A Meshfree Method for Korteweg-de Vries (KdV) Equation by A New Multiquadric Quasi-interpolation

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***Abstract*—The quasi-interpolation operator is widely used in numerical approximation and numerical solutions of differential equations. This paper proposes a new multiquadric(MQ) quasi-interpolate and formulates a meshfree method for the Korteweg-de Vries(KdV) equation based on the proposed multiquadric quasi-interpolate. More specifically, based on the multiquadric function, a new univariate multiquadric(MQ) quasi-interpolation scheme is structured, which possesses high accuracy, simple structure, and ease of programming. Moreover, the error estimation of the new quasi-interpolate is shown in detail. Next, a meshfree method for the Korteweg-de Vries (KdV) is proposed by using the novel multiquadric(MQ) quasi-interpolation operator. In the spatial direction, the derivative is approximated by the proposed multiquadric quasi-interpolate, and the forward divided difference approximates the temporal derivative. Several numerical examples are presented at the end of the paper to verify the expected approximation capability, and the experiment results show that the meshfree method (based on the new multiquadric(MQ) quasi-interpolation operator) is valid.**

***Index Terms*—Multiquadric (MQ) quasi-interpolation, Meshfree method, Korteweg-de Vries (KdV) equation**

I. INTRODUCTION

THE radial basis function (RBF) is a multivariate function generated by a univariate function. Due to its simple form and good approximation behavior, the radial basis functions (RBFs) method has become an effective tool in different fields, such as function approximation, neural networks, machine learning, and the numerical solutions of differential equations. Since the multiquadric (MQ) function, which is a type of radial basis function (RBF), was proposed by Hardy[2], multiquadric quasi-interpolation attracts a great deal of scholarly attention. Beaston and Powell[3] proposed three multiquadric (MQ) quasi-interpolant operators, namely $L_A f(x)$, $L_B f(x)$ and $L_C f(x)$, and the properties of the three operators can be found in [17]. Wu and Schaback [4] proposed a new MQ quasi-interpolant operator $L_D f(x)$ and proved that the approximation order is two at most. Chen et al. [21] defined a multiquadric quasi-interpolant operator $f^*(x)$, which is the generalization of $L_D f(x)$. Besides, there exists a large number of results in the study of multiquadric quasi-interpolation operators, such as multilevel quasi-interpolation operators in the papers [8,22,26], the quasi-interpolation operator for linear functional data [24] and quasi-interpolation operators based on the function cubic MQ functions in the papers [9,19,27]. Furthermore, many scholars have further discussed solving differential equations based on MQ quasi-interpolation [10,25,27,30,31,33].

Many practical problems are perceived as Dispersive Wave Equations. In this paper, we shall concentrate on the following nonlinear partial differential equation named the Korteweg-de Vries (KdV) equation, which is one of the well-known Dispersive Wave Equations:

$$\frac{\partial u(x,t)}{\partial t}+\varepsilon u(x,t)\frac{\partial u(x,t)}{\partial x}+\mu\frac{\partial^3 u(x,t)}{\partial x^3}=0$$

where ε and μ are positive real constants. The equation exhibits both dispersion and non-linearity. For appropriate initial conditions, Gardner et al. [1] has shown the existence and uniqueness of solutions of the KdV equation. In recent years, the numerical solutions of the KdV equation have attracted the interest of a large number of scholars. So far, there are many classical numerical methods have been used to solve the KdV equation, such as the finite-element method, finite-difference method, radial basis functions collation method, fourier spectral methods [6,11,13,14,16,18], heat balance integral method(HBIM) [7] and the HBIM' improvement without the exact solution of the KdV equation [5]. The exponential finite-difference method (EFDM) is given by Bahadir [12], which possesses higher accuracy for small time. Besides, Yan presented three approaches that focus on energy-preserving and momentum-preserving principles. These approaches are discussed in detail in references [28,29,32]. However, those methods require solving large scale linear systems of the KdV equation. Based on Chen and Wu's quasi-interpolation [21], Xiao [23] discussed a numerical scheme that overcomes the problem.

In this paper, based on the new MQ quasi-interpolant we constructed, we formulate a meshfree method for the KdV equation, which does not need to solve any linear system of equations. Specifically, in the spatial direction, the derivative is approximated by the proposed multiquadric quasi-interpolation. Besides, the temporal derivative is

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approximated by the forward divided difference.

This paper is organized as follows. Some preliminaries about the multiquadric quasi-interpolant operator and its correlation theorem are presented in Section II. In Section III, a novel MQ quasi-interpolant $Q_d f(x)$ is constructed, and the properties of the new operator are discussed. Based on the new MQ quasi-interpolation, the numerical scheme is presented to solve the KdV equation in Section IV. In Section V, numerical experiments are given to show that the new quasi-interpolation operator $Q_d f(x)$ has good approximation capability as the classical quasi-interpolation $L_D f(x)$, and verify the effectiveness of the meshfree method. Finally, some conclusions are given in Section VI.

II. MULTIQUADRIC QUASI-INTERPOLATION OPERATOR

For scattered points $a = x_0 < x_1 < \cdots < x_n = b$, the univariate multiquadric quasi-interpolation operator of a function $f:[a,b]\to R$ has the form

$$Qf = \sum f_j \psi_j(x) \tag{1}$$

where $\psi_j(x)$ is a linear combination of the function $\phi_j(x)=\sqrt{(x-x_j)^2+c^2}$ introduced by Hardy [2] with the shape parameter c.

The multiquadric quasi-interpolant $f^*(x)$ of Chen and Wu [21] is defined as follows

$$f^*(x)=\sum_{j=0}^{m} f_j\psi_j(x),$$

Where

$$\psi_j(x)=\frac{\varphi_{j+1}(x)-\varphi_j(x)}{2(x_{j+1}-x_j)}-\frac{\varphi_j(x)-\varphi_{j-1}(x)}{2(x_j-x_{j-1})}, 1\le j\le m,$$

$$\begin{cases}\varphi_{-1}(x)=\varphi_0(x)+x_0-x_{-1},\\ \varphi_m(x)=\varphi_0(x)-2x+x_m+x_0,\\ \varphi_{m+1}(x)=\varphi_m(x)+x_{m+1}-x_m.\end{cases} \tag{2}$$

Remark [21] The quasi-interpolant $f^*(x)$ becomes the quasi-interpolant operator $L_D f(x)$ whose approximation order is two at most, which is proposed by Schaback and Wu in [4] under the following condition

$$\phi_0(x)=x-x_0, \tag{3}$$

which means that the operator $f^*(x)$ is the generalization of $L_D f(x)$.

Furthermore, the error estimate of the multiquadric quasi-interpolation operator $f^*(x)$ is defined by (1) and (2) as follows (see[21] for details).

Theorem 1.[21]. For $f\in C^2[x_0,x_n]$, there exist positive constants C_0, C_1, C_2 and C_3 independent of h and c, the quasi-interpolation operator $f^*(x)$ satisfies an error estimate of the type

$$\| f^*(x)-f(x)\|_\infty \le C_0C_h + C_1h^2 + C_2ch + C_3c^2\log h,$$

where $C_h=\min\{c,\frac{c^2}{h}\}, h=\max(x_j-x_{j-1}), j=1,2,\cdots,n.$

III. A NOVEL MULTIQUADRIC (MQ) QUASI-INTERPOLATION OPERATOR $Q_d f(x)$

For scattered points $a = x_0 < x_1 < \cdots < x_n = b$, a novel multiquadric quasi-interpolation operator (based on multiquadric function) is constructed as follows

$$Q_d f(x) = f_0\beta_0(x)+f_1\beta_1(x)+\sum_{j=2}^{n-2} f_j\psi_j(x)+f_{n-1}\beta_{n-1}(x)+f_n\beta_n(x), \tag{4}$$

where

$$\beta_0=\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)}+\frac{\varphi_0(x)+\varphi_n(x)}{2(x_n-x_0)}$$

$$\beta_1=\frac{\phi_2(x)-\phi_1(x)}{2(x_2-x_1)}-\frac{\phi_1(x)-(x-x_0)}{2(x_1-x_0)},$$

$$\psi_j(x)=\frac{\phi_{j+1}(x)-\phi_j(x)}{2(x_{j+1}-x_j)}-\frac{\phi_j(x)-\phi_{j-1}(x)}{2(x_j-x_{j-1})}, 2\le j\le n-2,$$

$$\beta_{n-1}=\frac{(x_n-x)-\phi_{n-1}(x)}{2(x_n-x_{n-1})}-\frac{\phi_{n-1}(x)-\phi_{n-2}(x)}{2(x_{n-1}-x_{n-2})},$$

$$\beta_n=\frac{\phi_0(x)+\phi_n(x)}{2(x_n-x_0)}-\frac{(x_n-x)-\phi_{n-1}(x)}{2(x_n-x_{n-1})}. \tag{5}$$

Theorem 2. For $f\in C^2[x_0,x_n]$, let $C_1=\frac{(f_0+f_n)}{2(x_n-x_0)}$, there are positive constants C_2, C_3 and C_4 independent h and c, then the L_∞ error of the novel quasi-interpolation operator $Q_d f(x)$ defined by (4) and (5) satisfies

$$\| f(x)-Q_d f(x)\|_\infty \le C_1c + C_2h^2 + C_3ch + C_4c^2\log h.$$

Proof of Theorem 2. Let $x\in(a,b)$, according to equation (1)-(3), there is

$$|Q_d f(x)-L_D f(x)| = \left| f_0(\alpha_0(x)-\beta_0(x)) + f_n(\alpha_n(x)-\beta_n(x))\right|$$

$$=\left|\frac{(f_0+f_n)[\phi_0(x)+\phi_n(x)-x_n+x_0]}{2(x_n-x_0)}\right|$$

$$=\left|\frac{(f_0+f_n)}{2(x_n-x_0)}[\phi_n(x)-(x_n-x)+\phi_0(x)-(x-x_0)]\right|.$$

Combining the estimation

$$[\phi_n(x)-(x_n-x)+\phi_0(x)-(x-x_0)]\le 2c,$$

the following inequality holds

$$|Q_d f(x)-L_D f(x)| \le C_1c. \tag{6}$$

Theorem 2 holds from (6) and the following inequality(see [4]for details).

$$\| f(x)-L_D f(x)\|_\infty \le (k_1h^2+k_2ch+k_3c^2\log h)\| f''\|_\infty .$$

Remark The novel quasi-interpolant has excellent approximation accuracy, which is not inferior to the operator $f^*(x)$. In fact, our operator has almost the same accuracy as the quasi-interpolation operator $L_D f(x)$ in numerical experiments (see Section 5 for details).

IV. NUMERICAL SCHEME USING MQ QUASI-INTERPOLATION

In this section, we propose a numerical method for solving the KdV equation based on the novel MQ quasi-interpolation, which is a third-order nonlinear equation

$$\frac{\partial u(x,t)}{\partial t}+\varepsilon u(x,t)\frac{\partial u(x,t)}{\partial x}+\mu\frac{\partial^3 u(x,t)}{\partial x^3}=0, x\in\Omega=[a,b], t>0,$$

with the initial condition $u(x,0)=u^0(x)$, and boundary conditions $u(x,t)=f(t), x\in\partial\Omega, t>0$, $u_x(b,t)=g(t), t>0$.

Discretizing the KdV equation by using forward divided difference in the temporal direction with mesh length τ, we obtain the following formula

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\varepsilon u_j^n (u_x)_j^n+\mu (u_{xxx})_j^n=0,$$

i.e.,

$$u_j^{n+1}=u_j^n-\tau[\varepsilon u_j^n (u_x)_j^n+\mu (u_{xxx})_j^n],$$

where u_j^n is the approximation of $u(x_j,t_n),\ x_j=jh,\ t_n=n\tau$.

In the spatial direction, u_x is approximated by the derivatives of the new MQ quasi-interpolant. Meanwhile, the third derivative u_{xxx} is replaced using the following method.

$$(u_{xxx})_j^n=\frac{(u_x)_{j+1}^n-2(u_x)_j^n+(u_x)_{j-1}^n}{h^2}.$$

V. Numerical Experiments

This section can be divided into two parts. Firstly, we will be using the new quasi-interpolation operator $Q_d f(x)$ to approximate four classical functions. Secondly, we will be discussing a meshfree method for the KdV equation. This method is based on the new operator $Q_d f(x)$, which is defined by equations (4) and (5).

A. Numerical approximation

To begin the study, we will employ a novel quasi-interpolation called $Q_d f(x)$, defined by equations 4 and 5, to approximate four classical functions. Our aim is to assess the efficacy and accuracy of operator $Q_d f(x)$ by comparing the resulting L_∞ errors with those obtained using two existing operators (operator $f^*(x)$ proposed by Chen [21] and operator $L_D f(x)$ presented by Wu and Schaback in [4]). Through this comparison, we can better understand the effectiveness of the new quasi-interpolation $Q_d f(x)$.

Example 1. $f_1(x)=x^9,\ x\in[0,1]$.

Example 2. $f_2(x)=e^{-2^{16(x-0.8)^2}},\ x\in[0,1]$.

Example 3. $f_3(x)=\sin(\pi x)+0.1\sin(32\pi x),\ x\in[0,1]$.

Example 4. $f_4(x)=\arctan(100(x-0.3)),\ x\in[0,1]$.

The graphs of the test functions $f_i(x)\ (i=1,2,3,4)$ and $Q_d f(x)$ are shown in Fig. 1-Fig. 4 respectively. The L_∞ errors by using the operators $f^*(x)$, $L_D f(x)$ and $Q_d f(x)$ are given in TABLE I with $h=0.1$, and TABLE II with $h=0.01$.

TABLE I. The errors of operators $f^*(x)$, $L_D f(x)$ and $Q_d f(x)$ for $f_i(x)\ (i=1,2,3,4)$ with $h=0.1$.

c	0.001	0.01	0.03	0.05	0.1
$\|f^*(x)-f_1(x)\|_\infty$	0.0031	0.0304	0.0902	0.1485	0.2878
$\|L_D f(x)-f_1(x)\|_\infty$	0.0018	0.0185	0.0583	0.1018	0.2225
$\|Q_d f(x)-f_1(x)\|_\infty$	0.0018	0.0187	0.0608	0.1084	0.2459
$\|f^*(x)-f_2(x)\|_\infty$	5.8335e-04	0.0058	0.0175	0.0291	0.0614
$\|L_D f(x)-f_2(x)\|_\infty$	4.1021e-04	0.0044	0.0149	0.0275	0.0650
$\|Q_d f(x)-f_2(x)\|_\infty$	4.0988e-04	0.0043	0.0146	0.0267	0.0619
$\|f^*(x)-f_3(x)\|_\infty$	0.0031	0.0308	0.0913	0.1504	0.2893
$\|L_D f(x)-f_3(x)\|_\infty$	0.0021	0.0208	0.0603	0.0977	0.1886
$\|Q_d f(x)-f_3(x)\|_\infty$	0.0021	0.0208	0.0603	0.0977	0.1886
$\|f^*(x)-f_4(x)\|_\infty$	0.0071	0.0693	0.1979	0.3131	0.5480
$\|L_D f(x)-f_4(x)\|_\infty$	0.0071	0.0694	0.1979	0.3132	0.5485
$\|Q_d f(x)-f_4(x)\|_\infty$	0.0071	0.0694	0.1979	0.3132	0.5483

TABLE II. The errors of operators $f^*(x)$, $L_D f(x)$ and $Q_d f(x)$ for $f_i(x)\ (i=1,2,3,4)$ with $h=0.01$.

c	0.001	0.003	0.005	0.01	0.02
$\|f^*(x)-f_1(x)\|_\infty$	0.0043	0.0130	0.0216	0.0430	0.0855
$\|L_D f(x)-f_1(x)\|_\infty$	3.7950e-04	0.0014	0.0030	0.0085	0.0250
$\|Q_d f(x)-f_1(x)\|_\infty$	5.4746e-04	0.0019	0.0037	0.0095	0.0279
$\|f^*(x)-f_2(x)\|_\infty$	7.1753e-04	0.0022	0.0036	0.0072	0.0143
$\|L_D f(x)-f_2(x)\|_\infty$	7.2973e-05	2.8606e-04	5.8318e-04	0.0016	0.0051
$\|Q_d f(x)-f_2(x)\|_\infty$	1.0518e-04	3.1516e-04	5.8847e-04	0.0017	0.0050
$\|f^*(x)-f_3(x)\|_\infty$	0.0016	0.0046	0.0075	0.0141	0.0252
$\|L_D f(x)-f_3(x)\|_\infty$	0.0047	0.0143	0.0240	0.0500	0.1055
$\|Q_d f(x)-f_3(x)\|_\infty$	0.0047	0.0143	0.0240	0.0500	0.1055
$\|f^*(x)-f_4(x)\|_\infty$	0.0230	0.0675	0.1099	0.2049	0.3453
$\|L_D f(x)-f_4(x)\|_\infty$	0.0230	0.0675	0.1099	0.2049	0.3453
$\|Q_d f(x)-f_4(x)\|_\infty$	0.0230	0.0675	0.1099	0.2049	0.3453

Fig.1-Fig. 4 indicate that the new MQ quasi-interpolation operator $Q_d f(x)$ possesses satisfactory approximation capability. TABLE I-TABLE II demonstrate that the quasi-interpolations $Q_d f(x)$ and $L_D f(x)$ proposed by Wu and Schaback in[4] have almost the same accuracy for the test functions $f_i(x)\ (i=1,2,3,4)$. The approximation accuracy of operators $Q_d f(x)$ and $L_D f(x)$ is superior to $f^*(x)$ except in a few cases (For the function $f_3(x)$ with $h=0.01$). Meanwhile, TABLE I-TABLE II also show that the approximation capacity of the three quasi-interpolants is dependent on c and h. Moreover, the L_∞ errors of the $Q_d f(x)$ are smaller if the constants c and h are smaller. These numerical results are consistent with the error analysis.

B. Solving KdV equation by using the new operator $Q_d f(x)$

In this part, the novel quasi-interpolation operator defined in the previous section will be used to solve two types of the KdV equation and record the numerical solutions as MQQIs. Additionally, we will calculate the L_∞ error between the numerical solutions and the analytical solution of the equation and compare the results with the numerical solutions MQQI in [23]. The comparison results are shown in TABLE III-TABLE VIII. More specifically, these tables (TABLE III-TABLE VIII) consist of five columns. The first column, labeled "Exact", denotes actual data. The second column, labeled "MQQIs", represents the results of the meshfree method based on the novel operator A. The third column, labeled "Error1", shows the difference between the actual data and the data obtained through the meshfree method. The fourth column and the fifth column are the results of the MQQI method and its error (Error2) in [23].

Moreover, compared with **Example 5.** In this example, we consider the situation of propagation of single solitary wave [15]. let $\varepsilon=6$ and $\mu=1$ in the KdV equation. The initial condition is

$$u^0(x)=\frac{r}{2}\operatorname{sech}^2(\frac{\sqrt{r}}{2}x-7),t=0,r=0.5.$$

The exact solution is

$$u(x,t)=\frac{r}{2}\operatorname{sech}^2(\frac{\sqrt{r}}{2}(x-rt)-7),r=0.5,$$

and the boundary functions $f(t)$ and $g(t)$ can be obtained from the exact solution. In the numerical examples, we set the mesh length $\tau = 0.001$, $h = 0.2$ with the domain $0 \le x \le 40$ and $0 \le t \le 5$. TABLE III-TABLE V show the numerical solutions based on the novel operator MQQIs and the numerical solutions MQQI obtained in [23]. Error1 and Error2 are the L_∞ errors of the numerical solutions MQQIs based on the new operator $Q_d f(x)$ and the numerical solutions MQQI obtained in [23], respectively, with $c = 0.2799$ at $t = 1,3,5$.

TABLE III. THE L_∞ ERROR OF THE NUMERICAL SOLUTIONS MQQIS AND MQQI AT $t = 1$.

x	Exact	MQQIs	Error1	MQQI[23]	Error2
17	0.080625	0.081817	0.001192	0.081817	0.001192
18	0.137393	0.137899	0.000506	0.137899	0.000506
19	0.203886	0.204228	0.000342	0.204288	0.000402
20	0.247227	0.247960	0.000733	0.247960	0.000733
21	0.235251	0.235194	0.000057	0.235194	0.000057
22	0.177627	0.176306	0.001321	0.176306	0.001321
23	0.112353	0.110928	0.001425	0.110928	0.001425
24	0.063421	0.062590	0.000830	0.062591	0.000830
25	0.033545	0.033223	0.000322	0.033223	0.000322

TABLE IV. THE L_∞ ERROR OF THE NUMERICAL SOLUTIONS MQQIS AND MQQI AT $t = 3$.

x	Exact	MQQIs	Error1	MQQI[23]	Error2
17	0.043573	0.045284	0.001711	0.045284	0.001711
18	0.080625	0.082629	0.002004	0.082636	0.002011
19	0.137393	0.139849	0.002456	0.139860	0.002467
20	0.203886	0.206768	0.002882	0.206787	0.002901
21	0.247227	0.249016	0.001789	0.249011	0.001784
22	0.235251	0.233686	0.001565	0.233662	0.001589
23	0.177627	0.173619	0.004008	0.173616	0.004011
24	0.112353	0.108701	0.003652	0.108716	0.003637
25	0.063421	0.061288	0.002133	0.061304	0.002117

TABLE V. THE L_∞ ERROR OF THE NUMERICAL SOLUTIONS MQQIS AND MQQI AT $t = 5$.

x	Exact	MQQIs	Error1	MQQI[23]	Error2
17	0.022515	0.025975	0.003460	0.024489	0.001974
18	0.043573	0.045341	0.001768	0.045640	0.002067
19	0.080625	0.081410	0.000785	0.084426	0.003801
20	0.137393	0.141610	0.004217	0.142601	0.005208
21	0.203886	0.212876	0.008990	0.208712	0.004826
22	0.247227	0.250276	0.003049	0.249883	0.002656
23	0.235251	0.225279	0.009972	0.231660	0.003591
24	0.177627	0.177104	0.000523	0.171589	0.006038
25	0.112353	0.107337	0.005016	0.106221	0.006132

Fig. 5 indicates that the numerical solutions based on the new MQ quasi-interpolation operator $Q_d f(x)$ possess satisfactory approximation capability with $c = 0.2799, \tau = 0.001, h = 0.2$ at $t = 1$. It can be also observed from TABLE III-TABLE V that when the time layer remains unchanged, the error accuracy of the intermediate nodes in the spatial layer is lower than that of the boundary nodes. When the spatial layer remains unchanged, the error accuracy of the intermediate nodes in the temporal layer is also lower than that of the boundary nodes. The numerical solutions used in this paper and the errors are MQQIs and Error1, respectively. Meanwhile, the numerical solutions in [15] and the errors are MQQI [23] and Error2, respectively.

TABLE III-TABLE V show that the meshfree method proposed possesses almost the same approximation error as the numerical solutions MQQI in [23]. Moreover, the meshfree method is slightly superior to the numerical solutions MQQI method used in the article [23] except for the approximation results of the intermediate nodes.

Example 6. In this example, we consider the situation of propagation of two solitary waves [15]. Let $\varepsilon = 6$ and $\mu = 1$ in the KdV equation. The initial condition is

$$u^0(x) = 12\{\frac{3 + 4\cosh(2x) + \cosh(4x)}{[3\cosh(x) + \cosh(3x)]^2}\},$$

and the exact solution is

$$u(x,t) = 12\{\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}\}.$$

Considering $c = 0.01, \tau = 0.00001$ and $h = 0.1$ in the domain $-5 \le x \le 15,\ 0 \le t \le 0.1$. TABLE VI-TABLE VIII, which are containing six columns, demonstrate the effectiveness of the meshfree method we proposed. The numerical solutions MQQIs (the third column), which are based on our operator, and the numerical solutions MQQI (the fifth column), which are obtained in [23]. The fourth and sixth columns (Error1 and Error2) in the TABLE VI-TABLE VIII represent the L_∞ errors of the numerical solutions based on our operator MQQIs and the numerical solutions obtained in [23], respectively, with $c = 0.01$ at $t = 0.01, 0.05, 0.1$.

TABLE VI. THE L_∞ ERROR OF THE NUMERICAL SOLUTIONS MQQIS AND MQQI AT $t = 0.01$.

x	Exact	MQQIs	Error1	MQQI[23]	Error2
-3	0.054477	0.054448	0.000029	0.054358	0.000119
-2	0.382934	0.382850	0.000084	0.382756	0.000178
-1	2.084133	2.084222	0.000089	2.084206	0.000073
0	5.638245	5.640733	0.002488	5.640733	0.002488
1	3.192964	3.186663	0.006301	3.186663	0.006301
2	0.478495	0.478633	0.000138	0.478633	0.000138
3	0.064520	0.064569	0.000049	0.064570	0.000050
4	0.008723	0.008730	0.000007	0.008730	0.000007
5	0.001180	0.001181	0.000001	0.001181	0.000001

TABLE VII. THE L_∞ ERROR OF THE NUMERICAL SOLUTIONS MQQIS AND MQQI AT $t = 0.05$.

x	Exact	MQQIs	Error1	MQQI[23]	Error2
-3	0.039507	0.038499	0.001008	0.038461	0.001046
-2	0.275315	0.279317	0.004002	0.279287	0.003972
-1	1.390946	1.380456	0.010490	1.380403	0.010543
0	2.574829	2.597851	0.023022	2.597785	0.022956
1	6.881609	6.922285	0.040676	6.922214	0.040605
2	1.207024	1.192006	0.015018	1.191924	0.015100
3	0.100955	0.101539	0.000584	0.101458	0.000503
4	0.012239	0.012310	0.000071	0.012227	0.000012
5	0.001630	0.001645	0.000015	0.001559	0.000071

TABLE VIII. THE L_∞ ERROR OF THE NUMERICAL SOLUTIONS MQQIS AND MQQI AT $t=0.1$.

x	Exact	MQQIs	Error1	MQQI[23]	Error2
-3	0.026554	0.024052	0.002502	0.023935	0.002619
-2	0.188162	0.194745	0.006583	0.194571	0.006409
-1	1.045967	1.038892	0.007075	1.038671	0.007296
0	2.000572	1.990282	0.010290	1.990040	0.010532
1	1.717101	1.765233	0.048132	1.764999	0.047898
2	7.171392	7.139916	0.031476	7.139714	0.031678
3	0.464299	0.452859	0.011440	0.452621	0.011678
4	0.024308	0.024709	0.000401	0.024443	0.000135
5	0.002542	0.002585	0.000043	0.002261	0.000281

Fig. 6 indicates that the numerical solutions based on the new MQ quasi-interpolation operator $Q_d f(x)$ possess satisfactory approximation capability with $c=0.01, \tau=0.00001, h=0.1$ at $t=0.1$. In TABLE VI-TABLE VIII, the numerical solutions used in this paper and the errors are MQQIs and Error1, respectively. Meanwhile, the numerical solutions in [15] and the errors are MQQI [23] and Error2, respectively. From TABLE VI-TABLE VIII, it can be seen that the error accuracy for the middle node is relatively low in both the temporal and spatial layers. However, the approximation accuracy for other nodes is relatively high. Moreover, compared with the result of MQQI method and its error (Error2) in [23], the numerical solutions based on the new operator have higher accuracy except for a few points, which means that our meshfree method is acceptable and valid.

VI. CONCLUSION

This paper proposes an improved quasi-interpolation operator $Q_d f(x)$ based on the MQ function and achieves its error estimates. The quasi-interpolation operator possesses simple structure and easily performs programming. The numerical approximation of test functions validates that the operator $Q_d f(x)$ has almost the same accuracy as $L_D f(x)$, which is superior to $f^*(x)$ for the test functions except in a few cases. Meanwhile, a meshfree method by a new multiquadric quasi-interpolation for numerical solutions of the Korteweg-de Vries (KdV) equation is proposed. The numerical method has higher accuracy except in a few points, which means that the meshfree method based on our quasi-interpolation operator is acceptable and valid.

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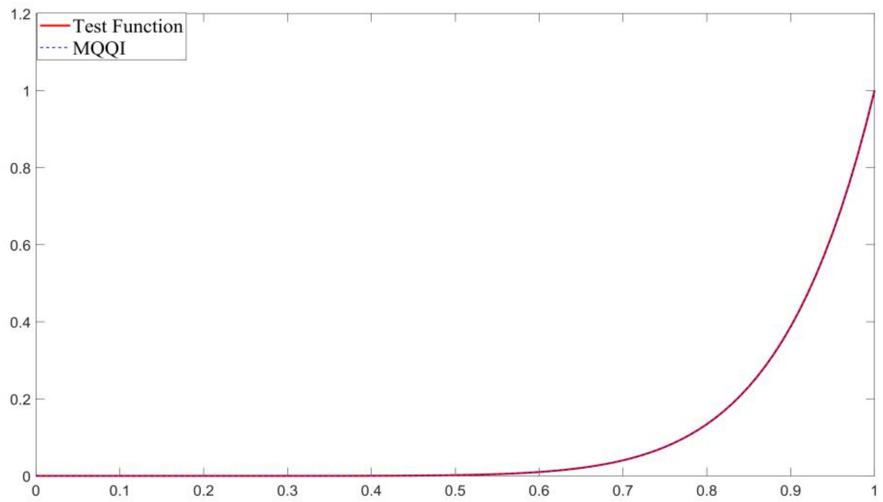
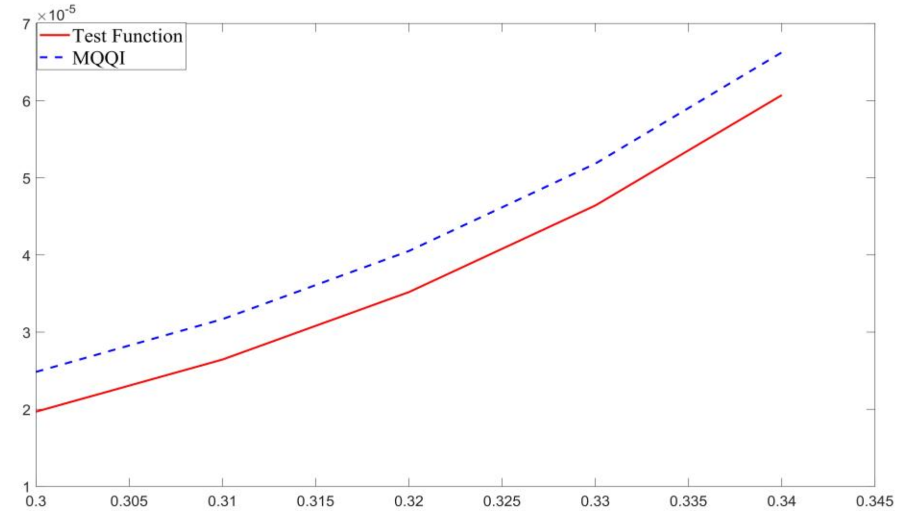
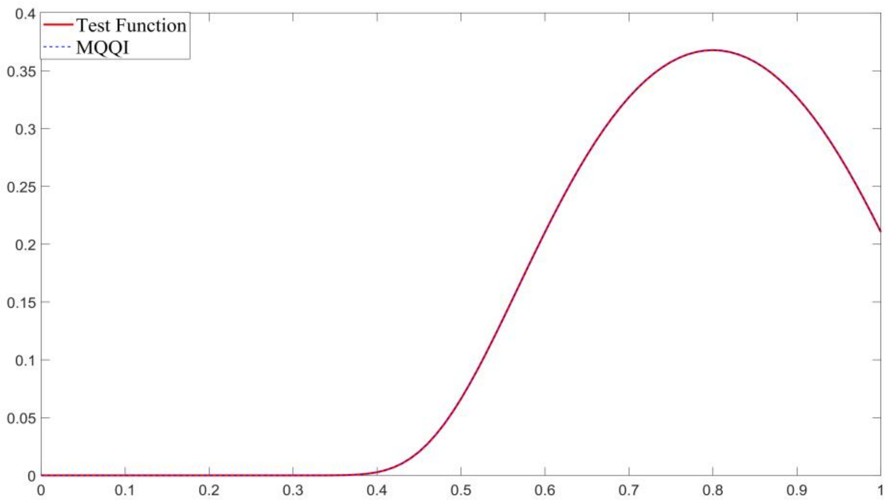


(a) The graphs of $f_1(x)$ and $Q_d f(x)$ with $h=0.01$ and $c=0.001$.

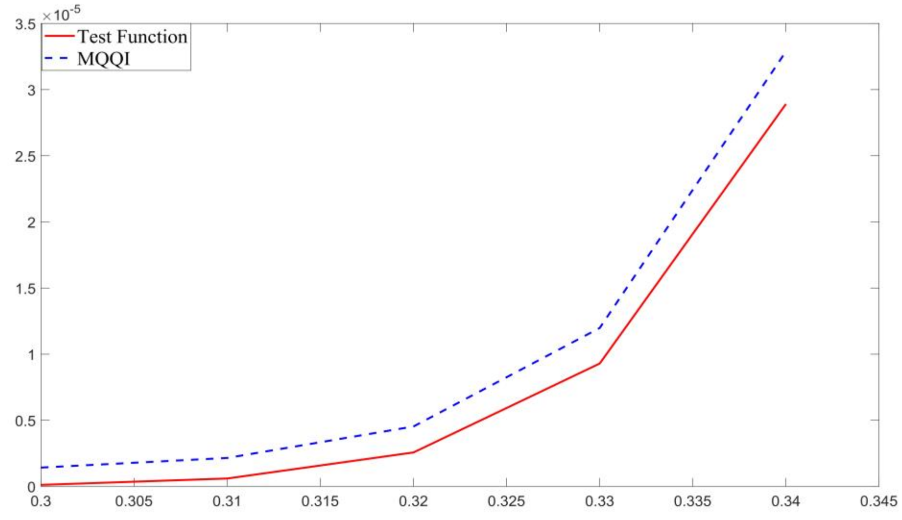


(b) The locally enlarged image of (a).

Fig. 1. The graphs of $f_1(x)$ and $Q_d f(x)$ and the enlarged image of them.

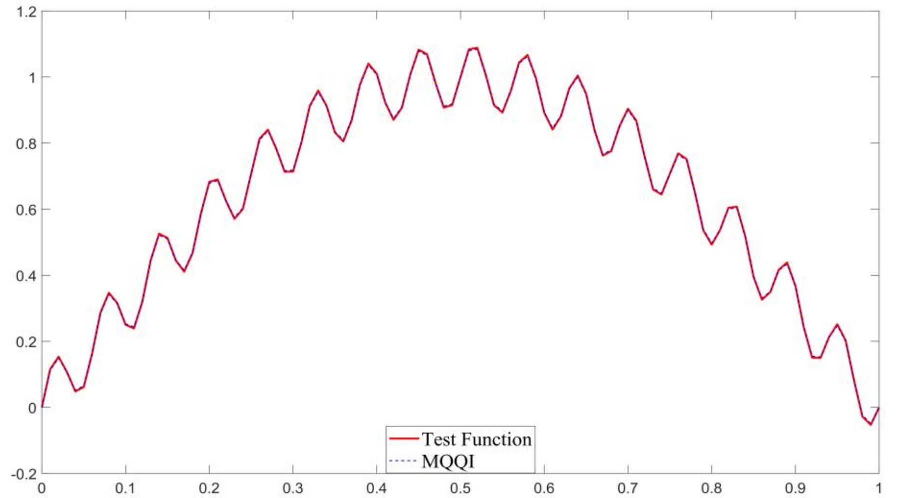


(a) The graphs of $f_2(x)$ and $Q_d f(x)$ with $h=0.01$ and $c=0.001$.



(b) The locally enlarged image of (a).

Fig. 2. The graphs of $f_2(x)$ and $Q_d f(x)$ and the enlarged image of them.



(a) The graphs of $f_3(x)$ and $Q_d f(x)$ with $h = 0.01$ and $c = 0.001$.

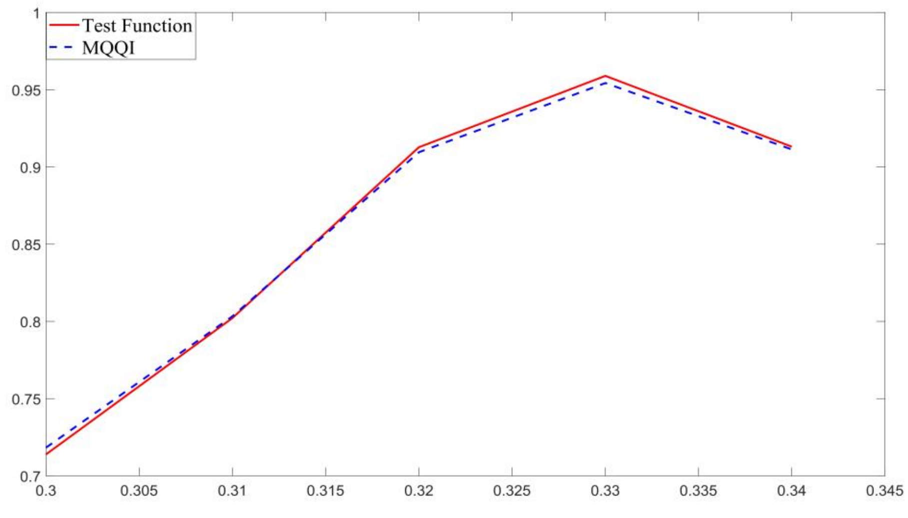


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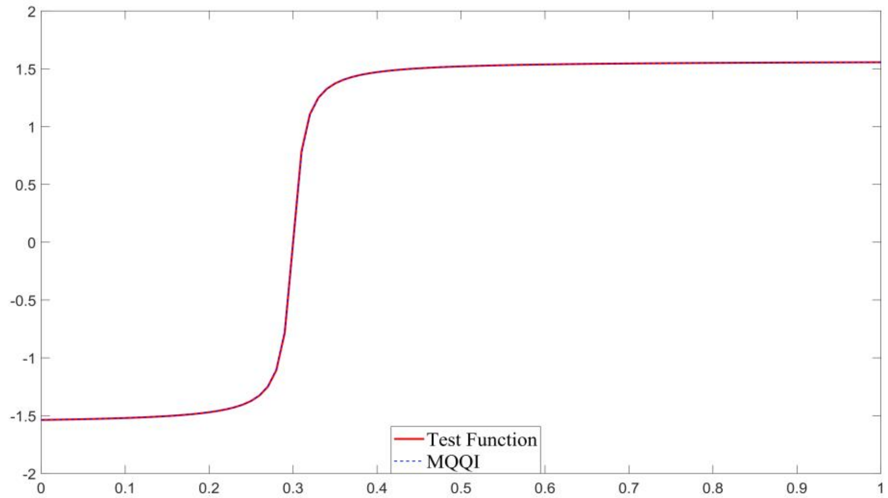
Fig. 3. The graphs of $f_3(x)$ and $Q_d f(x)$ and the enlarged image of them.



(a) The graphs of $f_4(x)$ and $Q_d f(x)$ with $h = 0.01$ and $c = 0.001$.

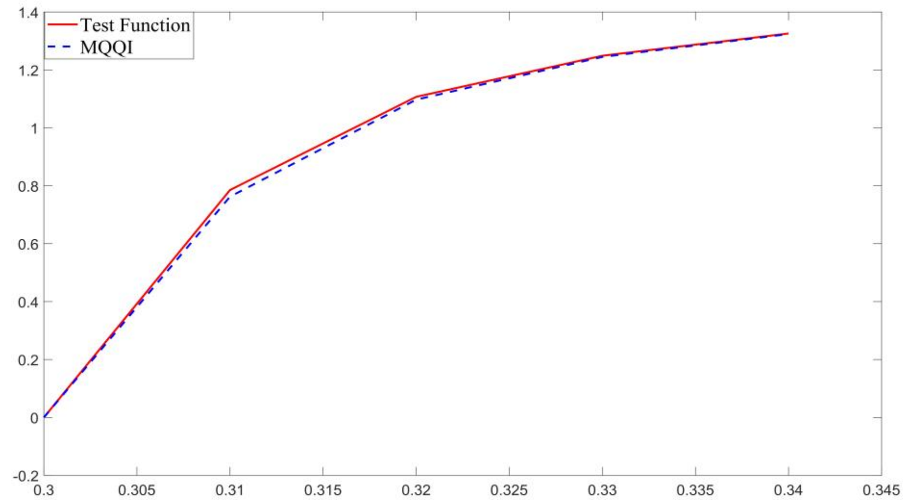


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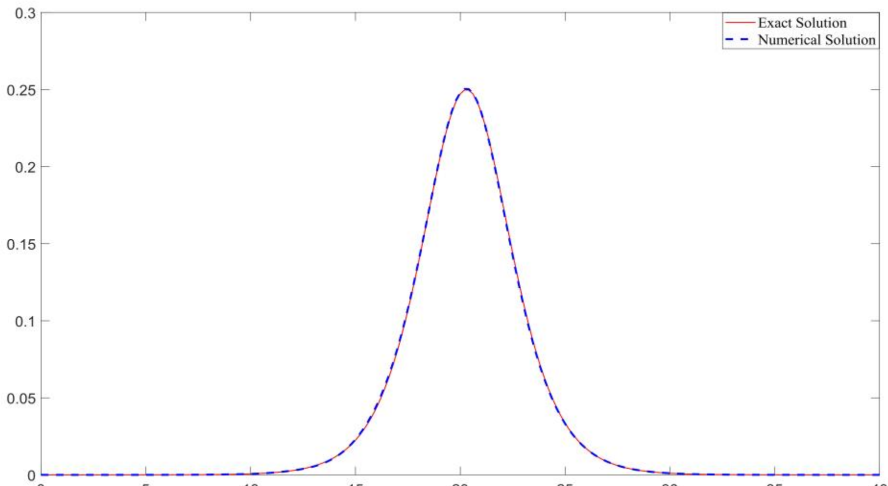
Fig. 4. The graphs of $f_4(x)$ and $Q_d f(x)$ and the enlarged image of them.



(a) The exact and numerical solutions of Korteweg-de Vries (KdV) equation with $c = 0.2799, \tau = 0.001, h = 0.2$ at $t = 1$.

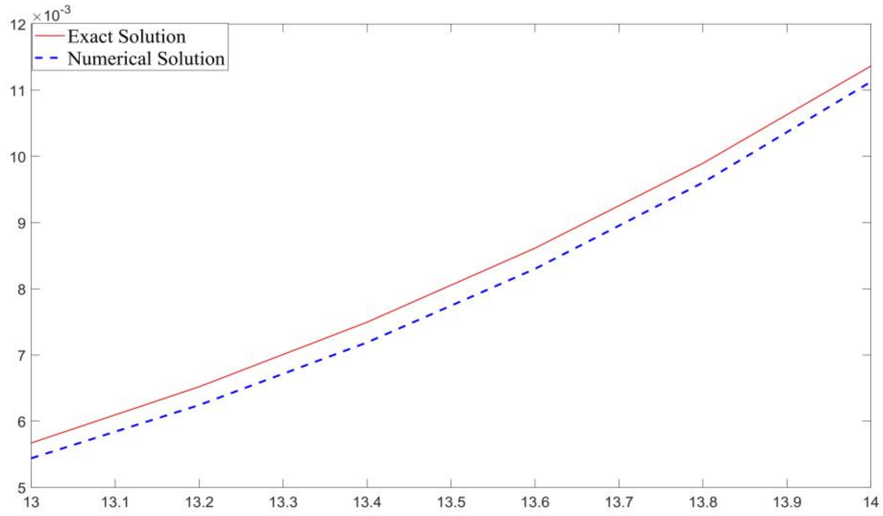


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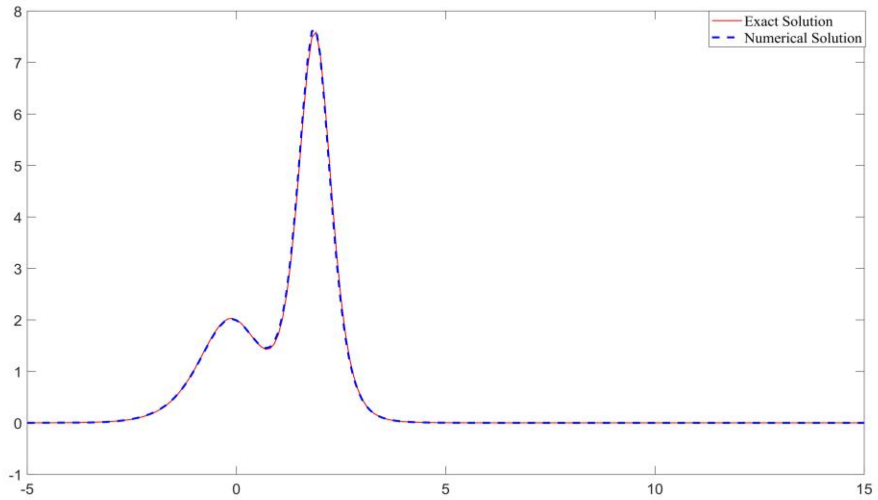
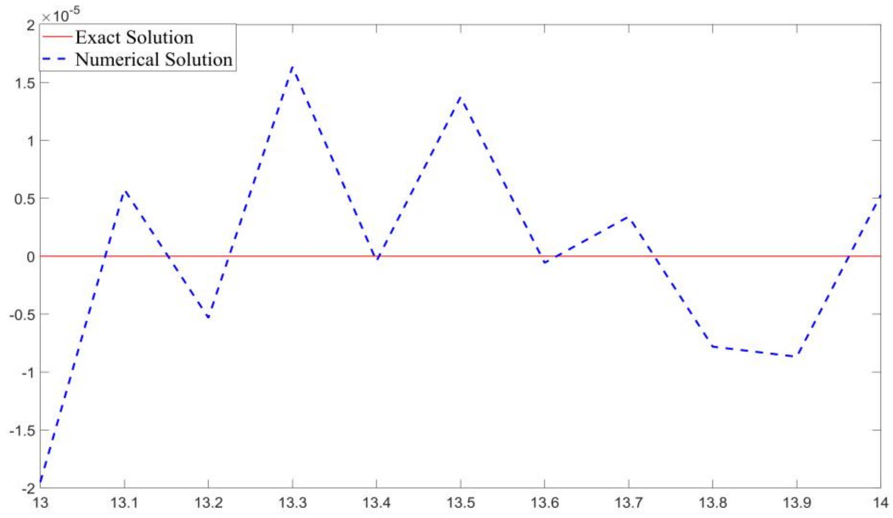
Fig. 5. The exact and numerical solutions of Korteweg-de Vries (KdV) equation and the enlarged image of them.



(a) The exact and numerical solutions of Korteweg-de Vries (KdV) equation with $c = 0.01, \tau = 0.00001, h = 0.1$ at $t = 0.1$.



(b) The locally enlarged image of (a).

Fig. 6. The exact and numerical solutions of Korteweg-de Vries (KdV) equation and the enlarged image of them.

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