditions are important properties for a CBF interior ideal of a semigroup which are shown in the following theorem.

Theorem 3.8. *A CBF set $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$ if and only if $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$ and $\mathfrak{F}^{RI} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{RI} \precsim \omega^{PN}$ where $\mathfrak{F}^{RI} = (\mathfrak{F}; \mathfrak{F}^{P}, \mathfrak{F}^{N}) = (\mathfrak{F}^{RP} + \iota\mathfrak{F}^{IP}, \mathfrak{F}^{RN} + \iota\mathfrak{F}^{IN})$ is a BCF set of $\mathfrak{F}$.*

Proof: Assume that $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. Then $\omega^{RI} = (\mathfrak{F}; \omega^{P}, \omega^{N}) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$. Thus by Theorem 2.11, $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$.
Let $h \in \mathfrak{F}$. Then $(\mathfrak{F}^{RI} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{PN})(h) = ((\mathfrak{F}^{RI} \circ \omega^{RI}) \circ \mathfrak{F}^{RI})(h)$.

If $A_h = \emptyset$, then it is easy to verify that, $(\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h) \leq \omega^{RP}(h)$, $(\mathfrak{F}^{IP} \circ \omega^{IP}) \circ \mathfrak{F}^{IP}(h) \leq \omega^{IP}(h)$ and $(\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h) \geq \omega^{RN}(h)$, $(\mathfrak{F}^{IN} \circ \omega^{IN}) \circ \mathfrak{F}^{IN}(h) \geq \omega^{IN}(h)$.

If $A_h \neq \emptyset$, then

$$\begin{aligned}
&(\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h) \\
&= \bigvee_{(k,o)\in A_h} \{(\mathfrak{F}^{RP} \circ \omega^{RP})(k) \wedge \mathfrak{F}^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_h} \{(\mathfrak{F}^{RP} \circ \omega^{RP})(k) \wedge \mathfrak{F}^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_h} \{ \bigvee_{(u,t)\in A_k} \{\mathfrak{F}^{RP}(u) \wedge \omega^{RP}(t)\} \wedge \mathfrak{F}^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_h} \{ \bigvee_{(u,t)\in A_k} \{1 \wedge \omega^{RP}(t)\} \wedge 1\} \\
&= \bigvee_{(k,o)\in A_h} \{\omega^{RP}(t)\} \leq \bigvee_{(k,o)\in A_h} \{\omega^{RP}(uto)\} \\
&= \omega^{RP}(h).
\end{aligned}$$

Thus, $(\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h) \leq \omega^{RP}(h)$. Similarly, we can show that $(\mathfrak{F}^{IP} \circ \omega^{IP}) \circ \mathfrak{F}^{IP}(h) \leq \omega^{IP}(h)$. And

$$\begin{aligned}
&(\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h) \\
&= \bigwedge_{(k,o)\in A_h} \{(\mathfrak{F}^{RN} \circ \omega^{RN})(k) \vee \mathfrak{F}^{RN}(o)\} \\
&= \bigwedge_{(k,o)\in A_h} \{ \bigwedge_{(u,t)\in A_k} \{\mathfrak{F}^{RN}(u) \vee \omega^{RN}(t)\} \vee \mathfrak{F}^{RN}(o)\} \\
&= \bigwedge_{(k,o)\in A_h} \{ \bigwedge_{(u,t)\in A_k} \{-1 \vee \omega^{RP}(t)\} \vee -1\} \\
&= \bigwedge_{(k,o)\in A_h} \{\omega^{RN}(t)\} \geq \bigwedge_{(k,o)\in A_h} \{\omega^{RN}(uto)\} = \omega^{RN}(h).
\end{aligned}$$

Thus, $(\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h) \geq \omega^{RN}(h)$. Similarly, we can show that $(\mathfrak{F}^{IN} \circ \omega^{IN}) \circ \mathfrak{F}^{IN}(h) \geq \omega^{IN}(h)$. Hence, $\mathfrak{F}^{PN} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{PN} \precsim \omega^{PN}$.

Conversely, suppose that $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$ and $\mathfrak{F}^{PN} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{PN} \precsim \omega^{PN}$. Let $h_1, h_2 \in \mathfrak{F}$.
Since $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$, we have $\omega^{RI} = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$, by Theorem 2.11.

Let $h_1, h_2, h_3 \in \mathfrak{F}$. Then by assumption, $(\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h_1h_2h_3) \leq \omega^{RP}(h_1h_2h_3)$, $(\mathfrak{F}^{IP} \circ \omega^{IP}) \circ \mathfrak{F}^{IP}(h_1h_2h_3) \leq \omega^{IP}(h_1h_2h_3)$, $(\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h_1h_2h_3) \geq \omega^{RN}(h_1h_2h_3)$ and $(\mathfrak{F}^{IN} \circ \omega^{IN}) \circ \mathfrak{F}^{IN}(h_1h_2h_3) \geq \omega^{IN}(h_1h_2h_3)$. Thus,

$$\begin{aligned}
&\omega^{RP}(h_1h_2h_3) \geq (\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h_1h_2h_3) \\
&= \bigvee_{(k,o)\in A_{h_1h_2h_3}} \{(\mathfrak{F}^{RP} \circ \omega^{RP}(k)) \wedge \mathfrak{F}^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_{h_1h_2h_3}} \{ \bigvee_{(u,t)\in A_k} \{\mathfrak{F}^{RP}(u) \wedge \omega^{RP}(t)\} \wedge \mathfrak{F}^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_{h_1h_2h_3}} \{ \bigvee_{(u,t)\in A_k} \{1 \wedge \omega^{RP}(t)\} \wedge 1\} \\
&= \bigvee_{(k,o)\in A_{h_1h_2h_3}} \{\omega^{RP}(t)\} \geq \omega^{RP}(h_2).
\end{aligned}$$

Hence, $\omega^{RP}(h_1h_2h_3) \geq \omega^{RP}(h_2)$. Similarly, we can show that $\omega^{IP}(h_1h_2h_3) \geq \omega^{IP}(h_2)$.
Therefore, $\omega^{P}(h_1h_2h_3) \geq \omega^{P}(h_2) \Rightarrow \omega^{RP}(h_1h_2h_3) \geq \omega^{RP}(h_2)$ and $\omega^{IP}(h_1h_2h_3) \geq \omega^{IP}(h_2)$.

$$\begin{aligned}
&\omega^{RN}(h_1h_2h_3) \leq (\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h_1h_2h_3) \\
&= \bigwedge_{(k,o)\in A_{h_1h_2h_3}} \{(\mathfrak{F}^{RN} \circ \omega^{RN}(k)) \vee \mathfrak{F}^{RP}(o)\} \\
&= \bigwedge_{(k,o)\in A_{h_1h_2h_3}} \{ \bigwedge_{(u,t)\in A_k} \{\mathfrak{F}^{RN}(u) \vee \omega^{RN}(t)\} \vee \mathfrak{F}^{RN}(o)\} \\
&= \bigwedge_{(k,o)\in A_{h_1h_2h_3}} \{ \bigwedge_{(u,t)\in A_k} \{-1 \vee \omega^{RN}(t)\} \vee -1\} \\
&= \bigwedge_{(k,o)\in A_{h_1h_2h_3}} \{\omega^{RN}(t)\} \leq \omega^{RN}(h_2).
\end{aligned}$$

Hence, $\omega^{RN}(h_1h_2h_3) \leq \omega^{RN}(h_2)$. Similarly, we can show that $\omega^{IN}(h_1h_2h_3) \leq \omega^{IN}(h_2)$.
Therefore, $\omega^{N}(h_1h_2h_3) \leq \omega^{N}(h_2) \Rightarrow \omega^{RN}(h_1h_2h_3) \leq \omega^{RN}(h_2)$ and $\omega^{IN}(h_1h_2h_3) \leq \omega^{IN}(h_2)$.

Consequently, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. ■

The following theorem is an important property for an equivalent of a BCF interior ideal of a left (right, intra-) regular semigroup.

Theorem 3.9. *For any left (right, intra-) regular semigroup $\mathfrak{F}$, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$ if and only if $\omega^{RI} \circledcirc \omega^{RI} = \omega^{RI}$ and $\mathfrak{F}^{RI} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{RI} = \omega^{RI}$.*

Proof: Assume that $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$ and let $h \in \mathfrak{F}$. Since $\mathfrak{F}$ is left regular, there exists $r \in \mathfrak{F}$ such that $h = rh^2 = r(hh) = (rh)h = (r(rh^2))h = (rrhh)h = ((r^2)hh)(rh^2) = (r^2hh)(rhh)$. Thus,

$$\begin{aligned}
&(\omega^{RP} \circ \omega^{RP})(h) = \bigvee_{(k,o)\in A_h} \{\omega^{RP}(k) \wedge \omega^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_{(r^2hh)(rhh)}} \{\omega^{RP}(k) \wedge \omega^{RP}(o)\} \\
&\geq (\omega^{RP}(r^2hh) \wedge \omega^{RP}(rhh)) \geq \omega^{RP}(h) \wedge \omega^{RP}(h) \\
&= \omega^{RP}(h).
\end{aligned}$$

Hence, $(\omega^{RP} \circ \omega^{RP})(h) \geq \omega^{RP}(h)$. Similarly, we can show that $(\omega^{IP} \circ \omega^{IP})(h) \geq \omega^{IP}(h)$. And

$$\begin{aligned}
&(\omega^{RN} \circ \omega^{RN})(h) = \bigwedge_{(k,o)\in A_h} \{\omega^{RN}(k) \vee \omega^{RN}(o)\} \\
&= \bigwedge_{(k,o)\in A_{(r^2hh)(rhh)}} \{\omega^{RN}(k) \vee \omega^{RN}(o)\} \\
&\leq (\omega^{RN}(r^2hh) \vee \omega^{RN}(rhh)) \\
&\leq \omega^{RN}(h) \vee \omega^{RN}(h) \bigwedge = \omega^{RN}(h).
\end{aligned}$$

Hence, $(\omega^{RN} \circ \omega^{RN})(h) \leq \omega^{RN}(h)$. Similarly, we can show that $(\omega^{IN} \circ \omega^{IN})(h) \leq \omega^{IN}(h)$.
Therefore, $\omega^{RI} \circledcirc \omega^{RI} \succsim \omega^{RI}$.

Since $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$, we have $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$, by Theorem 2.11. Thus, $\omega^{RI} \circledcirc \omega^{RI} = \omega^{RI}$. Since $\mathfrak{F}$ is left regular, there exists $r \in \mathfrak{F}$ such that $h = rh^2 = rhh = r(rhh)h$. Thus,

$$\begin{aligned}
&(\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h) = \bigvee_{(k,o)\in A_h} \{(\mathfrak{F}^{RP} \circ \omega^{RP}(k)) \\
&\wedge \mathfrak{F}^{RP}(o)\} = \bigvee_{(k,o)\in A_{r(rhh)h}} \{(\mathfrak{F}^{RP} \circ \omega^{RP}(k)) \wedge \mathfrak{F}^{RP}(o)\} \\
&\geq (\mathfrak{F}^{RP} \circ \omega^{RP})(r(rhh)) \wedge \mathfrak{F}^{RP}(h) \\
&= (\mathfrak{F}^{RP} \circ \omega^{RP})(r(rhh)) \wedge 1 = (\mathfrak{F}^{RP} \circ \omega^{RP})(r(rhh)) \\
&= \bigvee_{(k,o)\in A_{r(rhh)}} \{\mathfrak{F}^{RP}(k) \wedge \omega^{RP}(o)\} \geq \mathfrak{F}^{RP}(r) \wedge \omega^{RP}(rhh) \\
&= 1 \wedge \omega^{RP}(rhh) = \omega^{RP}(rhh \geq \omega^{RP}(h).
\end{aligned}$$

Hence, $(\mathfrak{F}^{RP} \circ \omega^{RP}) \circ \mathfrak{F}^{RP}(h) \geq \omega^{RP}(h)$. Similarly, we can show that $(\mathfrak{F}^{IP} \circ \omega^{IP}) \circ \mathfrak{F}^{IP}(h) \geq \omega^{IP}(h)$. And

$$\begin{aligned}
&(\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h) = \bigwedge_{(k,o)\in A_h} \{(\mathfrak{F}^{RN} \circ \omega^{RN}(k)) \\
&\vee \mathfrak{F}^{RN}(o)\} = \bigwedge_{(k,o)\in A_{r(rhh)h}} \{(\mathfrak{F}^{RN} \circ \omega^{RN}(k)) \vee \mathfrak{F}^{RN}(o)\} \\
&\leq (\mathfrak{F}^{RP} \circ \omega^{RN})(r(rhh)) \vee \mathfrak{F}^{RN}(h) \\
&= (\mathfrak{F}^{RN} \circ \omega^{RN})(r(rhh)) \vee -1 = (\mathfrak{F}^{RN} \circ \omega^{RN})(r(rhh)) \\
&= \bigwedge_{(k,o)\in A_{r(rhh)}} \{\mathfrak{F}^{RN}(k) \vee \omega^{RN}(o)\} \leq \mathfrak{F}^{RN}(r) \vee \omega^{RP}(rhh) \\
&= -1 \vee \omega^{RN}(rhh) = \omega^{RN}(rhh) \leq \omega^{RN}(h).
\end{aligned}$$

Hence, $(\mathfrak{F}^{RN} \circ \omega^{RN}) \circ \mathfrak{F}^{RN}(h) \leq \omega^{RN}(h)$. Similarly, we can show that $(\mathfrak{F}^{IN} \circ \omega^{IN}) \circ \mathfrak{F}^{IN}(h) \leq \omega^{IN}(h)$. Hence, $\mathfrak{F}^{RI} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{RI} \succsim \omega^{RI}$. By Theorem 3.8 we have $\mathfrak{F}^{RI} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{RI} \precsim \omega^{RI}$. Thus, $\mathfrak{F}^{RI} \circledcirc \omega^{RI} \circledcirc \mathfrak{F}^{RI} = \omega^{RI}$.

For the converse, it follows from Theorem 3.8.

Similarly, we can prove the other cases also. ■

Some equivalent conditions are important for a BCF interior ideal of semisimple and regular semigroups.

Theorem 3.10. *Let $\mathfrak{F}$ be a semisimple semigroup. If $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$, then $\omega^{RI} \circledcirc \omega^{RI} = \omega^{RI}$.*

Proof: Assume that $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$ and let $h \in \mathfrak{F}$. Since $\mathfrak{F}$ is semisimple, there exist $r, t, e, d \in \mathfrak{F}$ such that $h = rhtehd$. Thus,

$$\begin{aligned}
&(\omega^{RP} \circ \omega^{RP})(h) = \bigvee_{(k,o)\in A_h} \{\omega^{RP}(k) \wedge \omega^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_{(rht)(ehd)}} \{\omega^{RP}(k) \wedge \omega^{RP}(o)\} \\
&\geq (\omega^{RP}(rht) \wedge \omega^{RP}(ehd)) \geq \omega^{RP}(h) \wedge \omega^{RP}(h) \\
&= \omega^{RP}(h).
\end{aligned}$$

Hence, $(\omega^{RP} \circ \omega^{RP})(h) \geq \omega^{RP}(h)$. Similarly, we can show that $(\omega^{IP} \circ \omega^{IP})(h) \geq \omega^{IP}(h)$. And

$$\begin{aligned}
&(\omega^{RN} \circ \omega^{RN})(h) = \bigwedge_{(k,o)\in A_h} \{\omega^{RN}(k) \vee \omega^{RN}(o)\} \\
&= \bigwedge_{(k,o)\in A_{(rht)(ehd)}} \{\omega^{RN}(k) \vee \omega^{RN}(o)\} \\
&\leq (\omega^{RN}(rht) \vee \omega^{RN}(ehd)) \leq \omega^{RN}(h) \vee \omega^{RN}(h) \\
&= \omega^{RN}(h).
\end{aligned}$$

Hence, $(\omega^{RN} \circ \omega^{RN})(h) \leq \omega^{RN}(h)$. Similarly, we can show that $(\omega^{IN} \circ \omega^{IN})(h) \leq \omega^{IN}(h)$. Therefore, $\omega^{RI} \circledcirc \omega^{RI} \succsim \omega^{RI}$.

Since $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \imath\omega^{IP}, \omega^{RN} + \imath\omega^{IN})$ is a BCF subsemigroup of $\mathfrak{F}$, we have $\omega^{RI} \circledcirc \omega^{RI} \precsim \omega^{RI}$, by Theorem 2.11. Thus, $\omega^{RI} \circledcirc \omega^{RI} = \omega^{RI}$. ■

Theorem 3.11. *Let $\mathfrak{F}$ be a regular semigroup. If $\omega^{RI} = (\mathfrak{F};\omega^P,\omega^N) = (\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$, then $\mathfrak{F}^{RI}\circledcirc\omega^{RI}\circledcirc\mathfrak{F}^{RI}=\omega^{RI}$.*

Proof: Assume that $\omega^{RI} = (\mathfrak{F};\omega^P,\omega^N) = (\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$ and let $h\in\mathfrak{F}$. Since $\mathfrak{F}$ is regular, there exists $r\in\mathfrak{F}$ such that $h=hrh=(hrh)rh=h(rhr)h$. Thus,

$$\begin{aligned}
&(\mathfrak{F}^{RP}\circ\omega^{RP})\circ\mathfrak{F}^{RP}(h)\\
&= \bigvee_{(k,o)\in A_h}\{(\mathfrak{F}^{RP}\circ\omega^{RP}(k))\wedge\mathfrak{F}^{RP}(o)\}\\
&= \bigvee_{(k,o)\in A_{h(rhr)h}}\{(\mathfrak{F}^{RP}\circ\omega^{RP}(k))\wedge\mathfrak{F}^{RP}(o)\}\\
&\geq (\mathfrak{F}^{RP}\circ\omega^{RP})(r(rhr))\wedge\mathfrak{F}^{RP}(h)\\
&= (\mathfrak{F}^{RP}\circ\omega^{RP})(r(rhh))\wedge 1 = (\mathfrak{F}^{RP}\circ\omega^{RP})(r(rhh))\\
&= \bigvee_{(k,o)\in A_{r(rhh)}}\{\mathfrak{F}^{RP}(k)\wedge\omega^{RP}(o)\}\geq\mathfrak{F}^{RP}(r)\wedge\omega^{RP}(rhh)\\
&= 1\wedge\omega^{RP}(rhh)=\omega^{RP}(rhh)\geq\omega^{RP}(h).
\end{aligned}$$

Hence, $(\mathfrak{F}^{RP}\circ\omega^{RP})\circ\mathfrak{F}^{RP}(h)\geq\omega^{RP}(h)$.
Similarly, we can show that $(\mathfrak{F}^{IP}\circ\omega^{IP})\circ\mathfrak{F}^{IP}(h)\geq\omega^{IP}(h)$. And

$$\begin{aligned}
&(\mathfrak{F}^{RN}\circ\omega^{RN})\circ\mathfrak{F}^{RN}(h)\\
&= \bigwedge_{(k,o)\in A_h}\{(\mathfrak{F}^{RN}\circ\omega^{RN}(k))\vee\mathfrak{F}^{RN}(o)\}\\
&= \bigwedge_{(k,o)\in A_{h(rhr)h}}\{(\mathfrak{F}^{RN}\circ\omega^{RN}(k))\vee\mathfrak{F}^{RN}(o)\}\\
&\leq (\mathfrak{F}^{RP}\circ\omega^{RN})(r(rhr))\vee\mathfrak{F}^{RN}(h)\\
&= (\mathfrak{F}^{RN}\circ\omega^{RN})(r(rhh))\vee -1 = (\mathfrak{F}^{RN}\circ\omega^{RN})(r(rhh))\\
&= \bigwedge_{(k,o)\in A_{r(rhh)}}\{\mathfrak{F}^{RN}(k)\vee\omega^{RN}(o)\}\\
&\leq \mathfrak{F}^{RN}(r)\vee\omega^{RP}(rhh) = -1\vee\omega^{RN}(rhh)=\omega^{RN}(rhh)\\
&\leq\omega^{RN}(h).
\end{aligned}$$

Hence, $(\mathfrak{F}^{RN}\circ\omega^{RN})\circ\mathfrak{F}^{RN}(h)\leq\omega^{RN}(h)$. Similarly, we can show that $(\mathfrak{F}^{IN}\circ\omega^{IN})\circ\mathfrak{F}^{IN}(h)\leq\omega^{IN}(h)$. Hence, $\mathfrak{F}^{RI}\circledcirc\omega^{RI}\circledcirc\mathfrak{F}^{RI}\succsim\omega^{RI}$. By Theorem 3.8 we have $\mathfrak{F}^{RI}\circledcirc\omega^{RI}\circledcirc\mathfrak{F}^{RI}\precsim\omega^{RI}$. Thus, $\mathfrak{F}^{RI}\circledcirc\omega^{RI}\circledcirc\mathfrak{F}^{RI}=\omega^{RI}$. ■

IV. Characterize semisimple semigroups in terms of bipolar complex fuzzy interior ideal and bipolar complex fuzzy ideals.

In this topic, we will characterize a semisimple semigroup in terms of BCF interior ideals and BCF ideals.

Lemma 4.1. *If $\omega^{RI} = (\mathfrak{F};\omega^P,\omega^N) = (\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ is a BCF right ideal and $\psi^{RI} = (\mathfrak{F};\psi^P,\psi^N) = (\mathfrak{F};\psi^{RP}+\iota\psi^{IP},\psi^{RN}+\iota\psi^{IN})$ is a BCF left ideal of $\mathfrak{F}$, then $\omega^{RI}\circledcirc\psi^{RI}\precsim\omega^{RI}\cap\psi^{RI}$.*

Proof: Assume that $\omega^{RI} = (\mathfrak{F};\omega^P,\omega^N) = (\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ is a BCF right ideal and $\psi^{RI} = (\mathfrak{F};\psi^P,\psi^N) = (\mathfrak{F};\psi^{RP}+\iota\psi^{IP},\psi^{RN}+\iota\psi^{IN})$ is a BCF left ideal of $\mathfrak{F}$ and let $h\in\mathfrak{F}$.

If $A_h=\emptyset$, then it is easy to verify that, $(\omega^{RP}\circ\psi^{RP})(h)\leq(\omega^{RP}\wedge\psi^{RP})(h), (\omega^{IP}\circ\psi^{IP})(h)\leq(\omega^{IP}\wedge\psi^{IP})(h)$ and $(\omega^{RN}\circ\psi^{RN})(h)\geq(\omega^{RN}\vee\psi^{RN})(h), (\omega^{IN}\circ\psi^{IN})(h)\geq(\omega^{IN}\vee\psi^{IN})(h)$. If $A_h\neq\emptyset$, then

$$\begin{aligned}
&(\omega^{RP}\circ\psi^{RP})(h)\ \bigvee_{(k,o)\in A_h}\{\omega^{RP}(k)\wedge\psi^{RP}(o)\}\\
&\leq \bigvee_{(k,o)\in A_h}\{\omega^{RP}(ko)\wedge\psi^{RP}(ko)\}=\omega^{RP}(h)\wedge\psi^{RP}(h)\\
&= (\omega^{RP}\wedge\psi^{RP})(h).
\end{aligned}$$

Hence, $(\omega^{RP}\circ\psi^{RP})(h)\leq(\omega^{RP}\wedge\psi^{RP})(h)$. Similarly, we can show that $(\omega^{IP}\circ\psi^{IP})(h)\leq(\omega^{IP}\wedge\psi^{IP})(h)$. And

$$\begin{aligned}
&(\omega^{RN}\circ\psi^{RN})(h) = \bigwedge_{(k,o)\in A_h}\{\omega^{RN}(k)\vee\psi^{RN}(o)\}\\
&\geq \bigwedge_{(k,o)\in A_h}\{\omega^{RN}(ko)\vee\psi^{RN}(ko)\}=\omega^{RN}(h)\vee\psi^{RN}(h)\\
&= (\omega^{RN}\vee\psi^{RN})(h).
\end{aligned}$$

Hence, $(\omega^{RN}\circ\psi^{RN})(h)\geq(\omega^{RN}\vee\psi^{RP})(h)$. Similarly, we can show that $(\omega^{IN}\circ\psi^{IN})(h)\geq(\omega^{IN}\vee\psi^{IN})(h)$.

Thus, $(\omega^{RI}\circ\psi^{PN})(h)\leq(\omega^{RI}\wedge\psi^{PN})(h)$.
Hence, $\omega^{RI}\circledcirc\psi^{RI}\precsim\omega^{RI}\cap\psi^{RI}$. ■

This theorem is a tool of characterizations semisimple in terms of BCF interior ideals.

Theorem 4.2. *[17] Let $\mathfrak{K}$ and $\mathfrak{L}$ be a non-empty subsets of $\mathfrak{F}$. Then*

(1) $\chi^{RI}_{\mathfrak{K}}\circledcirc\chi^{RI}_{\mathfrak{L}}=\chi^{RI}_{\mathfrak{K}\mathfrak{L}}$.
(2) $\chi^{RI}_{\mathfrak{K}}\cap\chi^{RI}_{\mathfrak{L}}=\chi^{RI}_{\mathfrak{K}\cap\mathfrak{L}}$.

Lemma 4.3. *[19] For a semigroup $\mathfrak{F}$, the following statements are equivalent.*

(1) *$\mathfrak{F}$ is semisimple,*
(2) *Every interior ideal $\mathfrak{K}$ of $\mathfrak{F}$ is idempotent,*
(3) *Every ideal $\mathfrak{K}$ of $\mathfrak{F}$ is idempotent,*
(4) *For any interior ideals $\mathfrak{K}$ and $\mathfrak{L}$ of $\mathfrak{F}$, $\mathfrak{K}\cap\mathfrak{L}=\mathfrak{K}\mathfrak{L}$,*
(5) *For any ideals $\mathfrak{K}$ and $\mathfrak{L}$ of $\mathfrak{F}$, $\mathfrak{K}\cap L=\mathfrak{K}\mathfrak{L}$,*
(6) *For any interior $\mathfrak{K}$ and any ideal $\mathfrak{L}$ of $\mathfrak{F}$, $\mathfrak{K}\cap\mathfrak{L}=\mathfrak{K}\mathfrak{L}$,*
(7) *For any ideal $\mathfrak{K}$ and any interior ideal $\mathfrak{L}$ of $\mathfrak{F}$, $\mathfrak{K}\cap\mathfrak{L}=\mathfrak{K}\mathfrak{L}$.*

The following theorem shows an equivalent conditional statement for a semisimple semigroup.

Theorem 4.4. *Let $\mathfrak{F}$ be a semigroup. Then the following are equivalent:*

(1) *$\mathfrak{F}$ is semisimple,*
(2) *$\omega^{RI}\circledcirc\omega^{RI}=\omega^{RI}$, for every BCF interior ideals $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ of $\mathfrak{F}$,*
(3) *$\omega^{RI}\circledcirc\omega^{RI}=\omega^{RI}$, for every BCF ideals $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ of $\mathfrak{F}$,*
(4) *$\omega^{RI}\circledcirc\psi^{RI}=\omega^{RI}\cap\psi^{RI}$, for every BCF interior ideals $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ and $\psi^{RI}=(\mathfrak{F};\psi^P,\psi^N)=(\mathfrak{F};\psi^{RP}+\iota\psi^{IP},\psi^{RN}+\iota\psi^{IN})$ of $\mathfrak{F}$,*
(5) *$\omega^{RI}\circledcirc\psi^{RI}=\omega^{RI}\cap\psi^{RI}$, for every BCF ideals $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ and $\psi^{RI}=(\mathfrak{F};\psi^{RP}+\iota\psi^{IP},\psi^{RN}+\iota\psi^{IN})$ of $\mathfrak{F}$,*
(6) *$\omega^{RI}\circledcirc\psi^{RI}=\omega^{RI}\cap\psi^{RI}$, for every BCF interior ideal $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ of $\mathfrak{F}$ and every CBF ideal $\psi^{RI}=(\mathfrak{F};\psi^P,\psi^N)=(\mathfrak{F};\psi^{RP}+\iota\psi^{IP},\psi^{RN}+\iota\psi^{IN})$ of $\mathfrak{F}$,*
(7) *$\omega^{RI}\circledcirc\psi^{RI}=\omega^{RI}\cap\psi^{RI}$, for every BCF ideal $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ of $\mathfrak{F}$ and every BCF interior ideal $\psi^{RI}=(\mathfrak{F};\psi^P,\psi^N)=(\mathfrak{F};\psi^{RP}+\iota\psi^{IP},\psi^{RN}+\iota\psi^{IN})$ of $\mathfrak{F}$.*

Proof: (1) ⇒ (2) Suppose that $\omega^{RI}=(\mathfrak{F};\omega^P,\omega^N)=(\mathfrak{F};\omega^{RP}+\iota\omega^{IP},\omega^{RN}+\iota\omega^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. Then, by assumption and by Theorem 3.10, $\omega^{RI}\circledcirc\omega^{RI}=\omega^{RI}$.

(2) ⇒ (1) Let $\mathfrak{K}$ be an interior ideal of $\mathfrak{F}$. Then by Theorem 3.6, $\chi^{RI}_{\mathfrak{K}}=(\mathfrak{F},\chi^P_{\mathfrak{K}},\chi^N_{\mathfrak{K}})=(\mathfrak{F};\chi^{RP}_{\mathfrak{K}}+\iota\chi^{IP}_{\mathfrak{K}},\chi^{RN}_{\mathfrak{K}}+\iota\chi^{IN}_{\mathfrak{K}})$ is a BCF interior ideal of $\mathfrak{F}$. By supposition and

Lemma 4.2, we have $\chi^{RP}_{\mathfrak{K}^2}(h) = (\chi^{RP}_{\mathfrak{K}} \circ \chi^{RP}_{\mathfrak{K}})(h) = \chi^{RP}_{\mathfrak{K}}(h) = 1$, $\chi^{IP}_{\mathfrak{K}^2}(h) = (\chi^{IP}_{\mathfrak{K}} \circ \chi^{IP}_{\mathfrak{K}})(h) = \chi^{IP}_{\mathfrak{K}}(h) = \iota 1$. And $\chi^{RN}_{\mathfrak{K}^2}(h) = (\chi^{RN}_{\mathfrak{K}} \circ \chi^{RN}_{\mathfrak{K}})(h) = \chi^{RN}_{\mathfrak{K}}(h) = -1$, $\chi^{IN}_{\mathfrak{K}^2}(h) = (\chi^{IN}_{\mathfrak{K}} \circ \chi^{IN}_{\mathfrak{K}})(h) = \chi^{IN}_{\mathfrak{K}}(h) = -\iota 1$. Thus, $h \in \mathfrak{K}^2$. Hence, $\mathfrak{K}^2 = \mathfrak{K}$. By Lemma 4.3, we have $\mathfrak{F}$ is semisimple.

(1) $\Rightarrow$ (4) Let $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ and $\psi^{RI} = (\mathfrak{F}; \psi^P, \psi^N) = (\mathfrak{F}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be BCF interior ideals of $\mathfrak{F}$. Then by Theorem 3.5, $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ and $\psi^{RI} = (\mathfrak{F}; \psi^P, \psi^N) = (\mathfrak{F}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ are BCF ideals of $\mathfrak{F}$. Thus, by Lemma 4.1, $\omega^{RI} \circledcirc \psi^{RI} \precsim \omega^{RI} \cap \psi^{RI}$.
Let $h \in \mathfrak{F}$. Then there exist $r, t, e, d \in \mathfrak{F}$ such that $h = rhtehd$. Thus,

$$
\begin{aligned}
&(\omega^{RP} \circ \psi^{RP})(h) = \bigvee_{(k,o)\in A_h} \{\omega^{RP}(k) \wedge \psi^{RP}(o)\} \\
&= \bigvee_{(k,o)\in A_{(rht)(ehd)}} \{\omega^{RP}(k) \wedge \psi^{RP}(o)\} \\
&\geq (\omega^{RP}(rht) \wedge \psi^{RP}(ehd)) \geq (\omega^{RP}(h) \wedge \psi^{RP}(h)) \\
&= (\omega^{RP} \wedge \psi^{RP})(h).
\end{aligned}
$$

Hence $(\omega^{RP} \circ \psi^{RP})(h) \geq (\omega^{RP} \wedge \psi^{RP})(h)$. Similarly, we can show that $(\omega^{IP} \circ \psi^{IP})(h) \geq (\omega^{IP} \wedge \psi^{IP})(h)$. And

$$
\begin{aligned}
&(\omega^{RN} \circ \psi^{RN})(h) = \bigwedge_{(k,o)\in A_h} \{\omega^{RN}(k) \vee \psi^{RN}(o)\} \\
&= \bigwedge_{(k,o)\in A_{(rht)(ehd)}} \{\omega^{RP}(k) \vee \psi^{RN}(o)\} \\
&\leq (\omega^{RN}(rht) \vee \psi^{RN}(ehd)) \leq (\omega^{RN}(h) \vee \psi^{RP}(h)) \\
&= (\omega^{RN} \vee \psi^{RN})(h).
\end{aligned}
$$

Hence, $(\omega^{RN} \circ \psi^{RN})(h) \leq (\omega^{RN} \vee \psi^{RN})(h)$. Similarly, we can show that $(\omega^{IN} \circ \psi^{IN})(h) \leq (\omega^{IN} \vee \psi^{IN})(h)$. It implies that, $\omega^{RI} \circledcirc \psi^{RI} \succsim \omega^{RI} \cap \psi^{RI}$. Therefore, $\omega^{RI} \circledcirc \psi^{RI} = \omega^{RI} \cap \psi^{RI}$.

(4) $\Rightarrow$ (1) Let $\mathfrak{K}$ and $\mathfrak{L}$ be interior ideals of $\mathfrak{F}$. Then by Theorem 3.6, $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{F}, \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ and $\chi^{RI}_{\mathfrak{L}} = (\mathfrak{F}, \chi^P_{\mathfrak{L}}, \chi^N_{\mathfrak{L}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{L}} + \iota\chi^{IP}_{\mathfrak{L}}, \chi^{RN}_{\mathfrak{L}} + \iota\chi^{IN}_{\mathfrak{L}})$ are BCF interior ideals of $\mathfrak{F}$. By supposition and Lemma 4.2, we have
$\chi^{PN}_{\mathfrak{K}\mathfrak{L}}(h) = (\chi^{PN}_{\mathfrak{K}} \circ \chi^{PN}_{\mathfrak{L}})(h) = (\chi^{PN}_{\mathfrak{K}} \wedge \chi^{PN}_{\mathfrak{L}})(h) = 1$, $\chi^{IP}_{\mathfrak{K}\mathfrak{L}}(h) = (\chi^{IP}_{\mathfrak{K}} \circ \chi^{IP}_{\mathfrak{L}})(h) = (\chi^{IP}_{\mathfrak{K}} \wedge \chi^{IP}_{\mathfrak{L}})(h) = \iota 1$. And $\chi^{RN}_{\mathfrak{K}\mathfrak{L}}(h) = (\chi^{RN}_{\mathfrak{K}} \circ \chi^{RN}_{\mathfrak{L}})(h) = (\chi^{RN}_{\mathfrak{K}} \vee \chi^{RN}_{\mathfrak{L}})(h) = -1$, $\chi^{IN}_{\mathfrak{K}\mathfrak{L}}(h) = (\chi^{IN}_{\mathfrak{K}} \circ \chi^{IN}_{\mathfrak{L}})(h) = (\chi^{IN}_{\mathfrak{K}} \vee \chi^{IN}_{\mathfrak{L}})(h) = -\iota 1$. Thus, $h \in \mathfrak{K}\mathfrak{L}$. Hence, $\mathfrak{K} \cap \mathfrak{L} = \mathfrak{K}\mathfrak{L}$. By Lemma 4.3, $\mathfrak{F}$ is semisimple.

(1) $\Rightarrow$ (6) Let $\omega^{RI} = (\mathfrak{F}; \omega^P, \omega^N) = (\mathfrak{F}; \omega^{RP} + \iota\omega^{IP}, \omega^{RN} + \iota\omega^{IN})$ and $\psi^{RI} = (\mathfrak{F}; \psi^P, \psi^N) = (\mathfrak{F}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ be a BCF interior ideal and a BCF ideal of $\mathfrak{F}$ respectively. Then $\psi^{RI} = (\mathfrak{F}; \psi^P, \psi^N) = (\mathfrak{F}; \psi^{RP} + \iota\psi^{IP}, \psi^{RN} + \iota\psi^{IN})$ is a BCF interior ideal of $\mathfrak{F}$. Thus by (4), $\omega^{RI} \circledcirc \psi^{RI} = \omega^{RI} \cap \psi^{RI}$.

(6) $\Rightarrow$ (1) Let $\mathfrak{K}, \mathfrak{L}$ be an interior ideal and ideal of $\mathfrak{F}$ respectively. Then by Theorem 3.6 and 2.8, $\chi^{RI}_{\mathfrak{K}} = (\mathfrak{F}, \chi^P_{\mathfrak{K}}, \chi^N_{\mathfrak{K}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{K}} + \iota\chi^{IP}_{\mathfrak{K}}, \chi^{RN}_{\mathfrak{K}} + \iota\chi^{IN}_{\mathfrak{K}})$ and $\chi^{RI}_{\mathfrak{L}} = (\mathfrak{F}, \chi^P_{\mathfrak{L}}, \chi^N_{\mathfrak{L}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{L}} + \iota\chi^{IP}_{\mathfrak{L}}, \chi^{RN}_{\mathfrak{L}} + \iota\chi^{IN}_{\mathfrak{L}})$ is a BCF interior ideal and BCF ideal of $\mathfrak{F}$ respectively. Then by Theorem 3.3, $\chi^{RI}_{\mathfrak{L}} = (\mathfrak{F}, \chi^P_{\mathfrak{L}}, \chi^N_{\mathfrak{L}}) = (\mathfrak{F}; \chi^{RP}_{\mathfrak{L}} + \iota\chi^{IP}_{\mathfrak{L}}, \chi^{RN}_{\mathfrak{L}} + \iota\chi^{IN}_{\mathfrak{L}})$ is a BCF interior ideal of $\mathfrak{F}$. Similarly, from (4) $\Rightarrow$ (1), we have $\mathfrak{F}$ is semisimple.

So, (1) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (7) are Straightforward. ■

V. Conclusion

In this paper, we give the concept of bipolar complex fuzzy interior ideals in semigroups. We investigate the properties of bipolar complex fuzzy interior ideals and between relation interior ideals and bipolar complex fuzzy interior ideals. Additionally, we find conditions bipolar complex fuzzy ideals and bipolar complex fuzzy interior ideals coincide. Finally, we characterize a semisimple semigroup in the bipolar complex fuzzy interior ideal. In the further, we study bipolar complex fuzzy bi-ideals in semigroups or algebraic systems.

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